Imran NASIR, PhD Candidate
inasir@iba.edu.pk
School of Mathematics and Computer Science, Institute of Business Administration Karachi, University of Karachi, Karachi, Pakistan

Muhammad SHERAZ, PhD (corresponding author)
msheraz@iba.edu.pk
School of Mathematics and Computer Science, Institute of Business Administration Karachi, Karachi, Pakistan

Lévy Stochastic Models, GARCH-Type Volatility, and Heavy-Tailed Innovations: Pricing European Options under Randomized Quasi-Monte Carlo Methods

Abstract. *In this paper, we propose a new hybrid framework to evaluate European call option pricing using Monte Carlo (MC) and Randomized Quasi-Monte Carlo (RQMC) simulation methods, incorporating Halton and Sobol sequences to improve convergence and accuracy in Black-Scholes-type option-pricing models across expiry and moneyness levels, with variance-reduction techniques to minimise standard errors. The standard Black-Scholes (BS) model is modified by replacing constant volatility with GARCH (1,1). To determine whether improvement lies in volatility dynamics or innovation density, we consider four cases: (i) a baseline GARCH-BS model under Gaussian density, (ii) the same GARCH-BS model with a heavy-tailed density, (iii) Gaussian density with a best-fitted GARCH(1,1) dynamic replacing the standard GARCH(1,1), and (iv) a combination of the heavy-tailed density and best-fitted GARCH(1,1). Our proposed RQMC framework improves pricing accuracy and reduces errors compared to standard MC across all models; the GARCH (1,1)-NIG framework outperforms the BS model in accuracy and convergence, and standard errors decrease with more simulations, confirming robustness across scenarios.*

Keywords: *Black-Scholes option pricing models, GARCH models, heavy-tail distributions, Monte Carlo simulations.*

JEL Classification: G13, C58, C15, C63.

Received: 12 October 2025	**Revised: 3 June 2026**	**Accepted: 10 June 2026**

1. Introduction

The Black–Scholes model (Black and Scholes, 1973) stands as a foundational framework in option pricing, offering a tractable analytical solution. Its reliance on simplified assumptions such as constant volatility and normally distributed returns has, however, motivated the development of more realistic extensions. These shortcomings have encouraged the development of extended option pricing frameworks that more accurately reflect market dynamics. Since asset returns often

DOI: 10.24818/18423264/60.2.26.01

exhibit volatility clustering, leverage effects, and non-normal tails, models that accommodate such features are expected to yield more reliable prices. This motivates a systematic investigation into richer volatility structures and distributional assumptions within the Black–Scholes–GARCH setting (Gong et al., 2010). Building on this motivation, we incorporate non-standard GARCH volatility models and investigate several enhanced model specifications to improve pricing accuracy. The objectives of this study are fourfold: (i) to improve option pricing accuracy by extending the BS–GARCH framework with alternative GARCH-type volatility models such as APARCH (1,1), GJR-GARCH (1,1), EGARCH (1,1) and CSGARCH (1,1), (ii) to investigate the role of heavy-tailed innovation distributions, including Student-t (T), skewed Student-t (SST), Generalized Hyperbolic (GH), and Normal Inverse Gaussian (NIG), in improving option pricing accuracy, (iii) to enhance computational efficiency through a quasi–Monte Carlo (QMC) framework, incorporating randomised QMC with variance reduction and conditional Monte Carlo techniques, and to evaluate the performance of the proposed model across different moneyness levels and maturities., and (iv) to identify the source of pricing improvements by disentangling the effects of volatility dynamics and innovation distributions through the examination of four model specifications: (a) the baseline GARCH–BS model with Gaussian density, (b) the GARCH–BS model with heavy-tailed innovations, (c) the Gaussian density with best-fitted variance dynamics, and (d) the joint heavy-tail and variance dynamic specification.

To the best of our knowledge, this study is among the first to jointly examine volatility dynamics, innovation distributions, and advanced Monte Carlo techniques within a unified BS–GARCH option pricing framework while explicitly identifying the sources of pricing improvements. Our empirical results show that the proposed hybrid BS–GARCH specification substantially enhances model performance and pricing accuracy across maturities and moneyness levels. In particular, the BS–GARCH model with NIG innovations consistently yields the lowest pricing errors and the highest predictive accuracy across simulation methods. The results also indicate that randomised quasi–Monte Carlo techniques improve the convergence efficiency relative to standard Monte Carlo. Moreover, the comparative analysis reveals that innovation distributions contribute more significantly to pricing improvements than volatility dynamics alone. Overall, these findings demonstrate the effectiveness of combining flexible volatility specifications, heavy-tailed innovations, and advanced simulation techniques within a unified option pricing framework.

The paper is organised as follows: Section 2 reviews the literature. Section 3 presents the hybrid Black–Scholes framework under stochastic volatility and variance reduction via RQMC and CMC. Section 4 presents details of GARCH modelling, best-fit selection, and option-chain construction. Section 5 evaluates the pricing performance of the hybrid BS with GARCH-type volatility and reports a simulation study. Section 6 concludes with key findings and implications.

2. Literature Review

Black and Scholes derived the closed-form solution, now known as the Black–Scholes formula, demonstrating that option values depend on the strike price, underlying asset value, volatility, and time to maturity. Nonetheless, the assumptions of constant volatility and a constant risk-free rate, together with Gaussian returns, are at odds with empirical evidence showing time-varying interest rates and non-Gaussian features in financial returns (Cont, 2001; Sheraz et al., 2021). Non-Gaussian log-return distributions have been integrated into the Black–Scholes framework while retaining closed-form solutions. The normal inverse Gaussian (NIG) model captures asymmetric, heavy tails (Barndorff-Nielsen, 1997); the logistic distribution improves empirical fit (Levy and Levy, 2014); and the double exponential also yields closed-form pricing (Ramos et al., 2016). The q-Gaussian, rooted in the Tsallis entropy, also has broad financial relevance (Preda et al., 2014) and supports a closed-form option pricing formula validated with high-frequency data (Nayak et al., 2021).

Time-varying volatility is commonly modelled through generalised autoregressive conditional heteroskedastic (GARCH) processes (Bollerslev, 1986) and stochastic volatility frameworks. The standard GARCH model assumes symmetric effects and normal errors, often failing to capture skewness and kurtosis. Extensions such as EGARCH (Nelson, 1991) and APARCH (Ding et al., 1993) address asymmetry, leverage, and richer dynamics. In option pricing, closed-form solutions under GARCH include the local risk-neutral valuation framework (Duan, 1995). Similarly, the Heston–Nandi model (Heston and Nandi, 2000) improves S&P 500 option pricing, while Gong et al. (2010) shows that embedding GARCH into Black–Scholes enhances S&P 100 performance. Further refinements integrate implied volatility to capture heavy-tailed returns more effectively (Sheraz et al., 2014). Beyond closed-form models, simulation methods are central to option pricing. Monte Carlo (Boyle, 1977) remains standard for valuation and risk, using pseudorandom numbers with probabilistic error bounds. Quasi-Monte Carlo improves convergence via low-discrepancy sequences (Sobol, Halton, Faure) effective for exotic options (Joy et al., 1996), but limited randomness complicates error estimation and stopping rules, motivating partial randomisation (Tan and Boyle, 2000), especially in higher dimensions. Conditional Monte Carlo (Trotter and Tukey, 1956) accelerates convergence by variance reduction via the law of iterated expectations and performs well with correlated risk factors. Hybrid schemes further enhance efficiency, including control variates for stochastic-volatility and jump-diffusion settings (Dingeç and Hörmann, 2013) and martingale control variates (Liang and Xu, 2020).

