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Asset Selection for Stock Market Investment Using Bonferroni OWA-Based Penalty Function

Abstract. *In the decision-making process, it is common to encounter disagreements that can undermine the importance of information through penalties. This article proposes a new penalty function based on Bonferroni means and ordered weighted averaging (OWA), referred to as Bonferroni OWA-based penalties (BP-OWA). Likewise, this operator can be extended with induced variables, heavy weights, and weighted averages. This approach's main advantage is that it captures the importance, preferences, expectations, and attitudes associated with penalty gradation, the interrelationships among arguments, weighting vectors, and the reordering process, which considers approximate reasoning in decision-making. Finally, new methods are applied to asset selection for stock market investment using Yahoo Finance, with different penalty weights for stocks, generating new return expectations to be considered in the decision of which stock to include in the portfolio.*

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1. Introduction

The field of group decision-making has seen significant advancements in aggregation and operator functions. There is growing interest in optimising and developing novel approaches to aggregate diverse information types to obtain a single representative value (Beliakov et al., 2016). In research on aggregation operators, the Bonferroni means are a notable area of focus (Bonferroni, 1950). This operator is characterised by its capacity for multiple comparisons, thereby

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compensating for potential errors through the alpha significance level for each comparison (Barnett et al., 2022). Considering these distinctive attributes, a range of extensions have been developed by combining the Bonferroni means with other aggregation methods. For example, some proposals combine fuzzy aggregation information (Chiao, 2021), generalised (Yager, 2009), and so on. Among the extensions developed, we focus on those used in conjunction with OWA operators. This allows the information to be reordered to obtain results within a specified range of minimum and maximum values (Yager, 1988). This operator, along with some of its extensions, combined with the Bonferroni means, enables reordering by continuously and simultaneously considering interrelationships, thereby facilitating more complex decision-making (Blanco-Mesa et al., 2019).

One of the complex decision problems in the world of investments is selecting assets for an investment portfolio (Hui et al., 2009). This decision is linked to several elements. On the one hand, it is essential to understand the investor's profile, including their risk tolerance and investment horizon (De Bortoli et al., 2019). On the other hand, since there is such a large number of assets, generating an analysis of all these becomes highly complicated, so you must choose between stocks, bonds, exchange rates, commodities, and other types of assets and then select which one of this segment you want to include (Hung et al., 2024). In this way, by considering different information aggregators that account for historical data, expectations, knowledge, and the decision-maker's skills, this type of problem becomes a versatile tool (Aliyeva et al., 2024).

In this sense, this article aims to present a new aggregation operator using the elements of a penalty function, Bonferroni means, and OWA operators to be used in the forecasting of the return of different stocks to be included in an investment portfolio, considering not just the historical data, but also incorporating different subjective information obtained by the decision-maker. Therefore, this work proposed a new penalty function, called Bonferroni OWA-based penalties (BP-OWA), based on Bonferroni means and ordered weighted average (OWA). Likewise, this operator can be extended with induced variables, heavy weights, and weighted averages. This research makes a significant contribution to the field by employing the operator to analyse the interrelations among arguments using the Bonferroni mean, the ordered weighted average (OWA) family of operators, and a penalty function for criteria outside the established parameters. These new operators can simultaneously capture the importance, preferences, expectations, and attitudes associated with disagreement, which can then be considered alongside the decision-maker's approximate reasoning to evaluate and select assets for investment more efficiently. Finally, for the decision-making process, a dataset of the top five tech stocks from Yahoo Finance is used as a mathematical tool to make the most efficient asset selection for stock market investment.

The structure of the paper is the following. Section 2 presents basic definitions of the penalty function, OWA operator, Bonferroni means, and BON–OWA. Section 3 presents the proposition of the Penalty-based BON–OWA operator (BP–OWA) and its extensions. Section 4 presents a step-by-step process for asset selection in the investment process. Finally, Section 5 presents the conclusions of the work

2. Literature review

In this section, the main definitions necessary to understand the new propositions are presented. The first method presented is the penalty function, which measures disagreement or dissimilarity and has been adjusted to account for penalty depreciation during the aggregation of expert opinions (Calvo & Beliakov, 2010).

