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## Probability-Coordinate Analysis of Financial Returns: A Chart-Based Methodology for Tail Behaviour and Boundary Pressure

**Abstract.** *Financial returns are commonly analysed through volatility, quantile-based risk measures and tail diagnostics, all defined in the original value space. We propose a complementary benchmark-relative methodology that represents each return by its position under a chosen cumulative distribution chart. Unlike the classical probability integral transform, uniformity is expected only when the chart is the true distribution function. Under external charts, non-uniformity records the interaction between data and benchmark geometry. The principal descriptor is adaptive excess boundary pressure, which compares observed probability mass near chart boundaries with that generated by a Gaussian reference under the same chart. Daily returns for six assets from January 2015 to June 2026 are examined under Gaussian, Logistic, Student-t and Cauchy charts across pre-COVID, COVID and post-COVID regimes. Adaptive boundary pressure is associated with, but not fully explained by, volatility; its relationship with Hill estimates is weak and threshold-sensitive, and its ordering differs from standard-deviation tail counts. Moving-block bootstrap intervals preserve local temporal dependence and incorporate calibration uncertainty. The results support probability coordinates as an exploratory complement to established financial diagnostics.*

**Keywords:** *financial returns, probability coordinates, heavy tails, boundary pressure, chart-based analysis, financial risk, probability-coordinate fingerprints.*

**JEL Classification:** C14, C46, C58, G17.

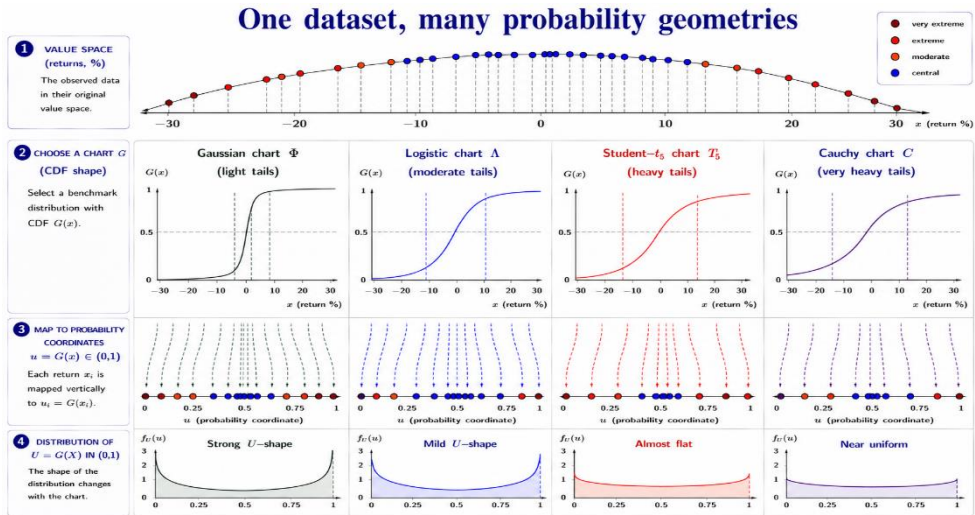
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### 1. Introduction

Financial returns exhibit heavy tails, excess kurtosis and volatility clustering across markets and asset classes (Cont, 2001; Embrechts et al., 1997; Fama, 1965; Mandelbrot, 1963; Tsay, 2010). These features have motivated an extensive toolbox ranging from volatility and higher-order moments to Value-at-Risk, Expected Shortfall and extreme-value methods (Cont, 2001; McNeil et al., 2015).

These methods remain indispensable, but they share a value-space starting point. We ask what can be learned when a return is represented instead by its position relative to a chosen cumulative distribution function. The data are unchanged and only the coordinate system in which they are examined is a design choice.

Different benchmark charts assign different probability coordinates to the same observation. Light-tailed charts place large returns closer to the boundaries of  $(0,1)$ , whereas heavier-tailed charts assign them less extreme positions. Figure 1 illustrates this chart dependence before the formal definitions are introduced. In addition, the illustration provides the conceptual map used throughout the paper: one return sample is viewed through several benchmark probability geometries.



**Figure 1. The same financial return series viewed through different benchmark probability charts (conceptual illustration)**

*Source:* Author's own creation using AI.

Changing  $G$  does not alter the underlying returns, but their probability-coordinate representation. The resulting boundary occupation is therefore explicitly chart-dependent and is interpreted only relative to the stated benchmark geometry.

The classical probability integral transform (PIT) is identified as the special intrinsic case  $G = F_X$ : if  $F_X$  is continuous, then  $F_X(X)$  is  $\text{Uniform}(0,1)$  (Rosenblatt, 1952). In the present framework  $G$  is generally an external benchmark chart. For  $G \neq F_X$ ,  $P(G(X) \leq u) = F_X(G^{-1}(u))$ , which is generally different than  $u$ . We therefore use the term probability-coordinate representation rather than treating every  $G(X)$  as a PIT value.

This interpretation follows the probability-coordinate framework of Cojoclea (2026), where cumulative distribution functions act as coordinate charts. The objective is not to identify one correct chart, but to compare how the same empirical distribution interacts with several reference geometries and to organise the resulting benchmark-relative tail information across a declared chart family.

The paper focuses on adaptive excess boundary pressure: observed boundary occupation is corrected by the occupation obtained when a fixed Gaussian reference distribution is represented through the same chart. The empirical analysis examines its relationship with volatility, the Hill tail index and conventional tail counts, and

separates the effects of robust calibration, chart-induced thresholds and multichart comparison. The Gaussian reference is motivated by the value of studying heavy-tailed behaviour from a light-tailed starting point, although it is not an absolute choice. Other reference distributions, including the Cauchy distribution, could be considered.

The study covers equity indices, commodities, foreign exchange and cryptocurrency returns. Because only 18 asset-regime configurations enter the cross-configuration comparisons, the results are interpreted as structured exploratory evidence.

## 2. Classical Analysis and Its Limitations

Let  $P_t$  be the asset price and  $R_t = (P_t - P_{t-1})/P_{t-1}$  the daily simple return. Simple returns are used because WTI futures reached negative prices in April 2020. Classical volatility, higher-order moments, VaR, ES, density estimates and tail diagnostics remain the value-space baseline. Probability coordinates add a benchmark-explicit view in which the same return may occupy different locations under different charts.

## 3. Probability-Coordinate Representation

The probability-coordinate framework introduced in Section 1 can now be formalized. Rather than representing observations by their numerical values, we represent them by their positions relative to a chosen benchmark cumulative distribution function. This leads naturally to the concept of a probability-coordinate chart.