3. Hybridising the Black-Scholes Framework with GARCH-Type Volatility

The BS model (Black and Scholes, 1973) is based on Geometric Brownian Motion (GBM), a stochastic differential equation for asset prices whose existence and uniqueness are well established and given by:

$$dS_t = (r - \delta)S_t\, dt + \sigma\, S_t\, dW_t \tag{1}$$

where S_t be a process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, representing the underlying asset price in a Black-Scholes framework. We also assume the existence of a risk-neutral probability measure $\mathbb{Q}$ equivalent to the physical probability measure $\mathbb{P}$ and r denotes the risk-free rate at time t, δ the dividend yield, σ the volatility, and W_t the Wiener process under the risk-neutral measure, $W_t \sim N(0, t)$ that follows a standard Brownian motion process. For a non-dividend paying stock ($\delta = 0$) and $t \leq T$, the change in the underlying asset price process follows a Gaussian distribution, and the solution of SDE (1) is given by:

$$S_t = S_0 \exp\big((r - 0.5\,\sigma^2)t + \sigma \sqrt{t}\, W_t \big) \tag{2}$$

In the BS model risk-free interest rate r and volatility σ are assumed to be constants. We suppose $\{R_t\}_{t \in T}$ represents the logarithmic returns of the underlying stochastic process and the frequently used sample standard deviation, referred to as volatility is given by:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} \frac{1}{t-1} (R_t - \mu)^2} \tag{3}$$

Suppose that volatility $\hat{\sigma}$ is replaced by a stochastic process θ_t defined by a GARCH (1,1)-type specification. Let the return process be defined as:

$$R_t = \mu + e_t, \tag{4}$$

where μ denotes the expected return, and e_t is a zero-mean, serially uncorrelated white noise process. Assume that two conditional moments exist, and that:

$$e_t = \theta_t\, z_t,\; z_t \sim \mathcal{N}(0,1) \tag{5}$$

where θ_t^2 is the conditional variance and e_t^2 is the squared error. The time-varying conditional variance θ_t^2 evolves according to the GARCH (1,1) process (Bollerslev, 1986) is given by:

$$\theta_t^2 = \omega + \alpha_1\, e_{t-1}^2 + \beta_1\, \theta_{t-1}^2,\; t \in \mathbb{Z} \tag{6}$$

where, $\omega \geq 0$ is the long-run average variance (variance intercept), $\alpha_1 \geq 0$ is the ARCH coefficient, $\beta_1 \geq 0$ is the GARCH coefficient, and $\alpha_1 + \beta_1 < 1$ to ensure strictly positive conditional variance ($\theta_t^2 > 0$) and weak stationarity.

In our proposed option pricing framework, we extend the BS-GARCH model (Gong et al., 2010) by incorporating GARCH variants such as APARCH(1,1) (Ding et al., 1993), its sub-model GJR-GARCH (1,1) (Glosten et al., 1993), along with EGARCH(1,1) (Nelson, 1991) and CSGARCH(1,1) (Engle, 1999), each combined with heavy-tailed conditional innovation distributions: Normal N(0,1), Student-t, Skewed Student-t, and Lévy-type Generalized Hyperbolic (GH) (Eberlein and Keller, 1995) and NIG (Barndorff-Nielsen, 1998). This structure captures key return features such as leptokurtosis, volatility clustering, long memory, and leverage effect. The APARCH (1,1) model (Ding et al., 1993) is defined as:

$$\theta_t^d = \omega + \alpha_1\, (\,|e_{t-1}| - \gamma_1\, e_{t-1}\,)^d + \beta_1\, \theta_{t-1}^d \tag{7}$$

where θ_t^d denotes the power-transformed conditional standard deviation at time t with power parameter $d \in \mathbb{R}^+$. The parameter $|\gamma_1| \leq 1$ captures leverage effects, i.e., the asymmetric impact of positive and negative shocks on volatility. If $\gamma_1 = 0$ the model reduces to a symmetric GARCH (1,1) model and for $\gamma_1 > 0$, negative shocks ($e_{t-1} < 0$) have a larger effect on volatility than positive shocks of equal magnitude. Special cases of this model include: SGARCH (1,1) when $\gamma_1 = 0, d = 2$ and GJR-GARCH when $\gamma_1 = 0, d = 2$.

The EGARCH (1,1) (Nelson, 1991) model is given by:

$$\log \theta_t^2 = \omega + \alpha_1 \left(|Z_{t-1}| - \mathbb{E}(|Z_{t-1}|) \right) + \beta_1 \log \theta_{t-1}^2 + \gamma_1 Z_{t-1} \tag{8}$$

where α_1 controls the size effect and β_1 is the persistence parameter, γ_1 denotes the Leverage parameter and $\alpha_1 + \beta_1 < 1$ shows the stationarity condition.

The CGARCH (1,1) (Engle, 1999) model segregates volatility into temporary and permanent components to capture both short-term and long-term effects is given by:

$$\theta_t^2 = \psi_t + \alpha_1 (e_{t-1}^2 - \psi_{t-1}) + \beta_1 (\theta_{t-1}^2 - \psi_{t-1}) \tag{9}$$
$$\psi_t = \omega + \rho\psi_{t-1} + \phi(e_{t-1}^2 - \theta_{t-1}^2)$$

where $\alpha_1 \geq 0; \beta_1 \geq 0; \psi \geq 0, \omega \geq 0$ and $\rho < 1$ to get persistence in long-term and short-term components.

3.1 Option Pricing and BS Model with GARCH-Type Volatility

For a European call option having time to expiration T and strike price K, the closed form solution of the BS model is given by:

$$\mathrm{Call_{BS}} = S_0\, \Phi\left(\frac{\log\frac{S_0}{K} + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) - K\, e^{-rT}\, \Phi\left(\frac{\log\frac{S_0}{K} + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{10}$$

where Φ denotes the standard normal cumulation distribution function and S_0 the given price of the process S_t. Gong et.al., (2010) assumed that the price process S_t follows the conditional Black-Scholes model by incorporating GARCH-type volatility and replacing σ with the GARCH volatility process θ_t, the solution of the call option under the modified BS model with GARCH volatility is given by:

$$\mathrm{Call_{BS-GARCH}} = S_0\, \mathbb{E}_{\theta_t}\, \Phi(\widetilde{d_1}) - K\, e^{-rT} \mathbb{E}_{\theta_t} \Phi(\widetilde{d_2}) \tag{11}$$

where

$$\widetilde{d_1} = \frac{\log\frac{S_0}{K} + rT + 0.5\theta_t^2\, T}{\theta_t \sqrt{T}}$$
$$\widetilde{d_2} = \widetilde{d_1} - \theta_t \sqrt{T}$$

where θ_t is a stationary process representing the best fitted GARCH-type volatility, S_t denotes the stock price process, K is the strike price, and r is the risk-free interest rate. For a European call option, the payoff is $V_t = \max(S_T - K, 0) = (S_T - K)^+$, and put prices can be obtained by using the Put-Call parity relationship. The introduction of the heavy-tail model by Mandelbrot for stock and other financial return series provides superior results by capturing the desired statistical properties (Mandelbrot, 1963). In this regard, we consider NIG as a heavy-tailed distribution. The density function of NIG, NIG (Θ) with parameters $\Theta = (\alpha^*, \beta^*, \delta^*, \mu)$ is given by:

$$\mathrm{NIG}(\Theta) = \frac{\alpha^* \, \delta^* \, e^{\delta^* \gamma^*}}{\pi} \cdot \frac{K_1\left(\alpha^* \sqrt{\delta^{*2} + (x-\mu)^2}\right)}{\sqrt{\beta^{*2} + (x-\mu)^2}} \cdot e^{\beta^*(x-\mu)}, x \in \mathbb{R} \tag{12}$$

The parameter α^* is the tail parameter that controls the kurtosis of the distribution. The small α^* corresponds to the heavier tails. The parameter β^* indicates skewness and $\beta^* < 0$ shows left skewed and $\beta^* > 0$ for right-skewed data. The parameter $\mu \in \mathbb{R}$ is the mean and δ^* denotes the scale parameter that plays an analogue role to the variance-term σ^2 in the normal distribution.