Definition 1. The function $P: R^{n+1} \to (0, \infty)$ is a penalty function if it satisfies the following conditions:

$P(x_i, y) \geq 0$ for all $i = 1, \dots, n$;

$P(x_i, y) = 0$ if $x_i = y$;

For all fixed x, the penalty value will be a singular element or an interval, where $x = (x_1, \dots, x_n)$ is an input vector, while y is an output value or a fused value. It is essential to note that, according to the first two requirements, negative penalty values are not accepted. If a complete consensus is achieved, then a zero penalty is obtained.

Additionally, the OWA operator (Yager, 1988) is an aggregation function that, based on a weighting vector and a reordering of the arguments and weights, allows the calculation of maximum and minimum operators. The following definition illustrates this concept:

Definition 2. An OWA operator of dimension n is a mapping $OWA: R^n \to R$ with an associated weight vector W, then,

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_i denotes the jth largest value in the collection $a_1, a_2, \dots, a_n$.

Another extension that will be considered is the induced OWA (IOWA) operator (Yager & Filev, 1999), which, by including induced values, allows specific weights to be assigned to arguments. The definition is

Definition 3. An IOWA operator of dimension n is a mapping $IOWA: R^n \times R^n \to R$ such that it has an associated weight vector W, such that_

$$IOWA(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \dots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

where b_j is the a_i value of the OWA pair $< u_i, a_i >$ having the jth largest u_i. u_i is the order-inducing variable, and a_i is the argument variable.

An additional extension of the OWA operator to be used is the Heavy OWA (HOWA) operator (Yager, 2002), which, with an unbounded weighted vector, allows for under- or overestimation in the analysis. The definition is:

Definition 4. A HOWA operator is a mapping $HOWA: R^n \to R$ associated with weight vector W, such that

$$HOWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j\, b_j, \tag{3}$$

where b_j is the jth largest element of the collection $a_1, a_2, \ldots, a_n$ and the sum of the weights w_j is bounded to n or can be unbounded if the weighting vector W, $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

The last extension of the OWA operator used in the article is the Ordered Weighted Averaging Weighted Average (OWAWA) operator (Hussain et al., 2021). The definition is as follows:

Definition 5. An OWAWA operator of dimension n is a mapping $OWAWA: R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \quad (4)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{j=1}^{n} v_j = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

Also, the Bonferroni means will be used to make new propositions. The main advantage of this aggregation operator is that it can incorporate the interrelationship of the arguments in the formulation, generating a more complex analysis of the data (Bonferroni, 1950). The formulation is:

Definition 6. The Bonferroni mean is a continuous aggregation function that is given by:

$$\mathrm{B}(a_1, a_2, \ldots, a_n) = \left(\sum_{k=1}^{n} a_i^r \left(\frac{1}{1-n} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q \right) \right)^{\frac{1}{r+q}} \quad (5)$$

Finally, the Bonferroni means have been combined with the OWA operator. In this operator, there is an interrelationship between the arguments via Bonferroni means and a weighting vector, with a reordering process based on the values of the arguments, allowing for different comparisons of the information (Blanco-Mesa et al., 2020). The definition is as follows.

Definition 7. The Bonferroni OWA (BON-OWA) is a mean-type continuous aggregation function that is given by:

$$BON - OWA(a_1, \ldots, a_n) = \left(\frac{1}{n} \sum_i a_i^r OWA_W(V^i) \right)^{\frac{1}{r+q}}, \quad (6)$$

where $OWA_W(V^i) = \left(\frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^q \right)$ and (V^i) is the vector for all a_j excluding a_i. w is a weighting vector associated with α_i and w_{ij} are the components of the OWA weights. Then, $OWA_W(V^i) = \left(\sum_{j=1}^{n-1} w_i a_{\pi_k(j)} \right)$, where $a_{\pi_k(j)}$ denotes the largest value in tuple V^i and $w_i = \frac{1}{n-1}$ for all i.