**Definition 1** (Probability coordinate chart). *Let  $\tilde{G}: \mathbb{R} \rightarrow [0,1]$  be a continuous cumulative distribution function. Suppose there exists a non-empty interval  $I = (a, b) \subseteq \mathbb{R}$ , such that:*

1. *the restriction  $G = \tilde{G}|_I$  is strictly increasing*
2.  *$\lim_{x \rightarrow a^+} G(x) = 0$  and  $\lim_{x \rightarrow b^-} G(x) = 1$ .*

*Then the restriction  $G: I \rightarrow (0,1)$  is called the **probability coordinate chart** induced by  $\tilde{G}$ .*

The probability coordinate chart establishes an invertible correspondence between the interval  $I$  and the probability interval  $(0,1)$  so, if the cumulative distribution function is strictly increasing, the two notions coincide. Consequently, every observation is represented not by its numerical magnitude, but by its probabilistic position relative to the chosen benchmark chart. Since the transformation is strictly increasing, the ordering of the observations is preserved while the geometry of the representation is fundamentally changed.

For a sample  $X_1, \dots, X_n$  and calibration  $\theta = (m, s)$ , define  $U_i = G_\theta(X_i) = G_\theta((X_i - m)/s)$ ,  $i = 1, \dots, n$ , where  $G_\theta$  is the standard chart. The transformation maps all observations to  $(0,1)$ , preserves their ordering and makes the choice of chart and calibration explicit.

**Remark 1** (Intrinsic benchmark). If  $G_\theta = F_X$ , continuity gives  $F_X(X) \sim \text{Uniform}(0,1)$ . This is the classical PIT. For an external benchmark chart, non-uniformity is expected and records the relation between the data distribution and the chosen geometry. So it is not a model failure but a feature.

In empirical finance, the true distribution of returns is unknown. The purpose of using several benchmark charts is therefore not to identify a single correct chart, but to study how the return series behaves relative to different probabilistic reference geometries.

#### 4. Benchmark Charts

We use Gaussian, Logistic, Student-t and Cauchy charts, spanning progressively heavier-tailed geometries. For a standard CDF  $G_0$  and  $\theta = (m,s)$ ,  $G_\theta(x) = G_0((x-m)/s)$ . The Student-t family  $v \in \{3,5,10\}$  supplies the primary neighbouring chart dictionary used in the empirical fingerprints.

A chart  $G$  induces the distance between  $x$  and  $y$  as  $|G(x) - G(y)|$ . Changing the chart therefore changes the metric representation of magnitudes while preserving the order of the returns. For the boundary functional studied below, this geometric change appears concretely through the value-space thresholds obtained by applying the inverse chart at  $\varepsilon$  and  $1-\varepsilon$ .

We use symmetric charts here to summarise total two-sided tail pressure without conflating it with directional asymmetry. This is a modelling choice rather than a claim that financial tails are symmetric. Asymmetric charts or separate lower- and upper-boundary components are natural extensions.

#### 5. Probability-Coordinate Indicators

##### 5.1 Basic Probability-Coordinate Descriptors

The probability-coordinate representation can be summarised through numerical descriptors analogous to those used in classical statistics. Throughout this paper, we consider three fundamental characteristics describing the probability-coordinate representation: its location, its dispersion, and its boundary occupation.

**Definition 2** (Probability-coordinate location). *The probability-coordinate location of  $X$  with respect to  $G$  is  $p_G = \mathbb{E}[G(X)]$ .*

*Its empirical counterpart is  $\hat{p}_G = \frac{1}{n} \sum_{i=1}^n G(X_i)$ .*

**Definition 3** (Probability-coordinate variance). *The probability-coordinate variance of  $X$  with respect to  $G$  is  $V_G = \text{Var}(G(X))$ .*

*Its empirical counterpart is  $\hat{V}_G = \frac{1}{n-1} \sum_{i=1}^n (G(X_i) - \hat{p}_G)^2$ .*

## 5.2 Boundary Pressure and Gaussian-Reference Excess

**Definition 4** (Boundary pressure). For  $\varepsilon \in (0, 1/2)$ , the boundary-pressure indicator is

$$B_G(\varepsilon) = \mathbb{P}(G(X) < \varepsilon) + \mathbb{P}(G(X) > 1 - \varepsilon). \quad (1)$$

Its empirical counterpart is

$$\hat{B}_G(\varepsilon) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{G(X_i) < \varepsilon\}} + \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{G(X_i) > 1 - \varepsilon\}} \quad (2)$$

Boundary pressure is the proportion of observations mapped into the two boundary intervals  $(0, \varepsilon)$  and  $(1 - \varepsilon, 1)$ . A high value therefore means that many observations are extreme relative to the thresholds induced by the stated benchmark chart.

## 5.3 Excess Boundary Pressure

Raw boundary pressure depends not only on the empirical distribution but also on the geometry of the chosen benchmark chart. Consequently, equal values of boundary pressure obtained under different charts are not directly comparable. To isolate the contribution of the data from that of the benchmark geometry, we compare the observed boundary pressure with the corresponding benchmark value.

Under the Gaussian reference distribution, the benchmark boundary pressure associated with the chart  $G$  is defined as

$$B_{\text{benchmark}}(G, \varepsilon) = \mathbb{P}(G(Z) < \varepsilon) + \mathbb{P}(G(Z) > 1 - \varepsilon), \quad Z \sim N(0, 1). \quad (3)$$

**Definition 5** (Excess boundary pressure)

The excess boundary pressure is then defined by

$$\text{Excess}_{B_G}(\varepsilon) = B_G(\varepsilon) - B_{\text{benchmark}}(G, \varepsilon). \quad (4)$$

The estimator of the excess boundary pressure is the plug-in estimator

$$\widehat{\text{Excess}}_{B_G}(\varepsilon) = \hat{B}_G(\varepsilon) - B_{\text{benchmark}}(G, \varepsilon). \quad (5)$$

Positive values indicate more boundary mass than expected under the benchmark, whereas negative values indicate less.

In the empirical analysis,  $\Delta_v(\varepsilon)$  denotes the estimated adaptive excess boundary pressure under the Student-t chart with  $v$  degrees of freedom; equivalently, it is the empirical boundary pressure under that chart minus the corresponding Gaussian-reference benchmark.

The Gaussian-reference term is not generally  $2\varepsilon$ . It equals  $2\varepsilon$  only for the Gaussian chart  $G = \Phi$ . For an arbitrary strictly increasing chart,  $B_{\text{benchmark}}(G, \varepsilon) = \Phi(G^{-1}(\varepsilon)) + 1 - \Phi(G^{-1}(1 - \varepsilon))$ . Thus, the paper compares the observed distribution with a Gaussian reference represented under the same chart.