3.2 Option Pricing and Monte Carlo Methods

Monte Carlo (MC) is a simulation-based method for modelling complex systems. In finance, it approximates derivative values – e.g., options under the Black–Scholes framework. For a random variable $S(\widetilde{\omega})$, MC estimates the expectation as the arithmetic mean of outcomes from many independent, identically distributed trials. Large simulations approximate the mathematical and statistical properties of $S(\widetilde{\omega})$. Accordingly, the MC estimate of an option's expected value using $S(\widetilde{\omega})$ is:

$$\mathbb{E}[S(\widetilde{\omega})] \approx \overline{S}(\widetilde{\omega}) = \frac{\sum_{i=1}^{N} s_i(\widetilde{\omega})}{N} \tag{13}$$

where s_i represent the simulated values of $S(\widetilde{\omega})$. For a sufficiently large number of simulations N, the law of large numbers and central limit theorem guarantee that $\mathbb{E}[S(\widetilde{\omega})]$ could be estimated to a significance level $1 - \epsilon$, i.e.

$$\lim_{N \to +\infty} P\left(\frac{|\bar{S}(\widetilde{\omega})| - \mathbb{E}[S(\widetilde{\omega})]}{\sigma\sqrt{N}}\right) \le M_{\frac{\epsilon}{2}} = 1 - \epsilon \tag{14}$$

Here, σ is the standard deviation of $S(\widetilde{\omega})$ and $\sigma/\sqrt{N}$ the standard error. From (14), the estimator of $S(\widetilde{\omega})$ is unbiased, and its variance governs accuracy; hence variance reduction improves the performance of MC. Since, standard MC converges at $O(N^{-1/2})$, we adopt QMC, proposed by Joy et al. (1996) to mitigate slow PRN convergence. QMC replaces PRNs with low-discrepancy QRNs (e.g., Halton, Sobol) that are more uniform, lower-dispersion, and faster. QMC preserves the MC form on the unit cube, using low-discrepancy points; equivalently, it approximates an integral on a d-dimensional hypercube over points $\{x_i\}_{i=1}^N$, given by:

$$\int_{[0,1]^d} g(u)du \approx \frac{1}{N}\sum_{i=1}^{N} g(x_i) \quad (15)$$

The approximated error estimation under QMC can be expressed in terms of discrepancy D_N^* of QRN and bounded by:

$$\left|\frac{1}{N}\sum_{i=1}^{N} g(x_i) - \int_{[0,1]^d} g(x)dx\right| \leq V(g)D_N^*\{x_i\}_{i=1}^{N} \quad (16)$$

where $V(g)$ is variation of the integrand g(x)(Hardy, 1906). The convergence rate of QMC is $O(N^{-1}\log N^d)$, which is affected by number of simulated paths N, and dimension d of the QRN. QMC provides better error convergence with an order of $O(N^{-1})$ as against the standard MC's convergence rate of $O(N^{-1/2})$. Classical QMC suffers from error bounds (and constants) that grow exponentially with dimension d. Randomized Quasi-Monte Carlo (RQMC) addresses this by (i) combining low-discrepancy sequences (Sobol and Halton) with MC's statistical properties, (ii) randomising these sequences via digital shifts, scrambling, random linear scrambling, or random starts to preserve low discrepancy and enable MC-style error estimation, (iii) performing multiple independent randomisations to generate distinct sample sets and evaluate the integrand, and (iv) averaging the estimates to obtain a variance-based error. RQMC typically outperforms MC, especially in high dimensions; its convergence resembles QMC, and despite added variance from randomisation, overall error can approach $O(N^{-1})$ (Owen, 1997). On the other hand, the Conditional Monte Carlo (CMC) method is a variance reduction technique that leverages conditional expectations to improve the efficiency and accuracy of MC simulations. The CMC estimator is unbiased, and variance follows the decomposition:

$$\mathbb{V}(X) = \mathbb{E}(\mathbb{V}(X|Y)) + \mathbb{V}(\mathbb{E}(X|Y)) \quad (17)$$

where $\mathbb{V}$ (X|Y) ≥ 0 and $\mathbb{E}(\mathbb{V}(X|Y)) > 0$, therefore $\mathbb{V}(X) \geq \mathbb{V}(\mathbb{E}(X|Y))$.

The CMC and RQMC methods can be combined to leverage the strengths of both techniques. Finally, to assess the option pricing accuracy performance of each underlying MC model, we compute the mean absolute percentage error (MAPE). The measure of accuracy performance is given as:

$$\text{MAPE} = \sum_{i=1}^{N} \frac{|F_e - F_o|}{|F_o|} \times 100\% \quad (18)$$

where, F_e, F_0 and n represent estimated or predicted, observed option prices and number of options in the option chain, respectively. We adopt the mean absolute percentage error (MAPE) as it helps to better compare option pricing accuracy by using relative, scale-invariant deviations.

4. GARCH Volatility Estimation and Constructing an Option Chain

We analyse options on the S&P 100 Index daily series, analogous to S&P 500 options for valuing European options. Returns from 18 January 2018 to 13 January 2023 (n = 1256) provide a diverse sample; data are from www.thomsonreuters.com. Table 1 reports descriptive statistics for daily closing S&P 100 log returns: the range −12.3% to 9.6% indicates high volatility, with asymmetric (left-skewed) behaviour and high kurtosis. The ADF test's null is rejected, confirming non-Gaussian behaviour, while significant LM and Box statistics indicate strong ARCH. Overall, S&P 100 returns display typical asset-return traits leftward skew, leptokurtosis, and serial correlation in volatility.

Table 1. Descriptive statistics of S&P 100 log-returns

Min	Max	Mean	Volatility	Skewness
-0.123	0.096	0.001	0.221	-0.654
Kurtosis	JB Test	ADF Test	LM Stat	BOX Stat
11.323	15918***	-13.40***	479.20 ***	47.11***

Note: Estimated values labelled with *** are statistically significant at 1% level

Source: Authors' own work.

State variables – interest rate, underlying asset price, and volatility are central to option valuation. Accordingly, we first examine S&P100 volatility using the SGARCH, GJR-GARCH(1,1), EGARCH(1,1), APARCH(1,1) and CSGARCH(1,1) each paired with its best-fit ARMA model. Conditional error distributions include Student's t (T), Skew Student's t (SST), Generalized Hyperbolic (GH), and Normal Inverse Gaussian (NIG). Model selection uses AIC, BIC, and HQC. The model with lowest sum ranks with respect to AIC, BIC and HQC is selected as the best-fit, which appears to be APARCH (1,1)-NIG for return volatility. Estimated parameters of the best -fitted model are shown in Table 2.