3. Bonferroni OWA-based penalties (BP-OWA)

This section presents Bonferroni OWA-based penalties (BP–OWA) that combine penalty aggregation functions and the Bonferroni OWA operator. The main advantage is that it can integrate a range of penalty-based functions, including the classical Bonferroni aggregation and the OWA operator. This allows incorporating expert opinions into the decision-making process, evaluating them based on factors such as disagreement or dissimilarity, multiple comparisons, and importance. The formulations are presented below for reference.

Proposition 1. The Bonferroni Penalty means is a mean-type continuous aggregation function that considers the disagreement and attitudinal character of expert opinions:

$$\mathrm{BP}(x_i,\mathrm{y}) = \left(\sum_{k=1}^{n} p(x_i,\mathrm{y})_i^r \left(\frac{1}{1-n}\sum_{\substack{j=1\\ j\neq i}}^{n} p(x_j,\mathrm{y})_j^q\right)\right)^{\frac{1}{r+q}}, \tag{7}$$

Proposition 2. The Bonferroni Penalty OWA (BP–OWA) is a mean-type continuous aggregation function that considers the disagreement and attitudinal character of expert opinions, defined as follows:

$$BP-OWA(x_i,\mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i,\mathrm{y})_i^r\, pOWA_W(V^i)\right)^{\frac{1}{r+q}}, \tag{8}$$

where $pOWA_W(V^i) = \left(\frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1\\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} p(x_j,\mathrm{y})_j^q\right)$ and (V^i) is the vector for all $p(x_j,\mathrm{y})$ excluding $p(x_i,\mathrm{y})$. w is a weighting vector associated with α_i and w_{ij} are the components of the OWA weights. Then, $pOWA_W(V^i) = \left(\sum_{\mathrm{j}=1}^{n-1} w_i p(x_j,\mathrm{y})_{\pi_k(j)}\right)$, where $p(x_j,\mathrm{y})_{\pi_k(j)}$ denotes the largest value in tuple V^i and $w_i = \frac{1}{n-1}$ for all i.

Let us discuss the properties of BP–OWA. Note that the proofs are in annex 1.

Theorem 1: (Commutativity-OWA). Assume f is the BP–OWA, operators, then $f(\boldsymbol{x_i},\mathrm{y}) = f(\boldsymbol{x_i},\mathrm{y})$.

Theorem 2: (Monotonicity). Assume f is the BP–OWA operators; if $(\boldsymbol{x_i},\mathrm{y}) \geq (\boldsymbol{c_i},\mathrm{y})$ for all i_i, then $f(\boldsymbol{x_i},\mathrm{y}) \geq f(\boldsymbol{c_i},\mathrm{y})$.

Theorem 3: (Bounded). Assume f is the BP–OWA operators, then: $min(\boldsymbol{x_i},\mathrm{y}) \leq f(\boldsymbol{x_i},\mathrm{y}) \leq max(\boldsymbol{x_i},\mathrm{y})$.

Theorem 4: (Idempotency). Assume f is the BP–OWA operators; if $(\boldsymbol{x_i},\mathrm{y}) = a$ for all i, then $f(\boldsymbol{x_i},\mathrm{y}) = a$.

Theorem 5: (Non-negativity). Assume f is the BP–OWA operators then: $f(\boldsymbol{x_i},\mathrm{y}) \geq 0$

Theorem 6: (Reflexivity). Assume f is the BP–OWA operators then: $f(\boldsymbol{x_i},\mathrm{y}) = 0$

In addition, if $q = 0$, then BP–OWA, they follow that:

$$\mathrm{BP-OWA}^{r,0}(\boldsymbol{x_i},\mathrm{y})=\left(\frac{1}{\mathrm{n}}\sum_{\mathrm{k=1}}^{\mathrm{n}} pb_{\mathrm{i}}^{\mathrm{r}}\left(\frac{1}{\mathrm{n-1}}\sum_{\substack{\mathrm{j=1}\\ \mathrm{j\neq k}}}^{\mathrm{n}} pb_{\mathrm{j}}^{0}\right)\right)^{\frac{1}{\mathrm{r+0}}}= \left(\frac{1}{\mathrm{n}}\sum_{\mathrm{k=1}}^{\mathrm{n}} pb_{\mathrm{i}}^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}}. \quad (9)$$