The theoretical benchmark is fixed once  $G$  and  $\varepsilon$  are chosen. Sampling uncertainty from estimated median-MAD calibration is not absorbed into this term, but it is propagated by recalculating the calibration within every moving-block bootstrap resample.

## 5.4 Relationship with Conventional Tail-Exceedance Probabilities

Let the lower and upper chart thresholds be the inverse-chart values at  $\varepsilon$  and  $1-\varepsilon$ , respectively. Since  $G$  is strictly increasing, the boundary event can be written exactly in the original value space:

$$B_G(\varepsilon) = \mathbb{P}(X < q_G^-(\varepsilon)) + \mathbb{P}(X > q_G^+(\varepsilon)).$$

Consequently, boundary pressure is a conventional two-sided tail-exceedance probability evaluated at chart-induced thresholds. For a symmetric location-scale chart, let the positive cut-off be the inverse of the standard chart at  $1-\varepsilon$ . Then:

$$B_{G_\theta}(\varepsilon) = \mathbb{P}\left(\left|\frac{X-m}{s}\right| > c_G(\varepsilon)\right).$$

Under adaptive median–MAD calibration, the corresponding empirical excess form is:

$$\hat{\Delta}_G^{\text{ad}}(\varepsilon) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\left\{\left|\frac{X_i - \hat{m}}{\hat{s}}\right| > c_G(\varepsilon)\right\} - 2[1 - \Phi(c_G(\varepsilon))].$$

Thus, for one fixed chart  $G$  and one fixed boundary level  $\varepsilon$ , adaptive excess boundary pressure is an affine recentering of the matched robust standardised tail-exceedance frequency. It contains no additional sample information beyond that exactly matched count. The methodological contribution lies instead in the explicit Gaussian-reference correction and the coordinated comparison across a declared chart family.

The vector  $(\Delta_3, \Delta_5, \Delta_{10})$  samples the robust two-sided survival profile at three chart-induced cut-offs, producing a benchmark-corrected finite-tail fingerprint. At  $\varepsilon = 0.05$ , the Student- $t_3$ , Student- $t_5$  and Student- $t_{10}$  charts induce positive cut-offs of approximately 2.353, 2.015 and 1.812 robust standardised units, respectively.

Contrasts among the three coordinates therefore show whether benchmark-relative excess is concentrated at the more extreme cut-off or persists as the cut-off moves inward. This cross-cut-off profile is the information retained by the multichart fingerprint beyond any one of its component counts.

## 6. Empirical Methodology

### 6.1 Data

Daily price data were collected from Yahoo Finance for six representative financial assets spanning major classes of financial markets: the S&P 500 index, the NASDAQ Composite index, Bitcoin, Gold futures, WTI crude oil futures, and the EUR/USD exchange rate. Together, these assets provide a diverse collection of return dynamics, ranging from relatively stable exchange-rate movements to highly volatile cryptocurrency and commodity markets.

For each asset, we retrieved the daily adjusted closing price from Yahoo Finance. The observation period begins on 1 January 2015 and ends between 17 and 21 June 2026, depending on the final available observation for each asset in the manuscript dataset. Exact sample sizes vary with trading calendars and data availability; missing observations were removed prior to the analysis.

Daily simple returns were computed as  $R_t = (P_t - P_{t-1})/P_{t-1}$ . Simple rather than logarithmic returns were used because WTI crude oil futures reached negative prices during the market disruption of April 2020, making logarithmic returns undefined. Using simple returns allowed all selected assets, including WTI, to be analysed consistently over the full observation period.

To investigate the effect of the COVID-19 market disruption, the sample was partitioned into three economically meaningful market regimes. The *Pre-COVID* period comprises all observations prior to 20 February 2020. The *COVID* period extends from 20 February 2020 through 31 December 2020 and captures the acute phase of the pandemic-induced financial turmoil. The remaining observations, from 1 January 2021 onwards, constitute the *Post-COVID* period. This partition allows us to compare both classical and probability-coordinate descriptors before, during, and after the initial market shock.

## 6.2 Classical Descriptive Statistics

Classical analysis comprised descriptive statistics, graphical diagnostics, and normality assessment. For each asset and market regime we computed summary statistics, Value-at-Risk, Expected Shortfall, kernel density estimates, boxplots, Q-Q plots, and Kolmogorov–Smirnov/Lilliefors tests. These analyses provide the value-space baseline against which the proposed probability-coordinate methodology is compared.

## 6.3 Probability-Coordinate Analysis

The probability-coordinate analysis uses the benchmark charts from Section 4, with the regime results centred on Student charts  $v \in \{3, 5, 10\}$ . The neighbouring Student family allows the chart geometry to vary while all other elements of the transformation remain fixed.

For every chart we computed probability-coordinate location, variance, boundary pressure and adaptive excess boundary pressure for the complete sample and for each market regime.

Adaptive calibration was performed separately within each asset–regime configuration:  $\hat{m} = \text{median}(X_1, \dots, X_n)$ ,  $\hat{s} = \text{MAD}(X_1, \dots, X_n)/\Phi^{-1}(0.75)$ , and  $U_i = G_0((X_i - \hat{m})/\hat{s})$ . Full-sample summaries were calibrated separately by asset. The same  $\hat{m}$  and  $\hat{s}$  were used across charts, so the Student-chart fingerprint changes  $G_0$ , not the standardised data. Under fixed calibration, the common values  $m_0 = 0$  and  $s_0 = 0.01$  preserve differences in return magnitude across assets.

Unless otherwise stated, the boundary level is  $\varepsilon = 0.05$  and adaptive calibration is the primary scheme.

## 7. Empirical Results

### 7.1 Daily Return Series

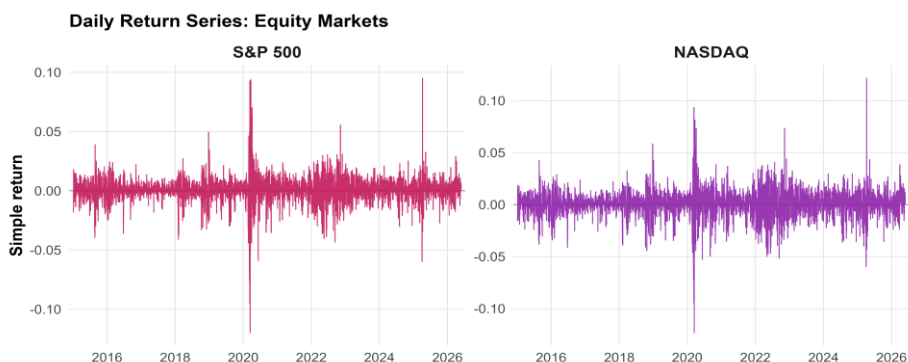
We begin with a direct examination of the daily simple return series. Before introducing probability-coordinate representations and probability-geometric descriptors, it is useful to understand the raw behaviour of the assets in value space. Visual inspection provides a first indication of volatility clustering, market disruptions, and extreme observations that may later influence both classical and probability-coordinate diagnostics.