Table 2. Parameter Estimates of ARMA (0,1)-APARCH (1,1)-NIG model employed to S&P100 returns are sampled from 18th January 2018 to 13th January 2023 containing 1256 observations.

ARMA(0,1)-APARCH (1,1)-NIG Parameter Estimates								
MA	μ	ω	α_1	β_1	γ_1	δ	Skew	Shape
-0.0516* **	0.0003***	0.001	0.112***	0.887***	0.910***	0.754***	-0.461***	3.102***
Normal Inverse Gaussian (NIG) Parameter Estimates								
	μ^*	α^*	β^*	δ^*				
	0.0017^{***}	-48.667^{***}	-8.016^{***}	0.008^{***}				

Note: Estimated values labelled with***, ** and * indicate p-value less than 0.01, 0.05 and 0.1, respectively.

Source: Authors' own work.

Estimated ARMA (0,1)–APARCH (1,1)–NIG parameters indicate volatility clustering, asymmetry, and heavy tails in S&P 100 returns. The ARMA (0,1) mean shows little autoregression; a small but significant μ implies slight positive drift and sensitivity to past shocks. Significant ARCH/GARCH coefficients indicate highly persistent volatility. The leverage term (γ_1) confirms stronger responses to negative shocks, while the power parameter $(\delta \neq 2)$ allows a flexible shock–volatility relation. The NIG estimated parameters exhibit negative skew and heavy tails, implying more frequent extreme losses than under normality.

In addition, we analyse the S&P 100 index call options from 1 January to 31 March 2023 (CBOE). The sample spans 99 strikes (900–1760) and eight maturities (8–50 days). To maintain an arbitrage-free set, options priced below intrinsic value are excluded. Pricing state variables are the interest rate r, underlying price S_t, and volatility σ; volatility is modelled via GARCH (1,1), and r is drawn from 4-, 8-, 13-, 17-, and 26-week T-bill rates (Extracted from https://home.treasury.gov). Options are grouped by moneyness $M = \frac{d(S_t - K)}{K}$, where $d = 1$ for call option and $d = -1$ for put option and $M: 0\% - 1\%$ as at-the-money (ATM) and $M > 1\%$ as in-the-money (ITM). This structure supports the analysis of pricing dynamics, implied-volatility patterns, and short-/medium-term behaviour. Table 3 reports counts, average prices, and implied volatilities: average price rises with T for ATM/OTM, while for ITM it first rises then falls; implied-volatility changes are most pronounced for ITM.

Table 3. S&P 100 options chain for three months data from January 1 to March 31, 2023

	Time to Expiry (Days)							
	8	15	22	29	36	43	50	Total
Numbers of options								
ATM	7	7	7	7	7	7	4	46
ITM	82	46	51	52	95	48	27	401
OTM	89	36	30	28	65	32	23	303
Total	178	89	88	87	167	87	54	750
Average Price								
ATM	26.2	31.3	40.0	45.0	51.4	54.7	59.7	43.0
ITM	317.4	148.3	164.5	170.9	337.7	171.6	183.2	237.9
OTM	7.8	11.5	10.0	13.1	12.4	17.4	20.6	11.9
Total	151.2	83.8	102.0	110.0	199.1	105.4	104.8	134.6
Implied Volatility								
ATM	0.25	0.22	0.23	0.22	0.22	0.22	0.22	0.22
ITM	1.04	0.4	0.37	0.35	0.54	0.31	0.31	0.54
OTM	0.02	0.07	0.13	0.15	0.09	0.17	0.15	0.09
Total	0.5	0.25	0.28	0.28	0.35	0.25	0.23	0.34

Source: Authors' own work.

5. Pricing under Black-Scholes Model with GARCH Volatility

We assess our proposed BS model with APARCH (1,1)–NIG volatility versus standard BS under MC/RQMC, using CMC for variance reduction. Options are grouped by moneyness and maturity; $T \leq 22$ days is short-term, $T > 22$ long-term. The accuracy is benchmarked by MAPE relative to the standard BS prices. Four cases: normal vs NIG innovations with/without ARMA (0,1)–APARCH (1,1) are run via MC and RQMC (Halton, Sobol), denoted RQMC(H) and RQMC(S). For $T \leq 22$ days, NIG-based GARCH outperforms normal, indicating better volatility capture. ARMA (0,1)–APARCH (1,1)–NIG attains the lowest error (6.51%). The results by moneyness appear in Table 4.

Table 4. European call option pricing errors for the S&P 100 across maturities. Cases 1–4 denote: (i) BS–GARCH (1,1) with normal innovations; (ii) BS–GARCH (1,1) with NIG innovations; (iii) BS–GARCH with ARMA (0,1)–APARCH (1,1) and normal innovations; (iv) BS–GARCH with ARMA (0,1)–APARCH (1,1) and NIG innovations. MC, RQMC(H), and RQMC(S) indicate standard MC, RQMC with Halton, and RQMC with Sobol, respectively.

Maturity (Days)	Case 1	Case 2	Case 3	Case 4
Monte Carlo (MC)				
8	5.65%	4.10%	4.40%	4.32%
15	8.39%	7.18%	7.73%	8.15%
22	8.96%	6.23%	6.95%	7.33%
29	9.18%	7.41%	8.37%	8.56%
36	5.82%	5.05%	5.50%	5.77%
43	10.07%	10.20%	11.34%	11.93%
50	9.56%	9.69%	10.53%	11.00%
Overall	7.70%	6.51%	7.16%	7.43%
RMQC (H)				
8	4.12%	4.09%	4.32%	4.40%
15	7.27%	7.22%	7.84%	8.05%
22	6.25%	6.14%	6.89%	7.19%
29	7.50%	7.42%	8.25%	8.55%
36	5.06%	4.98%	5.51%	5.69%
43	10.37%	10.15%	11.34%	11.74%
50	9.55%	9.37%	10.44%	10.94%
Overall	6.56%	6.46%	7.12%	7.37%
RMQC (S)				
8	4.11%	4.10%	4.34%	4.38%
15	7.28%	7.21%	7.85%	8.04%
22	6.26%	6.17%	6.93%	7.22%
29	7.55%	7.43%	8.25%	8.52%
36	5.05%	4.99%	5.50%	5.68%
43	10.32%	10.13%	11.36%	11.83%
50	9.48%	9.34%	10.50%	10.92%
Overall	6.55%	6.46%	7.14%	7.37%

Source: Authors' own work.

Table 5 extends the analysis across moneyness levels to assess MC methods. For at-the-money options, the standard BS–GARCH with a normal distribution performs best (MAPE ≈ 18.26%), whereas BS–GARCH with ARMA (0,1)–APARCH (1,1)–NIG performs worst under MC, RQMC(H), and RQMC(S). Overall, Case 2 yields the lowest MAPE, suggesting that replacing the NIG distribution with a normal distribution improves pricing in the standard BS–GARCH framework. In MC and in RQMC (Halton/Sobol), Case 3 follows Case 2. See Appendix A (Tables A1) for details.