If $r=2$ and $q=0$, then BP–OWA reduces to square mean:

$$\mathrm{BP-OWA}^{2,0}(\boldsymbol{x_i},\mathrm{y})=\left(\frac{1}{\mathrm{n}}\sum_{\mathrm{k=1}}^{\mathrm{n}} pb_{\mathrm{i}}^{2}\right)^{\frac{1}{2}}. \quad (10)$$

If $r=1$ and $q=0$, then BP–OWA, reduces to the usual average:

$$\mathrm{BP-OWA}^{1,0}(\boldsymbol{x_i},\mathrm{y})=\frac{1}{\mathrm{n}}\sum_{\mathrm{k=1}}^{\mathrm{n}} pb_i \quad (11)$$

If $r\to+\infty$ and $q=0$, then BP–OWA, reduces to the max operator

$$\lim_{\mathrm{r}\to+\infty}\mathrm{BP-OWA}^{\mathrm{r},0}(\boldsymbol{x_i},\mathrm{y})=max\{pb_i\} \quad (12)$$

If $r\to 0$ and $q=0$, then BP–OWA reduces to the geometric mean:

$$\lim_{\mathrm{r}\to+\infty}\mathrm{BP-OWA}^{\mathrm{r},0}(\boldsymbol{x_i},\mathrm{y})=\left(\prod_{i=1}^{n} pb_{\mathrm{i}}\right)^{\frac{1}{\mathrm{n}}}. \quad (13)$$

If $r=q=1$, then BP–OWA, reduces to the following expression:

$$\mathrm{BP-OWA}^{1,1}(\boldsymbol{x_i},\mathrm{y})=\left(\frac{1}{\mathrm{n(1-n)}}\sum_{\substack{\mathrm{k,j=1}\\ \mathrm{j\neq k}}}^{\mathrm{n}} pb_i pb_j\right)^{\frac{1}{2}}. \quad (14)$$

Additionally, a new extension of the BP-OWA can be achieved by considering induced values, a heavy-weighting vector, and a new weighted average vector (See Proposition 3-5, respectively). These three propositions incorporate the main characteristics of their respective base definitions, namely Definitions 3, 4, and 5. Finally, the properties of these new definitions follow the same explanations as the BP-OWA operator, and to avoid redundancy, they are not detailed.

Proposition 3. A Bonferroni Penalty Induced OWA (BP–IOWA) is a mean-type of continuous aggregation operator that considers the disagreement, induced variable, and attitudinal character of expert opinions, defined as follows:

$$BP-IOWA(\langle \boldsymbol{x},u,\mathrm{y}\rangle)=\left(\frac{1}{n}\sum_i pb_i^r pIOWA_W\left(V^i\right)\right)^{\frac{1}{r+q}}, \quad (15)$$

where pb_i is the pa_i value of the BP–IOWA pair $<\boldsymbol{x_i},u_i,\mathrm{y}>$ having the jth largest u_i and $pIOWA_W\left(V^i\right)=\left(\frac{1}{\mathrm{n-1}}\sum_{\substack{\mathrm{j=1}\\ \mathrm{j\neq i}}}^{\mathrm{n}} pb_j^q\right)$ with $\left(V^i\right)$ being the vector of all pb_j except pb_i and w being an $n-1$ vector W_i associated with α_i whose components w_{ij} are the OWA weights.

According to 27 W be an OWA weighing vector of dimension $n-1$ with components $w_i\in[0,1]$ when $\sum_i w_i=1$, where the weights are associated according to the largest value of u_i and u_i is the order-inducing variable.

Proposition 4. A Bonferroni Penalty Heavy OWA (BP–HOWA) is a mean-type of continuous aggregation operator that considers the disagreement and attitudinal character of expert opinions, where the sum of the weights w_j is bounded to n, defined as follows:

$$BP - HOWA(\langle \boldsymbol{x}, \mathrm{y}\rangle) = \left(\frac{1}{n}\sum_i pb_i^r pHOWA_W\left(V^i\right)\right)^{\frac{1}{r+q}}, \qquad (16)$$

where pb_i is the pa_i value of the BP–HOWA pair $< \boldsymbol{x_i}, \mathrm{y} >$ having the jth largest $pHOWA_W\left(V^i\right) = \left(\frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} pb_j^q\right)$ with $\left(V^i\right)$ being the vector of all pb_j except pb_i and w being an $n-1$ vector W_i associated with α_i whose components w_{ij} are the OWA weights.