To facilitate comparison, the assets are grouped into three economically meaningful pairs. The first pair consists of the S&P 500 and NASDAQ, representing broad equity markets. The second pair contains Gold and EUR/USD, two traditional financial assets frequently associated with safe-haven behaviour and macroeconomic conditions. The third pair contains Bitcoin and WTI crude oil, assets that have experienced some of the most pronounced market dislocations during the sample period.

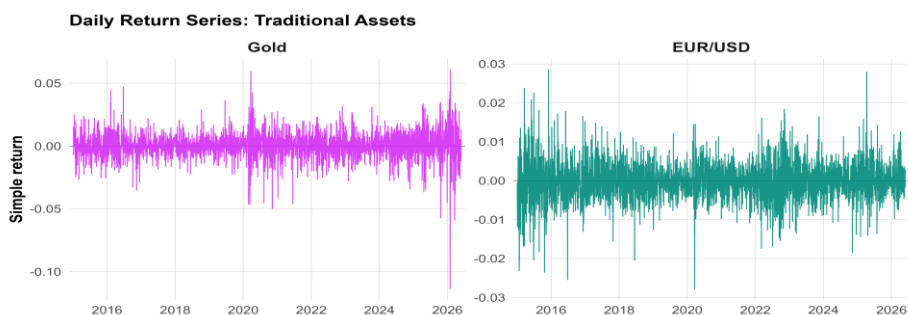
Figures 2–4 display the daily simple return series for all six assets. Each asset is shown on its own vertical scale to preserve visibility of both ordinary fluctuations and extreme events. This choice is particularly important because the magnitude of return variability differs substantially across asset classes.

Several features are immediately visible. Equity markets exhibit periods of elevated volatility associated with major market events, most notably during the COVID-19 period. Gold and EUR/USD display comparatively moderate fluctuations, although episodes of increased uncertainty remain apparent. Bitcoin exhibits substantially larger day-to-day variability than the traditional assets, while WTI crude oil contains the extraordinary negative-price episode of April 2020, which generated one of the most extreme return observations in the sample.

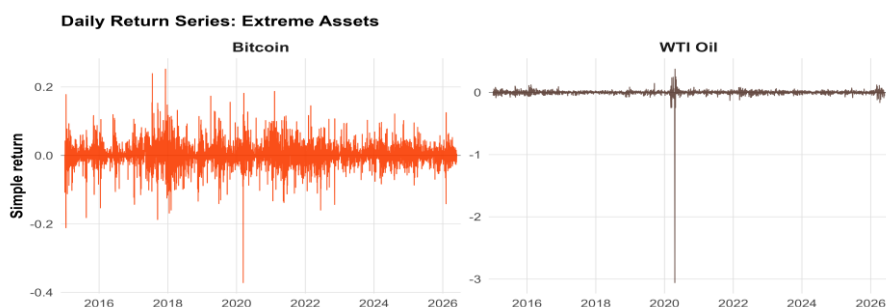
At this stage, the figures serve a descriptive purpose. They illustrate the diversity of return behaviour present in the dataset and motivate the subsequent search for descriptors capable of distinguishing different forms of tail activity beyond simple measures of volatility.



**Figure 2. Daily simple returns for the S&P 500 and NASDAQ indices**  
*Source:* Author's computations using R Studio, based on Yahoo Finance data.



**Figure 3. Daily simple returns for Gold and EUR/USD**  
*Source:* Author's computations using R Studio, based on Yahoo Finance data.



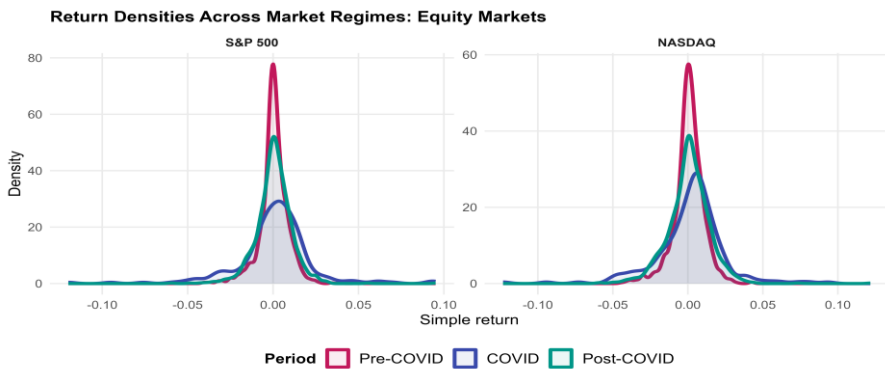
**Figure 4. Daily simple returns for Bitcoin and WTI crude oil**  
*Source:* Author's computations using R Studio, based on Yahoo Finance data.

## 7.2 Distributional Analysis Across Market Regimes

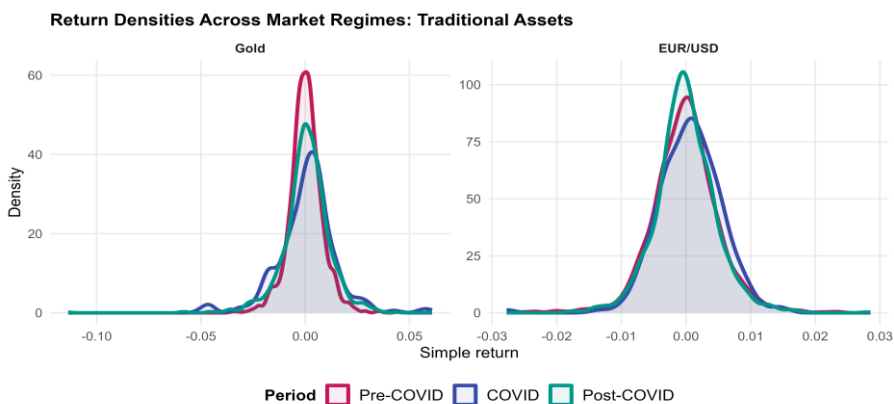
While the return series illustrate the temporal evolution of financial markets, the corresponding return densities reveal changes in the underlying distribution across market regimes. Figures 5-7 compare the empirical return densities during the Pre-COVID, COVID, and Post-COVID periods.