Moreover, we applied Repeated Measures ANOVA (RM-ANOVA) with contrast analysis, which is a parametric test assuming normally distributed residuals and sphericity. The contrasts show that Case 2 is statistically superior ($p < 0.05$) to the other cases across all three simulation methods (MC, RQMC(H), RQMC(S)); under RQMC(H), pairwise p-values are: Case 2 vs. Case 3, 0.0055; Case 2 vs. Case 4, 0.0064; and Case 1 vs. Case 2, 0.0383. Accordingly, Case 2—BS-GARCH (1,1) with NIG innovation achieves significantly lower MAPE and the best predictive accuracy, yielding the most reliable option-pricing estimates. Consistently, it records the lowest MAPE across methods (MC: 6.51%; RQMC(H): 6.46%; RQMC(S): 6.46%).

Table 5. Pricing errors for the S&P 100 option chain across moneyness levels. Cases 1–4 denote: (1) BS–GARCH (1,1) with normal innovations; (2) BS–GARCH (1,1) with NIG innovations; (3) BS–GARCH with ARMA (0,1)–APARCH (1,1) and normal innovations; (4) BS–GARCH with ARMA (0,1)–APARCH (1,1) and NIG innovations. MC, RQMC(H), and RQMC(S) indicate standard MC, RQMC with Halton, and RQMC with Sobol, respectively.

Moneyness	Case 1	Case 2	Case 3	Case 4
Monte Carlo (MC)				
0 to 5%	19.59%	18.26%	20.09%	20.90%
5 to 10%	5.69%	3.61%	4.17%	4.45%
10 to 20%	2.23%	1.37%	1.37%	1.31%
20% or more	0.92%	0.90%	0.90%	0.90%
Overall	7.70%	6.51%	7.16%	7.43%
RMQC (H)				
0 to 5%	18.38%	18.13%	20.05%	20.71%
5 to 10%	3.74%	3.61%	4.17%	4.42%
10 to 20%	1.31%	1.32%	1.31%	1.31%
20% or more	0.89%	0.89%	0.89%	0.89%
Overall	6.56%	6.46%	7.12%	7.37%
RMQC (S)				
0 to 5%	18.37%	18.10%	20.07%	20.71%
5 to 10%	3.72%	3.65%	4.20%	4.43%
10 to 20%	1.30%	1.32%	1.30%	1.30%
20% or more	0.90%	0.90%	0.89%	0.89%
Overall	6.55%	6.46%	7.14%	7.37%

Source: Authors' own work.

5.1 Error Rates and Regression Analysis under BS-GARCH Model

We now fit regression equation to further analyse the relationship between error rates, maturity (time to expiry), level of moneyness, and price of the options given by:

$$\text{MAPE} = a_0 + a_1 \ln M + a_2 \ln T + \epsilon \tag{19}$$

This regression framework assesses pricing errors and informs option-pricing accuracy by regressing MAPE on moneyness (M) and time to expiry (*T*). The intercept a_0 is the baseline at $M = T = 0$; a_1 measures the effect of M and a_2 the effect of *T*, with $\varepsilon \sim N(0, \sigma^2)$. Table 6 reports estimates for the four cases. The coefficients on M and *T* are statistically significant in all models. The coefficient of M is negative, and accuracy improves as M decreases – whereas *T*'s coefficient is positive, suggesting long-term options are priced more accurately than short-term. Case 2 (BS–GARCH with NIG) has the smallest M and *T* coefficients, consistent with its improved accuracy.

Table 6. Regression analysis results for Mean Absolute Percentage Errors (MAPEs) against log of moneyness (lnM) and log of time to maturity (lnT) where *, **, and * indicate p-value less than 0.01, 0.05, and 0.1, respectively.**

Model	**Coefficients Estimates**		
MC	a_0	a_1	a_2
Case 1	-0.1266***	-0.0735***	0.0045*
Case 2	-0.1624***	-0.0740***	0.0117***
Case 3	-0.1811***	-0.0813***	0.0138***
Case 4	-0.1937***	-0.0846***	0.0160***
RQMC(H)			
Case 1	-0.1639***	-0.0749***	0.0117***
Case 2	-0.1608***	-0.0740***	0.0111***
Case 3	-0.1823***	-0.0815***	0.0139***
Case 4	-0.1900***	-0.0840***	0.0151***
RQMC(S)			
Case 1	-0.1637***	-0.0749***	0.0115***
Case 2	-0.1599***	-0.0738***	0.0110***
Case 3	-0.1823***	-0.0815***	0.0139***
Case 4	-0.1903***	-0.0839***	0.0153***

Source: Authors' own work.

5.2 Simulations Study: BS-APARCH (1,1)-NIG Model

We present new simulation results for the proposed Black–Scholes model with GARCH (1,1)-type volatility. To assess the proposed RQMC method's error-convergence efficiency, we use the BS–GARCH framework across the four previously defined cases and compare their relative convergence. The study replicates an S&P 100 option chain with maturities 8, 15, 22, 29, 36, 43, and 50 days. As a case study, we analyse at-the-money options with $S = 1767.67$ and $K = 1765$. The GARCH (1,1) volatility specifications for the four cases are:

Case 1 and Case 2: Standard ARMA (0,1)-GARCH (1,1) with Normal and NIG Innovations.

$$R_t = 0.0009 + e_t \text{ where } e_t = \theta_t z_t$$
$$\theta_t^2 = 0.000004 + 0.193116 e_{t-1}^2 + 0.794179 \theta_{t-1}^2$$

where z_t follows Normal distribution in Case 1 and NIG in Case 2 and, $\theta_0 = 0.014$.

Case 3 and Case 4: ARMA (0,1)-APARCH (1,1) with Normal and NIG Innovations.

$$R_t = 0.00028 + e_t \text{ where } e_t = \theta_t z_t$$

$$\theta_t^{0.754} = 0.0013 + 0.1125(|e_{t-1}| - 0.9105 e_{t-1})^{0.754} + 0.8874\theta_{t-1}^{0.754}$$

where z_t follows Normal distribution in Case 3 and NIG in Case 4 and, $\theta_0 = 0.0081$ and estimates of the NIG model used are: $(\mu^*, \alpha^*, \beta^*, \delta^*) = (0.00017, -48.667, -8.016, 0.008)$.

Table 7 reports standard errors for BS–GARCH (1,1) under three methods: Monte Carlo (MC), RQMC(H), and RQMC(S). Overall, Case 2 (standard BS–GARCH (1,1) with NIG innovations) yields the lowest errors, followed by Case 1 (normal), indicating that NIG improves convergence. MC shows notably higher errors than both RQMC variants (lower sampling efficiency); RQMC(H) improves on MC via Halton sequences, while RQMC(S) generally performs best. The gap is between RQMC(H) and RQMC(S) is quite marginal, yet RQMC(S) typically leads. MC rises to 2.4903 at 50k, RQMC(H) remains relatively stable (1.0723–2.4242), and RQMC(S) stays lowest (1.0836–2.4770). Across the cases, Case 2 has the smallest errors, and Case 4 the largest, implying greater sensitivity/complexity with higher case numbers. Hence, sampling efficiency ranks RQMC(S) above RQMC(H) followed by MC, confirming RQMC's superior accuracy and convergence for BS–GARCH (1,1) under NIG.

Table 7. Standard errors across maturities T (days) and replications N_s. Cases 1–4: (i) standard BS–GARCH (1,1) with normal innovations; (ii) standard BS–GARCH (1,1) with NIG innovations; (iii) BS–GARCH (1,1) with ARMA (0,1)–APARCH (1,1) and normal innovations; (iv) BS–GARCH (1,1) with ARMA (0,1)– APARCH (1,1) and NIG innovations. Methods: MC = standard Monte Carlo; RQMC(H) = RQMC with Halton sequence; RQMC(S) = RQMC with Sobol sequence.