Proposition 5. A Bonferroni Penalty Ordered Weighted Average Weighted Average (BP–OWAWA) is a mean-type of continuous aggregation operator that considers the disagreement, weighted average vector, and attitudinal character of expert opinions, defined as follows:

$$BON - OWAWA(a_1, \dots, a_n) = \upbeta \times \left(\frac{1}{\mathrm{n}}\sum_{\mathrm{i}} b_i^r OWA_W\left(V^i\right)\right)^{\frac{1}{\mathrm{r+q}}} + (1 - \upbeta) \times \left(\frac{1}{\mathrm{n}}\sum_{i=1}^{n} b_i^r\, WA_{V_i}\left(V^i\right)\right)^{\frac{1}{\mathrm{r+q}}}, \qquad (17)$$

where b_i is the a_i value of the BON-OWAWA and $OWA_W\left(V^i\right) = \left(\frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} b_j^q\right)$ $and\ WA_{V_i}\left(V^i\right) = \frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} b_j^q$ where $\left(V^i\right)$ being the vector of all b_j except b_i and w being an $n-1$ vector W_i associated with α_i whose components w_{ij} are the OWA weights. Let W be an OWA weighing vector of dimension $n-1$ with components $w_i \in [0,1]$ when $\sum_i w_i = 1$ and each argument b_i has an associated weight (WA) v_i with $\sum_{j=1}^{n} v_j = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta)v_j$ with $\beta \in [0,1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the b_i.

Proposition 6. A Bonferroni Penalty OWAWA (BP–OWAWA) is a mean-type of continuous aggregation operator that considers the disagreement and attitudinal character of expert opinions, defined as follows:

$$BP - OWAWA(\langle \boldsymbol{x}, \mathrm{y}\rangle) = \left(\frac{1}{n}\sum_i pb_i^r pOWAWA_W\left(V^i\right)\right)^{\frac{1}{r+q}}, \qquad (18)$$

where pb_i is the pa_i value of the BP–OWAWA and $pOWA_W\left(V^i\right) = \left(\frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} pb_j^q\right)$ $and\ pWA_{V_i}\left(V^i\right) = \frac{1}{\mathrm{n}-1}\sum_{\substack{\mathrm{j}=1 \\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} pb_j^q$ where $\left(V^i\right)$ being the vector of all pb_j except pb_i and w being an $n-1$ vector W_i associated with α_i whose components w_{ij} are the OWA weights

To better understand the differences between the propositions presented, a specific analysis of their main properties will be conducted. The BP operator generates the interrelationship of the arguments (by using the Bonferroni means) and includes the disagreement or dissimilarity (by using the Penalty function). The BP–OWA operator includes two new characteristics: the first one is the use of a weighting vector, and the second one is the reordering process based on the value of the arguments, with which it is possible to generate a new range of results based on different reordering processes, like the maximum or minimum. The reason to use this is to incorporate the expert's expectations and knowledge into the calculation process, based on the decisions made in the weighting vector and the reordering process, potentially over- or underestimating the traditional results based on the BP operator.

The BP–IOWA operator introduces a new possibility in the formulation: the introduction of induced values, which will serve as the basis for the reordering process. So, instead of doing it based on the values of the arguments, the arrangement between the weights and arguments will be done using that induced vector. The main idea is to develop a new rearrangement process based on the expert's or decision maker's information. The BP–HOWA operator uses an unbounded weighting vector, that is, the sum of the weights is not equal to one; they can be higher or lower. This idea presents a new range of possibilities based on expectations of the future, because the unbounded weighting vector can lead to over- or underestimation of the results. Finally, the BP–OWAWA operator uses two sets of weighing vectors to include different weight scenarios.