The COVID regime generally exhibits wider tails and lower central peaks, reflecting increased uncertainty and a greater frequency of extreme observations. In contrast, the pre-COVID period is typically more concentrated, while the post-COVID period occupies an intermediate position. Across all assets, the principal differences arise from changes in dispersion and tail behaviour rather than shifts in location, with sample means and medians remaining close to zero.

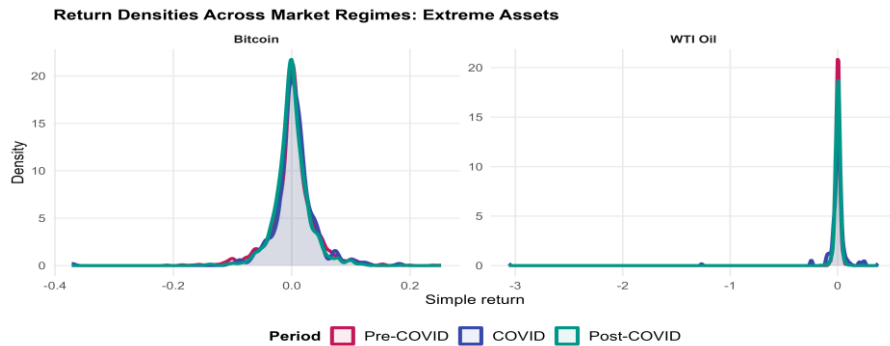
Normality was assessed using the Lilliefors correction of the Kolmogorov–Smirnov test. The null hypothesis was rejected for the complete return series of all six assets and for every market regime except EUR/USD during the COVID period ( $p = 0.605$ ). These results indicate that the observed non-Gaussian behaviour cannot be attributed solely to the aggregation of heterogeneous market regimes and provide statistical support for the probability-coordinate analysis developed in the following sections.



**Figure 5. Return densities across the three market regimes for the equity assets**  
*Source:* Author's computations using R Studio, based on Yahoo Finance data.



**Figure 6. Return densities across the three market regimes for Gold and EUR/USD**  
*Source:* Author's computations using R Studio, based on Yahoo Finance data.



**Figure 7. Return densities across the three market regimes for Bitcoin and WTI crude oil**

*Source:* Author's computations using R Studio, based on Yahoo Finance data.

### 7.3 Descriptive Statistics and Tail Event Frequencies

Following the graphical analysis, towards establishing a classical statistical baseline, Table 1 summarises the principal classical descriptive statistics.

**Table 1. Classical Distributional Diagnostics across Market Regimes**

| Asset   | Period     | <i>n</i> | SD     | Skewness | Kurtosis | VaR <sub>5</sub> | ES <sub>5</sub> |
|---------|------------|----------|--------|----------|----------|------------------|-----------------|
| Bitcoin | Pre-COVID  | 1875     | 0.0386 | 0.111    | 8.38     | -0.0604          | -0.0921         |
| Bitcoin | COVID      | 316      | 0.0391 | -2.297   | 30.83    | -0.0464          | -0.0855         |
| Bitcoin | Post-COVID | 1996     | 0.0302 | 0.126    | 6.90     | -0.0462          | -0.0694         |
| EUR/USD | Pre-COVID  | 1333     | 0.0052 | 0.068    | 5.94     | -0.0078          | -0.0116         |
| EUR/USD | COVID      | 226      | 0.0049 | -0.703   | 7.26     | -0.0066          | -0.0099         |
| EUR/USD | Post-COVID | 1418     | 0.0047 | 0.213    | 4.99     | -0.0074          | -0.0102         |
| Gold    | Pre-COVID  | 1284     | 0.0081 | 0.240    | 5.92     | -0.0130          | -0.0179         |
| Gold    | COVID      | 220      | 0.0143 | -0.036   | 6.28     | -0.0202          | -0.0333         |
| Gold    | Post-COVID | 1370     | 0.0115 | -0.978   | 12.23    | -0.0179          | -0.0278         |
| NASDAQ  | Pre-COVID  | 1290     | 0.0102 | -0.431   | 6.05     | -0.0164          | -0.0256         |
| NASDAQ  | COVID      | 220      | 0.0238 | -0.667   | 8.64     | -0.0379          | -0.0596         |
| NASDAQ  | Post-COVID | 1370     | 0.0141 | 0.170    | 8.10     | -0.0229          | -0.0320         |
| S&P 500 | Pre-COVID  | 1290     | 0.0084 | -0.462   | 6.78     | -0.0143          | -0.0216         |
| S&P 500 | COVID      | 220      | 0.0231 | -0.510   | 9.69     | -0.0340          | -0.0593         |
| S&P 500 | Post-COVID | 1370     | 0.0106 | 0.125    | 9.84     | -0.0164          | -0.0242         |
| WTI Oil | Pre-COVID  | 1286     | 0.0238 | 0.344    | 6.04     | -0.0389          | -0.0523         |
| WTI Oil | COVID      | 220      | 0.2322 | -10.984  | 139.45   | -0.0829          | -0.5143         |
| WTI Oil | Post-COVID | 1370     | 0.0251 | -0.537   | 7.02     | -0.0381          | -0.0603         |

*Source:* Author's computations based on Yahoo Finance data.

Table 1 confirms the substantial regime dependence of financial returns. Volatility and kurtosis increase markedly during the COVID period, particularly for the equity indices and WTI crude oil, while Bitcoin remains persistently heavy-tailed across all regimes.

**Table 2. Standard-Deviation Tail-Event Frequencies across Market Regimes.  
Gaussian Benchmark Probabilities are Shown Beneath each Threshold**