		MC				RQMC(H)				RQMC(S)			
N_s	T	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
50,000	8	0.9418	0.944	0.9555	0.9832	0.9429	0.9445	0.9716	0.9796	0.9446	0.9478	0.969	0.9862
50,000	15	1.2538	1.2298	1.2782	1.2765	1.2431	1.2299	1.2692	1.2817	1.2499	1.2329	1.2734	1.2981
50,000	22	1.5076	1.4865	1.516	1.5052	1.4738	1.4761	1.5096	1.534	1.4958	1.4694	1.534	1.5219
50,000	29	1.6902	1.7311	1.7016	1.7274	1.7048	1.678	1.7305	1.7523	1.6801	1.6536	1.7046	1.7344
50,000	36	1.9065	1.8322	1.9272	1.9189	1.8674	1.8563	1.9126	1.936	1.9105	1.8643	1.9142	1.9300
50,000	43	2.0532	2.0392	2.1209	2.1423	2.0479	2.0439	2.1028	2.124	2.0327	2.0289	2.0866	2.1012
50,000	50	2.1809	2.2097	2.2263	2.2732	2.2036	2.1939	2.2802	2.2734	2.2155	2.1733	2.2427	2.2763
100,000	8	1.0352	1.005	1.0485	1.0529	1.0164	1.0176	1.047	1.0536	1.0311	1.0095	1.049	1.0445
100,000	15	1.3508	1.3356	1.3428	1.3575	1.3538	1.3302	1.3718	1.3722	1.3396	1.3364	1.374	1.3747
100,000	22	1.5731	1.6044	1.621	1.6356	1.5752	1.5716	1.6185	1.6357	1.579	1.5862	1.6185	1.6204
100,000	29	1.8086	1.8223	1.8308	1.8259	1.7946	1.7874	1.8206	1.8498	1.808	1.8019	1.839	1.8394
100,000	36	2.032	1.9641	2.0031	2.0508	1.9927	1.9733	2.0352	2.0438	2.0067	1.9878	2.0214	2.0354
100,000	43	2.1853	2.1639	2.1953	2.2212	2.1619	2.1334	2.2195	2.2126	2.1446	2.1495	2.2127	2.2406
100,000	50	2.2834	2.3193	2.3711	2.3239	2.3443	2.3101	2.4079	2.3717	2.297	2.3071	2.3675	2.4051
500,000	8	1.066	1.0723	1.1002	1.1256	1.0601	1.0551	1.0866	1.0943	1.058	1.0569	1.0836	1.0976
500,000	15	1.4234	1.391	1.4473	1.4478	1.4159	1.3975	1.4335	1.4431	1.4132	1.4002	1.4301	1.4523

		MC				**RQMC(H)**				**RQMC(S)**			
N_s	*T*	**Case 1**	**Case 2**	**Case 3**	**Case 4**	**Case 1**	**Case 2**	**Case 3**	**Case 4**	**Case 1**	**Case 2**	**Case 3**	**Case 4**
500,000	22	1.6643	1.6391	1.7343	1.7542	1.6648	1.6629	1.7026	1.7088	1.6675	1.6653	1.7067	1.7252
500,000	29	1.8922	1.8889	1.9636	1.959	1.8863	1.8803	1.9271	1.9508	1.8935	1.8751	1.9313	1.935
500,000	36	2.0864	2.088	2.1219	2.1643	2.0783	2.0681	2.122	2.1491	2.0725	2.0687	2.1031	2.1408
500,000	43	2.2413	2.2431	2.2832	2.2982	2.2462	2.248	2.3015	2.3213	2.2807	2.2611	2.3139	2.3219
500,000	50	2.4099	2.3898	2.4903	2.4504	2.4242	2.4255	2.4671	2.5018	2.4266	2.3808	2.477	2.5056

Source: Authors' own work.

6. Conclusions

This study assesses a modified Black–Scholes (BS) option-pricing model that incorporates GARCH (1,1) volatility under both Gaussian and heavy-tailed innovations. Analysis of the S&P 100 option data reveals pronounced skewness, excess kurtosis, and ARCH effects, thereby motivating heavy-tailed return modelling.

Building on Gong et al. (2010), we estimate SGARCH, GJR-GARCH, EGARCH, APARCH, and CSGARCH specifications each paired with a best-fit ARMA component and consider Student-t, skewed-t, GH, and NIG error distributions. The ARMA (0,1)–APARCH (1,1) with NIG emerges as the best-fitting specification, capturing leverage and nonlinear return–volatility relations. It serves as the basis for simulations using standard Monte Carlo (MC) and randomised quasi-Monte Carlo with Halton [RQMC(H)] and Sobol [RQMC(S)] sequences. Across maturities and moneyness, GARCH-based BS models outperform fixed-volatility benchmarks, while RQMC(S) consistently outperforms RQMC(H).

In pricing European call option errors for the S&P 100 across maturities, the Shapiro–Wilk test confirms the normality of Case-2 (BS–GARCH with NIG) errors (p = 0.719), permitting paired t-tests. The results indicate statistically significant differences relative to Cases 1, 3, and 4 (p = 0.013, 0.00017, 0.00033), leading to rejection of the null hypothesis of equal performance. Case 2 therefore delivers lower pricing errors than both the Gaussian benchmark and the APARCH-augmented variants.

A regression of MAPE on time-to-maturity and moneyness shows that at-the-money options are the most challenging to price accurately, whereas deep-in-the-money contracts exhibit greater precision. The standard errors for MC, RQMC(H), and RQMC(S) decline approximately exponentially as the number of simulations increases.

References

[1] Barndorff-Nielsen, O.E. (1997), *Processes of normal inverse Gaussian type. Finance and stochastics*, 2, 41-68.

[2] Black, F., Scholes, M. (1973), *The pricing of options and corporate liabilities. Journal of political economy*, 81(3), 637-654.

[3] Bollerslev, T. (1986), *Generalized autoregressive conditional heteroskedasticity. Journal of econometrics*, 31(3), 307-327.

[4] Boyle, P.P. (1977), *Options: A Monte Carlo approach. Journal of financial economics*, 4(3), 323-338.

[5] Cont, R. (2001), *Empirical properties of asset returns: stylized facts and statistical issues. Quantitative finance*, 1(2), 223.

[6] Ding, Z., Granger, C.W., Engle, R.F. (1993), *A long memory property of stock market returns and a new model. Journal of empirical finance*, 1(1), 83-106.

[7] Dingeç, K.D., Hörmann, W. (2013), *Control variates and conditional Monte Carlo for basket and Asian options. Insurance: Mathematics and Economics*, 52(3), 421-434.

[8] Duan, J.C. (1995), *The GARCH option pricing model. Mathematical finance*, 5(1), 13-32.

[9] Eberlein, E., Keller, U. (1995), *Hyperbolic distributions in finance. Bernoulli*, 1(3), 281-299.

[10] Engle, R.F. (1999), *A long-run and short-run component model of stock return volatility. Cointegration, Causality, and Forecasting*, 475-497.

[11] Gong, H., Thavaneswaran, A., Singh, J. (2010), *A Black-Scholes Model with GARCH Volatility. Mathematical Scientist*, 35(1).