As can be seen, each proposition has different uses, and none is better than the others. The main idea is that, depending on the problem's complexity, we can use different operators; for example, if the problem does not involve disagreement or dissimilarity, the Penalty function isn't needed. The same idea: if the expert or decision maker indicates that the weighting vector must be bounded because the expectation of the future indicates that it won't present a radical change in the trend, then the use of the BP–HOWA is not needed. In this sense, the use of a specific operator depends mainly on the characteristics of the problem to be solved and its complexity, and based on that, the use or non-use of different operators will be justified.

4. Application in the asset selection process for investment

In the market, there are various ways and instruments by which a return can be obtained (Goyal et al., 2023). In a complex and changing financial environment, selecting assets for investment is an important challenge for portfolio management (Baker et al., 2020), with the main objective of maximising returns. This is a complex task, so making timely decisions helps anticipate changes in market trends, allowing investors to make relevant adjustments (Markowitz, 1952). Over the years, various contributions to science have improved the performance of statistical models (Torra et al., 2018). Several approaches have been used for analysis in the search for profitability maximisation, which help in selecting assets (Wang, 2019). One of them is multiple linear regression analysis where price is used as a dependent variable (Ge & Wu, 2020), factorial models that provide Markowitz's theory with additional factors for the explanation of returns such as the size of the company and growth trends (Fama & French, 1993), also, value at risk (VaR) models that take into account the risk analysed in the determined time horizon (Afzal et al., 2021). In the search for better results, various options for more complex tools in asset selection have been proposed, such as machine and deep learning models that learn from relationships between vectors and their characteristics (Zhang et al., 2020).

These techniques enable modelling complex, nonlinear patterns in financial data, overcoming some of the limitations of traditional models (Gu et al., 2020), such as autoregressive integrated moving averages and partial least squares. This allows us to compare performance for efficient decision-making (Peng et al., 2015), among others. From the point of view of Clinton and Lewis (2008), traditional statistical techniques offer an opportunity because they analyse historical information and do not consider decision makers, expectations, or expert knowledge. This section presents a step-by-step process for using the proposed aggregation operators in an investment selection process, specifically for comparing stocks based on expected returns.

Step 1. The initial process involves selecting the assets to be analysed; in this case, the top five tech stocks were considered (Yahoo Finance, 2024): Apple Inc., Microsoft Corporation, NVIDIA Corporation, Alphabet Inc., and Amazon.com Inc. For this analysis, information was obtained from January 1, 2024, to June 30, 2024, and extracted from Yahoo Finance. The daily returns were also obtained, and the data will be analysed in the following steps. This step was performed using R; the code is presented in Annex 2.

It is crucial to include assets, in this case stocks, that will have similar behaviours because of the effect of the penalty function and weighting vector will have in the results, if a large set of stocks of different sectors is used, it will be harder to decide the values of the penalty function and weighting vector values for the investor, because the trends and expectations for one sector it can be very different from another. Because of that, in this paper, only five stocks from the most important technological companies were used, as the trends and expectations for all of them are nearly the same, according to the investor consulted for the information.

Another element that has to be asked is the timeframe to be used for the calculation. In this case, 6 months were used because the investor believed a longer timeframe would impact the result, given the sector's high volatility and the fact that all stock values had been increasing since 2020. Considering that, if a more extended timeframe is used, the results will show a strong positive trend.

Step 2. Because it is challenging to obtain or understand the daily weightings from the expert, investor, or decision-maker, the average monthly return will be used instead (see Table 1). With that information, an investor was asked to give a weight to each month, considering that the company may have similar or related results; with that in mind, the weighting vector is $W = (0.20, 0.15, 0.15, 0.20, 0.10, 0.20)$. Additionally, the investor considered a penalty value of 0.10 for NVDA, as the decision maker believes the company's value will not remain stable in the second half of the year; this is evident in the cumulative returns for the first half of the year. For GOOG and AMZN, it will be 0.05, and for AAPL and MSFT, it will be 0.02.

Also, the induced values are $U = (3, 4, 5, 2, 1, 6)$, the heavy weighting vector is $H = (0.20, 0.15, 0.20, 0.25, 0.15, 0.20)$ and the second weighting vector is $W_2 = (0.15, 0.25, 0.20, 0.20, 0.10, 0.10)$ The weighting vectors are 70% and 30%, respectively.