| Asset   | Period     | <i>n</i> | P( R  > 2SD) (%)<br>Gaussian: 4.55% | P( R  > 3SD) (%)<br>Gaussian: 0.27% | P( R  > 4SD) (%)<br>Gaussian: 0.006% |
|---------|------------|----------|-------------------------------------|-------------------------------------|--------------------------------------|
| Bitcoin | Pre-COVID  | 1875     | 6.13                                | 1.76                                | 0.69                                 |
| Bitcoin | COVID      | 316      | 4.43                                | 1.27                                | 0.63                                 |
| Bitcoin | Post-COVID | 1996     | 5.81                                | 2.05                                | 0.50                                 |
| EUR/USD | Pre-COVID  | 1333     | 4.95                                | 1.35                                | 0.45                                 |
| EUR/USD | COVID      | 226      | 3.54                                | 0.44                                | 0.44                                 |
| EUR/USD | Post-COVID | 1418     | 5.08                                | 1.13                                | 0.07                                 |
| Gold    | Pre-COVID  | 1284     | 5.69                                | 1.17                                | 0.47                                 |
| Gold    | COVID      | 220      | 5.45                                | 2.73                                | 0.45                                 |
| Gold    | Post-COVID | 1370     | 5.69                                | 1.17                                | 0.36                                 |
| NASDAQ  | Pre-COVID  | 1290     | 5.97                                | 1.24                                | 0.47                                 |
| NASDAQ  | COVID      | 220      | 5.00                                | 2.73                                | 0.45                                 |
| NASDAQ  | Post-COVID | 1370     | 4.74                                | 0.88                                | 0.29                                 |
| S&P 500 | Pre-COVID  | 1290     | 5.66                                | 1.40                                | 0.54                                 |
| S&P 500 | COVID      | 220      | 5.45                                | 2.73                                | 1.82                                 |
| S&P 500 | Post-COVID | 1370     | 5.11                                | 1.02                                | 0.36                                 |
| WTI Oil | Pre-COVID  | 1286     | 5.60                                | 1.24                                | 0.31                                 |
| WTI Oil | COVID      | 220      | 0.91                                | 0.91                                | 0.91                                 |
| WTI Oil | Post-COVID | 1370     | 4.89                                | 1.24                                | 0.58                                 |

*Source:* Author's computations based on Yahoo Finance data.

At the 2SD threshold, most configurations remain relatively close to the Gaussian benchmark of 4.55%, whereas the frequencies become more differentiated at 3SD and 4SD. The rounded 3SD frequencies in Table 2 provide the classical ranking used in Table 5.

Standard-deviation tail counts tell a different story from boundary pressure. Most strikingly, WTI-COVID records only 0.91% of returns beyond two regime-specific standard deviations because the exceptional April 2020 observation inflates the threshold itself.

#### 7.4 Probability-Coordinate Analysis

The classical analysis establishes that the return distributions are non-Gaussian, heavy-tailed, and strongly regime dependent. However, all classical descriptors quantify extremeness directly in the original value space. In contrast, adaptive excess boundary pressure evaluates how observations occupy the boundary regions of a benchmark-induced probability-coordinate system after robust location-scale calibration.

The empirical discussion uses Student charts  $v \in \{3, 5, 10\}$  under adaptive calibration and focuses on  $\Delta v(0.05)$ . Fixed calibration is retained only as a secondary magnitude-sensitive comparison.

## 8. What Does Adaptive Excess Boundary Pressure Measure?

### 8.1 Relationship with Classical Diagnostics

The comparisons below use only 18 asset–regime configurations and are exploratory. With all configurations included, adaptive excess boundary pressure is positively associated with standard deviation, although the linear relationship is relatively weak (Pearson  $r = 0.370$ ,  $R^2 = 0.137$ ,  $p = 0.131$ ; Spearman  $\rho = 0.638$ ,  $p = 0.004$ ). WTI-COVID is an influential leverage point. After excluding it, Pearson  $r$  increases to 0.660 ( $R^2 = 0.436$ ,  $p = 0.004$ ), while Spearman  $\rho$  remains positive (0.584,  $p = 0.014$ ). Volatility is therefore related to the descriptor but does not fully summarise its cross-configuration variation.

Associations with the Hill estimator are weak and unstable across the 10%, 5% and 2% thresholds (Pearson correlations from  $-0.268$  to  $0.130$ ; Spearman correlations from  $-0.371$  to  $-0.029$ ). This is consistent with the different targets of the two quantities: the Hill estimator estimates asymptotic Pareto-type tail decay from upper-order statistics (Clauaset et al., 2009; Newman, 2005), whereas  $\Delta v$  is a two-sided finite-threshold functional. The comparison supports complementarity in this sample and does not establish the superiority of either descriptor.

The classical comparator used here is the frequency beyond two regime-specific sample standard deviations, which is essentially unrelated to  $\Delta s$  (Pearson  $r = 0.067$ ,  $p = 0.792$ ; Spearman  $\rho = 0.252$ ,  $p = 0.313$ ), consistent with the threshold-inflation effect noted for WTI-COVID in Section 7.3. This is not the matched robust count in Section 5.4: it uses mean–SD calibration and a fixed multiplier of 2, whereas  $\Delta s$  uses median–MAD calibration, the 0.95 quantile of the Student- $t_s$  distribution (approximately 2.015), and Gaussian-reference recentering. The empirical difference therefore identifies the consequences of calibration and threshold construction, but it is not evidence that  $\Delta s$  contains information beyond the exactly matched robust count.

### 8.2 Robustness Across Student Charts

Table 3 reports the mean and range of  $\Delta v$  across the neighbouring Student charts  $v \in \{3, 5, 10\}$ : the asset-level ordering is preserved, and the within-asset ranges remain small. At  $\varepsilon = 0.05$ , the three components correspond to robust exceedance frequencies at the 0.95 quantiles of the three Student distributions, each recentered by its Gaussian-reference probability. The observed stability therefore concerns the ordering of the finite-tail profile across neighbouring cut-offs; it is local robustness, not chart invariance.

**Table 3. Mean and range of adaptive excess boundary pressure across neighbouring Student charts**

| Asset   | Mean $\Delta v$ | Range  |
|---------|-----------------|--------|
| Bitcoin | 0.112           | 0.0170 |
| S&P 500 | 0.0827          | 0.0195 |
| NASDAQ  | 0.0785          | 0.0136 |
| Gold    | 0.0666          | 0.0154 |
| WTI Oil | 0.0456          | 0.0008 |
| EUR/USD | 0.0346          | 0.0055 |

*Source:* Author's computations based on Yahoo Finance data.

### 8.3 Statistical Uncertainty

Table 4 reports 95% percentile intervals from 5,000 moving-block bootstrap replications (Künsch, 1989). Within each configuration, overlapping blocks are resampled using a block length  $\ell$  equal to the smallest integer not below the cube root of  $n$ ; the median, MAD, Student- $t_s$  coordinates and  $\Delta_s$  are recalculated. Only EUR/USD during COVID has an interval containing zero; this pattern is unchanged for block lengths 5, 10 and 20.