[12] Glosten, L.R., Jagannathan, R., Runkle, D.E. (1993), *On the relation between the expected value and the volatility of the nominal excess return on stocks*. *The Journal of Finance*, 48(5), 1779-1801.

[13] Hardy, G.H. (1905), *On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters. Quarterly Mathematics*, 37(1) (1906), 53-79.

[14] Heston, S.L., Nandi, S. (2000), *A closed-form GARCH option valuation model*. *The Review of Financial Studies*, 13(3), 585-625.

[15] Joy, C., Boyle, P.P., Tan, K.S. (1996), *Quasi-Monte Carlo methods in numerical finance. Management science*, 42(6), 926-938.

[16] Levy, M., Levy, H. (2024), *Option pricing with the logistic return distribution*. *Journal of Risk and Financial Management*, 17(2), 67, https://doi.org/10.3390/jrfm17020067.

[17] Liang, Y., Xu, C. (2020), *An efficient conditional Monte Carlo method for European option pricing with stochastic volatility and stochastic interest rate. International Journal of Computer Mathematics*, 97(3), 638-655.

[18] Mandelbrot, B. (1963), *New methods in statistical economics. Journal of political economy*, 71(5), 421-440.

[19] Nayak, G., Singh, A.K., Senapati, D. (2021), *Computational modelling of non-Gaussian option price using non-extensive Tsallis' entropy framework. Computational Economics*, 57(4), 1353-1371.

[20] Nelson, D.B. (1991), *Conditional heteroskedasticity in asset returns: A new approach. Econometrica: Journal of the econometric society*, 347-370.

[21] Owen, A.B. (1997), *Monte Carlo variance of scrambled net quadrature. SIAM Journal on Numerical Analysis*, 34(5), 1884-1910.

[22] Preda, V., Dedu, S., Sheraz, M. (2014), *New measure selection for Hunt–Devolder semi-Markov regime switching interest rate models. Physica A: Statistical Mechanics and its Applications*, 407, 350-359.

[23] Ramos, A.M., Carvalho, J.A., Vasconcelos, G.L. (2016), *Exponential model for option prices: Application to the Brazilian market. Physica A: Statistical Mechanics and Its Applications*, 445, 161-168.

[24] Sheraz, M., Preda, V. (2014), *Implied volatility in black-scholes model with GARCH volatility. Procedia Economics and Finance*, 8, 658-663.

[25] Sheraz, M., Nasir, I., Dedu, S. (2021), *Extreme Value Analysis and Risk Assessment: A Case of Pakistan Stock Market. Economic Computation and Economic Cybernetics Studies and Research*, 3, 5-20.

[26] Tan, K.S., Boyle, P.P. (2000), *Applications of randomized low discrepancy sequences to the valuation of complex securities. Journal of Economic Dynamics and Control*, 24(11-12), 1747-1782.

[27] Trotter,H.F., Tukey, J.W. (1956), *Conditional Monte Carlo methods. Annals of Mathematical Statistics*, 27(3), 602-614.

Appendix A

Table A1: This table presents a snapshot of the comparative analysis of call price performance of S&P 100 option chain where Case 1, Case 2, Case 3 and Case 4 represent the standard BS-GARCH model with normal innovation, the standard BS-GARCH model with NIG distribution, the BS-GARCH framework with AR(0,1)-APARCH(1,1) model and normal innovation, and the BS-GARCH framework with AR(0,1)-APARCH(1,1) model with NIG innovations, respectively. The MC, RQMC(H) and RQMC(S), respectively represent the simulations performed using the standard MC, RQMC with Halton sequence and RQMC with Sobol sequence.