Table 1. Average monthly return for different stocks

Month	AAPL	MSFT	NVDA	GOOG	AMZN
January	-0.005015	0.070702	0.249927	0.020100	0.036404
February	-0.018130	0.042986	0.269394	-0.011919	0.135256
March	-0.050185	0.018713	0.142071	0.088215	0.021714
April	-0.004419	-0.075418	-0.030768	0.085559	-0.027309
May	0.125223	0.067521	0.249911	0.056222	0.009660
June	0.095483	0.074625	0.130042	0.055297	0.093347
Volatility	0.069690	0.057278	0.114300	0.038762	0.059291

Source: Authors' own creation.

Step 3. Based on the data presented in Step 2, the subsequent calculations will use the specified aggregation operators. In this instance, the operators in question are BP–OWA, BP–IOWA, BP–HOWA, and BP–OWAWA. The resulting data are presented in Table 3.

Table 2. Results using different aggregation operators.

	Average	BP–OWA	BP–IOWA	BP–HOWA	BP–WA	BP–OWAWA
AAPL	2.3826%	5.4338%	5.3452%	5.7928%	5.0002%	5.3037%
MSFT	3.3188%	4.3439%	4.0661%	4.5691%	4.2883%	4.3272%
NVDA	16.8430%	10.0836%	9.8544%	10.5545%	9.9246%	10.0359%
GOOG	4.8912%	2.5454%	2.6687%	2.7005%	2.6222%	2.5685%
AMZN	4.4845%	4.5311%	4.4873%	4.7254%	4.3867%	4.4878%

Source: Authors' own creation.

Step 4. Considering Table 2 it is possible to visualise different elements that help us to understand the possible outcomes based on the different operators. First, it is possible to visualise that based on the average NVDA is the stock with the highest return, after that is GOOG, AMZN, AAPL, and MSFT, respectively. But these results change when the different aggregation operators were used. One interesting result is that NVDA remains the stock with the highest return even when it received the higher penalty, but in the case of GOOG the use of the penalty function (considering that GOOG and AMZN are the stocks with the second highest penalty) and the interrelationship of the values based on the Bonferroni means consider an important decrease in expected return. Also, we can visualise an increase in the values of APPL and MSFT, considering that they had the lowest penalty value. All this can be visualised in Graph 1.

Also, the results will be analysed using the information presented in Table 3 to facilitate their visualisation. Graph 1 is presented for illustrative purposes. The data indicates that, it is recommended that NVDA be considered as the primary investment option. After that, there is a difference between the ranking provided by the average and the aggregation operators that uses penalty functions, Bonferroni means, OWA operators and its extensions. As can be seen in Table 3, it is possible to visualise the change of GOOG from rank 2 in the average to rank 5 in all other operators, changing its position with AAPL. Also, in all the operators AMZN and MSFT remain in third and fourth position respectively, but decreasing its distance between the results, in the case of the average the difference between AMZN and MSFT is 1.17% and with the use of the other aggregation operators it has an average difference of 0.20%.

Table 3. Ranking of the stocks based on the aggregation operators

Aggregation Operators	Ranking
Average	$NVDA > GOOG > AMZN > MSFT > AAPL$
BP–OWA	$NVDA > AAPL > AMZN > MSFT > GOOG$
BP–IOWA	$NVDA > AAPL > AMZN > MSFT > GOOG$
BP–HOWA	$NVDA > AAPL > AMZN > MSFT > GOOG$
BP–WA	$NVDA > AAPL > AMZN > MSFT > GOOG$
BP–OWAWA	$NVDA > AAPL > AMZN > MSFT > GOOG$

Source: Authors' own creation.

A specific analysis of each result is presented in Graph 2. For all stocks, each method offers different results. Notably, BP–HOWA provides the best return expectations. This is possible because of the heavy operator: when the heavy vector is greater than 1, the results will be overestimated.