**Table 4. Adaptive  $\Delta_s$  with 95% moving-block bootstrap percentile intervals**

| Asset   | Regime     | Adaptive $\Delta_s$ | MBB 95% CI       |
|---------|------------|---------------------|------------------|
| Bitcoin | Pre-COVID  | 0.140               | (0.114, 0.159)   |
| Bitcoin | COVID      | 0.124               | (0.080, 0.165)   |
| Bitcoin | Post-COVID | 0.090               | (0.075, 0.106)   |
| EUR/USD | Pre-COVID  | 0.041               | (0.021, 0.057)   |
| EUR/USD | COVID      | 0.005               | (-0.031, 0.031)† |
| EUR/USD | Post-COVID | 0.040               | (0.024, 0.061)   |
| Gold    | Pre-COVID  | 0.066               | (0.047, 0.086)   |
| Gold    | COVID      | 0.088               | (0.029, 0.152)   |
| Gold    | Post-COVID | 0.060               | (0.039, 0.083)   |
| NASDAQ  | Pre-COVID  | 0.082               | (0.058, 0.106)   |
| NASDAQ  | COVID      | 0.106               | (0.047, 0.174)   |
| NASDAQ  | Post-COVID | 0.055               | (0.033, 0.082)   |
| S&P 500 | Pre-COVID  | 0.105               | (0.079, 0.129)   |
| S&P 500 | COVID      | 0.106               | (0.042, 0.170)   |
| S&P 500 | Post-COVID | 0.063               | (0.036, 0.087)   |
| WTI Oil | Pre-COVID  | 0.052               | (0.032, 0.073)   |
| WTI Oil | COVID      | 0.111               | (0.052, 0.174)   |
| WTI Oil | Post-COVID | 0.033               | (0.012, 0.054)   |

*Note:* † The 95% moving-block bootstrap interval includes zero; adaptive excess boundary pressure is not distinguishable from the Gaussian-reference value for this configuration at the stated interval level.

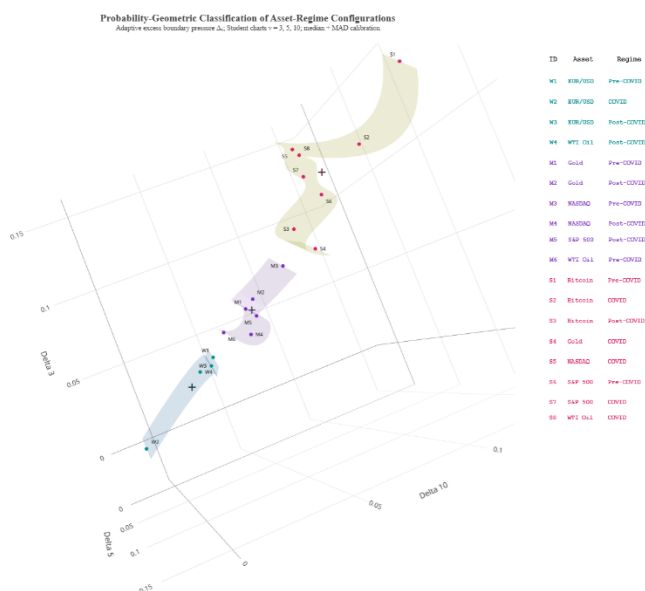
*Source:* Author's computations based on Yahoo Finance data.

### 8.4 Probability-Geometric Classification of Asset-Regime Configurations

Each configuration is represented by its Student-chart fingerprint ( $\Delta_3, \Delta_5, \Delta_{10}$ ), interpreted here as a benchmark-corrected finite-tail profile sampled at three Student-chart cut-offs. Before clustering, each coordinate was standardised across the 18 configurations to zero mean and unit sample variance.

The classification problem is therefore not posed as a search for permanent asset types. It asks whether the 18 asset-regime fingerprints can be organised into a small number of recurring geometric patterns. The construction arose from the empirical observation that  $\Delta_3$ ,  $\Delta_5$  and  $\Delta_{10}$  vary coherently across neighbouring Student charts: rather than treating the three coordinates as separate rankings, we retain each triplet as a single point in the descriptor space.

The reported classification was obtained by k-means with  $k = 3$ , 1,000 random starts and random seed 2406. The three raw clusters were ordered by their mean pressure, computed as the mean of the three unstandardised  $\Delta$  coordinates, and were then labelled weak, moderate and strong tail-pressure geometry. The choice  $k = 3$  provides an interpretable descriptive partition, but it is not presented as a population-optimal number of classes. With only 18 configurations, the labels summarise this sample and do not define a validated taxonomy. Because the coordinate standardisation and cluster centroids are estimated from the same 18 configurations, the assignments are sample-dependent: adding new configurations may change both the scaling and the centroids. No out-of-sample classification rule or predictive accuracy is claimed.



**Figure 8. Probability-geometric classification of the 18 asset-regime configurations in the descriptor space ( $\Delta_3, \Delta_5, \Delta_{10}$ ). Each point represents one asset within one market regime under adaptive median-MAD calibration**

*Source:* Author's computations using R Studio, based on Yahoo Finance data.

The resulting classes were difficult to display with standard cluster graphics. Ellipsoids, convex hulls and volumetric envelopes suggest that a class fills a compact three-dimensional region, whereas the observed points lie along narrow trajectories. To quantify this feature, a separate principal component analysis was performed within each class on the centred but unscaled original fingerprints. The first principal component explains 99.7% of within-class variation for the weak class, 89.6% for the moderate class and 86.2% for the strong class, supporting a trajectory-like rather than volumetric representation.

Figure 8 therefore uses semi-transparent ribbons aligned with the first within-class principal-component direction. The arbitrary sign of each direction was oriented so that its  $\Delta_s$  component is positive. Only this centreline direction is data-derived. Ribbon width and transparency are graphical choices used to keep the trajectories visible and separate. Accordingly, the ribbons are not confidence regions, decision boundaries or estimates of population support.

Bitcoin occupies the strong region across regimes, whereas EUR/USD lies in the weak region. Several COVID configurations move towards stronger boundary pressure. This ordering differs from the classical 3SD tail hierarchy, most visibly for WTI-COVID. Because the two rankings use different calibrations and cut-offs, their comparison shows the practical consequences of those choices but does not establish informational independence.