			Standard MC				RQMC (H)				RQMC (S)			
T	Strike	Option Price	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
8	1200	579.9	568.9 (1.9)	570.3 (1.7)	569.3 (1.8)	568.7 (1.9)	569.1 (1.9)	569.2 (1.9)	569.2 (1.8)	568.9 (1.9)	569.0 (1.9)	569.2 (1.9)	569.3 (1.8)	569.0 (1.9)
8	1180	592.0	589.0 (0.5)	589.7 (0.4)	590.1 (0.3)	588.9 (0.5)	589.0 (0.5)	589.2 (0.5)	589.0 (0.5)	588.9 (0.5)	588.9 (0.5)	588.9 (0.5)	589.1 (0.5)	589.1 (0.5)
8	1160	620.5	608.9 (1.9)	608.2 (2.0)	609.6 (1.8)	609.0 (1.9)	609.1 (1.8)	609.2 (1.8)	608.8 (1.9)	609.0 (1.9)	608.9 (1.9)	608.8 (1.9)	609.1 (1.8)	608.6 (1.9)
8	1140	635.1	629.1 (0.9)	628.9 (1.0)	628.9 (1.0)	628.6 (1.0)	628.9 (1.0)	629.0 (1.0)	629.0 (1.0)	628.9 (1.0)	628.9 (1.0)	629.0 (1.0)	628.9 (1.0)	629.0 (1.0)
8	1120	655.1	648.9 (1.0)	648.3 (1.0)	649.3 (0.9)	648.7 (1.0)	648.9 (0.9)	649.0 (0.9)	648.9 (0.9)	648.9 (0.9)	648.9 (1.0)	648.8 (1.0)	649.1 (0.9)	649.0 (0.9)
8	1100	678.4	668.8 (1.4)	668.8 (1.4)	669.1 (1.4)	668.1 (1.5)	668.8 (1.4)	669.0 (1.4)	669.0 (1.4)	669.0 (1.4)	669.1 (1.4)	668.9 (1.4)	668.9 (1.4)	669.0 (1.4)
8	1080	695.1	689.0 (0.9)	688.7 (0.9)	689.8 (0.8)	689.1 (0.9)	688.7 (0.9)	688.9 (0.9)	688.9 (0.9)	689.0 (0.9)	688.9 (0.9)	688.9 (0.9)	688.8 (0.9)	688.9 (0.9)
8	1060	715.1	708.4 (0.9)	709.1 (0.8)	709.0 (0.9)	708.9 (0.9)	709.0 (0.9)	708.9 (0.9)	708.8 (0.9)	709.0 (0.8)	708.9 (0.9)	709.2 (0.8)	709.1 (0.8)	708.9 (0.9)
8	1040	732.0	728.9 (0.4)	729.2 (0.4)	729.4 (0.4)	728.2 (0.5)	729.0 (0.4)	728.8 (0.4)	728.9 (0.4)	728.9 (0.4)	728.8 (0.4)	728.8 (0.4)	728.7 (0.4)	728.8 (0.4)
8	1020	752.0	748.8 (0.4)	748.7 (0.4)	748.2 (0.5)	748.3 (0.5)	748.8 (0.4)	748.6 (0.5)	748.8 (0.4)	749.1 (0.4)	748.7 (0.4)	748.6 (0.5)	748.9 (0.4)	748.8 (0.4)
8	1000	782.0	768.9 (1.7)	769.2 (1.6)	767.6 (1.8)	768.9 (1.7)	768.7 (1.7)	768.9 (1.7)	768.8 (1.7)	768.8 (1.7)	768.7 (1.7)	768.7 (1.7)	768.9 (1.7)	769.0 (1.7)
8	980	795.1	789.2 (0.7)	787.9 (0.9)	790.0 (0.6)	788.5 (0.8)	788.6 (0.8)	789.0 (0.8)	789.1 (0.8)	788.8 (0.8)	788.9 (0.8)	788.5 (0.8)	788.7 (0.8)	788.9 (0.8)
8	960	815.1	808.9 (0.8)	808.8 (0.8)	809.2 (0.7)	808.9 (0.8)	808.9 (0.8)	809.0 (0.8)	808.8 (0.8)	809.0 (0.8)	808.7 (0.8)	808.5 (0.8)	808.8 (0.8)	808.9 (0.8)
8	940	835.1	828.7 (0.8)	828.9 (0.7)	829.6 (0.7)	829.6 (0.7)	828.9 (0.7)	828.6 (0.8)	828.7 (0.8)	828.7 (0.8)	828.9 (0.7)	828.7 (0.8)	828.9 (0.7)	828.8 (0.8)
8	920	855.1	848.5 (0.8)	847.9 (0.8)	848.1 (0.8)	847.8 (0.9)	848.8 (0.7)	849.0 (0.7)	848.9 (0.7)	848.7 (0.8)	848.7 (0.8)	848.6 (0.8)	848.7 (0.7)	848.7 (0.8)
8	900	875.1	868.7 (0.7)	868.3 (0.8)	867.3 (0.9)	868.7 (0.7)	868.7 (0.7)	868.9 (0.7)	868.8 (0.7)	868.7 (0.7)	868.7 (0.7)	868.8 (0.7)	868.8 (0.7)	868.5 (0.8)
36	1390	389.3	385.5 (1.0)	386.2 (0.8)	386.3 (0.8)	384.4 (1.3)	385.6 (0.9)	385.9 (0.9)	385.9 (0.9)	385.7 (0.9)	386.0 (0.8)	385.9 (0.9)	385.5 (1.0)	386.1 (0.8)
36	1380	399.2	395.5 (0.9)	394.8 (1.1)	396.3 (0.7)	394.8 (1.1)	395.7 (0.9)	395.5 (0.9)	395.9 (0.8)	396.0 (0.8)	396.3 (0.7)	395.5 (0.9)	395.7 (0.9)	395.8 (0.9)
36	1370	409.1	405.7 (0.8)	405.3 (0.9)	404.6 (1.1)	406.2 (0.7)	405.6 (0.9)	405.5 (0.9)	405.6 (0.9)	405.2 (0.9)	405.5 (0.9)	405.7 (0.8)	405.8 (0.8)	405.3 (0.9)
36	1360	419.1	415.0 (1.0)	414.1 (1.2)	415.5 (0.9)	414.4 (1.1)	415.7 (0.8)	415.5 (0.8)	415.8 (0.8)	415.5 (0.9)	415.2 (0.9)	415.4 (0.9)	415.6 (0.8)	415.5 (0.9)
36	1340	439.3	435.2 (0.9)	433.9 (1.2)	435.2 (0.9)	435.3 (0.9)	435.2 (0.9)	435.4 (0.9)	435.1 (0.9)	435.5 (0.9)	435.4 (0.9)	434.6 (1.1)	435.1 (1.0)	435.0 (1.0)
36	1320	458.9	455.1 (0.8)	456.3 (0.6)	456.9 (0.4)	453.8 (1.1)	455.4 (0.8)	454.7 (0.9)	455.0 (0.9)	455.2 (0.8)	455.1 (0.8)	454.6 (0.9)	455.1 (0.8)	455.2 (0.8)
36	1300	478.8	473.9 (1.0)	475.7 (0.6)	477.0 (0.4)	474.6 (0.9)	474.9 (0.8)	475.0 (0.8)	474.7 (0.8)	474.6 (0.9)	474.5 (0.9)	474.7 (0.9)	475.0 (0.8)	475.4 (0.7)
36	1280	498.7	494.9 (0.8)	494.7 (0.8)	494.6 (0.8)	493.8 (1.0)	494.7 (0.8)	495.2 (0.7)	494.8 (0.8)	494.6 (0.8)	495.0 (0.7)	494.9 (0.8)	494.6 (0.8)	494.6 (0.8)
36	1260	518.7	514.5 (0.8)	515.1 (0.7)	512.7 (1.2)	511.4 (1.4)	514.8 (0.7)	514.9 (0.7)	514.8 (0.7)	515.1 (0.7)	514.6 (0.8)	514.6 (0.8)	514.2 (0.9)	514.6 (0.8)
36	1240	538.6	534.4 (0.8)	533.8 (0.9)	534.7 (0.7)	534.3 (0.8)	534.6 (0.7)	534.3 (0.8)	534.7 (0.7)	534.2 (0.8)	535.0 (0.7)	534.7 (0.7)	534.6 (0.7)	534.1 (0.8)
36	1220	558.8	553.9 (0.9)	556.2 (0.5)	553.9 (0.9)	554.2 (0.8)	554.3 (0.8)	554.0 (0.8)	554.7 (0.7)	554.6 (0.7)	554.2 (0.8)	554.2 (0.8)	554.7 (0.7)	554.3 (0.8)
36	1200	578.8	574.2 (0.8)	573.3 (1.0)	574.3 (0.8)	573.1 (1.0)	574.7 (0.7)	574.5 (0.7)	574.6 (0.7)	574.3 (0.8)	573.8 (0.9)	574.5 (0.7)	574.6 (0.7)	574.3 (0.8)
36	1180	598.3	593.8 (0.7)	593.9 (0.7)	593.6 (0.8)	595.5 (0.5)	594.3 (0.7)	594.0 (0.7)	594.3 (0.7)	594.3 (0.7)	594.5 (0.6)	594.1 (0.7)	594.1 (0.7)	594.0 (0.7)
36	1160	618.6	613.6 (0.8)	614.4 (0.7)	616.5 (0.3)	613.6 (0.8)	613.9 (0.8)	614.4 (0.7)	613.9 (0.8)	614.2 (0.7)	613.9 (0.8)	614.0 (0.7)	613.6 (0.8)	614.3 (0.7)
36	1140	639.0	633.9 (0.8)	633.7 (0.8)	632.5 (1.0)	633.0 (0.9)	634.0 (0.8)	633.8 (0.8)	633.2 (0.9)	634.1 (0.8)	633.7 (0.8)	634.4 (0.7)	634.2 (0.8)	633.9 (0.8)
36	1120	658.1	653.5 (0.7)	651.7 (1.0)	651.5 (1.0)	657.1 (0.1)	653.8 (0.7)	654.0 (0.6)	654.0 (0.6)	653.5 (0.7)	653.4 (0.7)	653.6 (0.7)	653.5 (0.7)	654.2 (0.6)
36	1100	678.0	673.8 (0.6)	674.2 (0.6)	675.7 (0.3)	674.4 (0.5)	673.9 (0.6)	673.7 (0.6)	673.7 (0.6)	673.6 (0.6)	673.7 (0.6)	673.6 (0.6)	673.6 (0.6)	674.2 (0.6)
36	1080	698.4	693.7 (0.7)	691.6 (1.0)	694.4 (0.6)	691.9 (0.9)	693.7 (0.7)	693.4 (0.7)	693.6 (0.7)	693.8 (0.7)	693.5 (0.7)	693.5 (0.7)	693.7 (0.7)	693.6 (0.7)
36	1060	717.9	713.2 (0.7)	715.4 (0.4)	713.5 (0.6)	713.1 (0.7)	713.6 (0.6)	713.4 (0.6)	713.8 (0.6)	713.4 (0.6)	713.2 (0.7)	713.5 (0.6)	713.0 (0.7)	713.7 (0.6)