**Table 5. Classical 3SD Exceedance-Frequency and Adaptive  $\Delta_s$  Rankings across the 18 Asset-Regime Configurations**

| Asset-Regime       | Classical Rank | Probability Rank | Difference |
|--------------------|----------------|------------------|------------|
| Bitcoin Pre-COVID  | 5              | 1                | 4          |
| Bitcoin COVID      | 8              | 2                | 6          |
| Bitcoin Post-COVID | 4              | 7                | -3         |
| EUR/USD Pre-COVID  | 7              | 15               | -8         |
| EUR/USD COVID      | 18             | 18               | 0          |
| EUR/USD Post-COVID | 14             | 16               | -2         |
| Gold Pre-COVID     | 12             | 10               | 2          |
| Gold COVID         | 1              | 8                | -7         |
| Gold Post-COVID    | 13             | 12               | 1          |
| NASDAQ Pre-COVID   | 9              | 9                | 0          |
| NASDAQ COVID       | 2              | 4                | -2         |
| NASDAQ Post-COVID  | 17             | 13               | 4          |
| S&P 500 Pre-COVID  | 6              | 6                | 0          |
| S&P 500 COVID      | 3              | 5                | -2         |
| S&P 500 Post-COVID | 15             | 11               | 4          |
| WTI Oil Pre-COVID  | 10             | 14               | -4         |
| WTI Oil COVID      | 16             | 3                | 13         |
| WTI Oil Post-COVID | 11             | 17               | -6         |

*Source:* Author's computations based on Yahoo Finance data.

Table 5 confirms that the probability-coordinate and classical 3SD hierarchies are not identical. The difference establishes a distinct empirical ordering in this sample, but rank disagreement alone does not prove that either representation contains information absent from the other. It reflects, at least in part, the replacement of mean–SD standardisation and a 3SD cut-off by robust median–MAD calibration and a chart-induced cut-off.

### 8.5 Geometric Role of Boundary Occupation

Among the three probability-coordinate descriptors examined--location, variance and boundary occupation-- the boundary component displayed the clearest variation across asset–regime configurations. Table 6 reports the three descriptors under the Student- $t_s$  chart at  $\varepsilon = 0.05$  with adaptive median–MAD calibration.

**Table 6. Probability-Coordinate Descriptors under the Student- $t_s$  chart at  $\varepsilon = 0.05$  with Adaptive Median–MAD Calibration**

| Asset   | Regime     | $p_G$ | $V_G$ | $B_G$ |
|---------|------------|-------|-------|-------|
| Bitcoin | Pre-COVID  | 0.506 | 0.090 | 0.183 |
| Bitcoin | COVID      | 0.517 | 0.087 | 0.168 |
| Bitcoin | Post-COVID | 0.507 | 0.084 | 0.134 |
| EUR/USD | Pre-COVID  | 0.500 | 0.078 | 0.085 |
| EUR/USD | COVID      | 0.499 | 0.072 | 0.049 |
| EUR/USD | Post-COVID | 0.506 | 0.077 | 0.084 |
| Gold    | Pre-COVID  | 0.504 | 0.080 | 0.110 |
| Gold    | COVID      | 0.489 | 0.085 | 0.132 |
| Gold    | Post-COVID | 0.505 | 0.080 | 0.104 |
| NASDAQ  | Pre-COVID  | 0.503 | 0.083 | 0.126 |
| NASDAQ  | COVID      | 0.473 | 0.084 | 0.150 |
| NASDAQ  | Post-COVID | 0.497 | 0.079 | 0.099 |
| S&P 500 | Pre-COVID  | 0.507 | 0.086 | 0.149 |
| S&P 500 | COVID      | 0.482 | 0.082 | 0.150 |
| S&P 500 | Post-COVID | 0.497 | 0.081 | 0.107 |
| WTI Oil | Pre-COVID  | 0.488 | 0.079 | 0.096 |
| WTI Oil | COVID      | 0.495 | 0.085 | 0.155 |
| WTI Oil | Post-COVID | 0.491 | 0.076 | 0.077 |

*Source:* Author's computations based on Yahoo Finance data.

Table 6 supports this choice quantitatively. Probability-coordinate location remains close to 0.5 after median-based centring, while the coefficient of variation of boundary occupation is approximately 14 times that of location and 5.5 times that of probability-coordinate variance. Variance and boundary occupation are strongly associated in this sample (Pearson  $r = 0.966$ ; Spearman  $\rho = 0.950$ ). These descriptive

comparisons motivated the focus on adaptive excess boundary pressure; they are not presented as a formal feature-selection test. At a single boundary level, boundary pressure remains an exceedance probability, while the geometric role of  $G$  is to select and coordinate thresholds across the chart family.

## 9. Limitations

The study contains six assets and 18 asset–regime configurations; correlations and clustering are exploratory. Results depend on the regime split, chart, boundary level and Gaussian reference. The charts are symmetric, while financial tails may not be. Moving-block intervals preserve local dependence and recalculated calibration, but remain conditional on block length and finite-sample conditions. The principal fingerprint and classification fix  $\varepsilon = 0.05$ . Stability across  $v \in \{3, 5, 10\}$  is therefore sensitivity to neighbouring chart geometries at one boundary level, not robustness to the choice of  $\varepsilon$ . The regime boundaries are imposed exogenously. Reported correlation p-values are unadjusted and interpreted descriptively and no multiple-testing correction is applied.

For a fixed chart and boundary level, adaptive excess boundary pressure is a one-to-one affine transformation of the matched robust tail-exceedance count. Claims of added value therefore concern benchmark-relative interpretation, coordinated chart comparison and the resulting fingerprint, rather than new raw sample information at a single threshold.

## 10. Conclusions

This paper develops a benchmark-relative probability-coordinate methodology for financial returns. The classical PIT appears only in the intrinsic case  $G = F_X$ , the empirical framework deliberately uses external charts, so non-uniformity is part of the information rather than a defect.

Adaptive excess boundary pressure compares observed boundary occupation with the occupation generated by a fixed Gaussian reference distribution under the same chart. For a fixed chart and boundary level, it is a recentered robust standardised tail-exceedance probability. The chart-specific Gaussian reference equals  $2\varepsilon$  only when  $G$  is Gaussian.

Across the configurations studied, the descriptor is related to but not fully summarised by volatility and targets a different feature from the Hill estimator. Its empirical ordering differs from mean–SD tail counts because the calibration and cut-offs differ. The multichart vector provides a stable local, benchmark-corrected finite-tail profile and is proposed as an exploratory complement to established financial diagnostics.

Moving-block bootstrap intervals retain the principal zero-exclusion pattern after allowing for local temporal dependence and recalculated calibration. The three-group fingerprint classification remains exploratory; with only 18 configurations, it is not a population taxonomy.

Future work should examine asymmetric boundary components, alternative reference distributions, alternative block-selection schemes and dynamic or multivariate fingerprints. These extensions would test which conclusions remain stable across charts, samples and market conditions.

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**Data and code availability:** Daily market data were obtained from Yahoo Finance as described in Section 6.1. The R code used to produce the numerical results, tables and data-driven figures is available at <https://github.com/simonacojocea/Probability-Coordinate-Analysis-of-Financial-Returns>.

A tagged release corresponding to the submitted version will be archived upon acceptance.

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