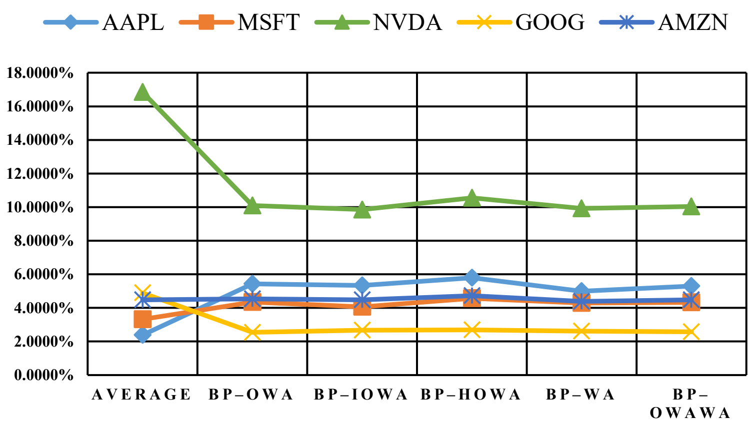
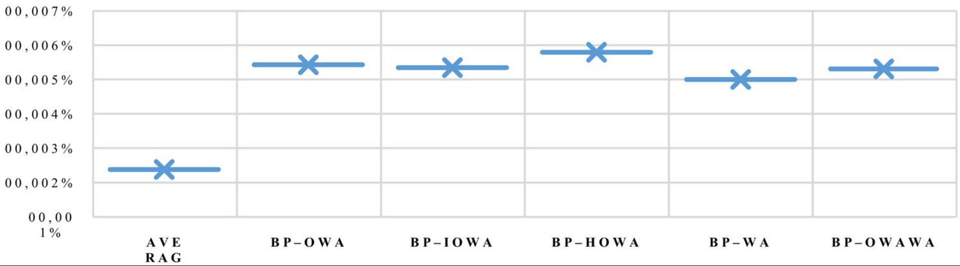


Figure 1. Results of the return of stocks using Penalty-based B-OWA's operators
Source: Authors' own creation.

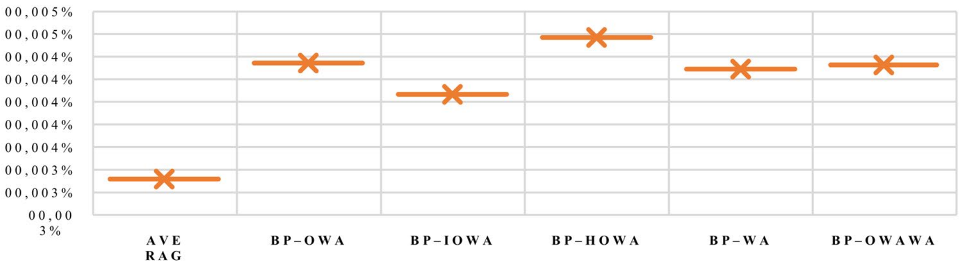
Thus, in the expectation of stock opportunities, there may be a trade-off between the penalty and the importance given to the argument to be considered. It is also noteworthy that BP–IOWA exhibits considerable variability in the expectation results. GOOG yields the optimal return, while MSFT and NVDA yield the least favourable returns, and AAPL and AMZN yield intermediate returns. This variable behaviour in the results occurs because the ordering of each variable is determined by an induced variable that represents the decision-maker's preferences.

Considering BP–OWAWA, the results appear relatively stable, exhibiting only slight variations with a slight tendency towards less favourable outcomes. In this context, the expectation of returns is influenced by both the weight vector and the ordered weighted average, with the β factor representing the decision maker's expectation. When considering only BP–WA, the ordering of the arguments associated with the weight vector is inconsequential; a direct aggregation of the weighting is made based on the penalty characteristics and the Bonferroni mean. Their results indicate the least favourable returns, which appear conservative or pessimistic. Notably, except for GOOG, BP–OWA presents intermediate returns. This enables the presentation of the ordered weighted average, thereby facilitating comparison with other extensions that consider under- or overestimation. When different positions on the same decision are considered, the results transform. Hence, it is noteworthy that penalisation can be incorporated into many investment decisions. For example, in this case, only tech companies were included; however, in a portfolio that considers different industries, penalisation can be applied not only to individual companies but also to the sector as a whole. With that in mind, penalties are closely related to the investor and their profile, and they are an element that can be easily applied to this kind of problem.

AAPL



MSFT



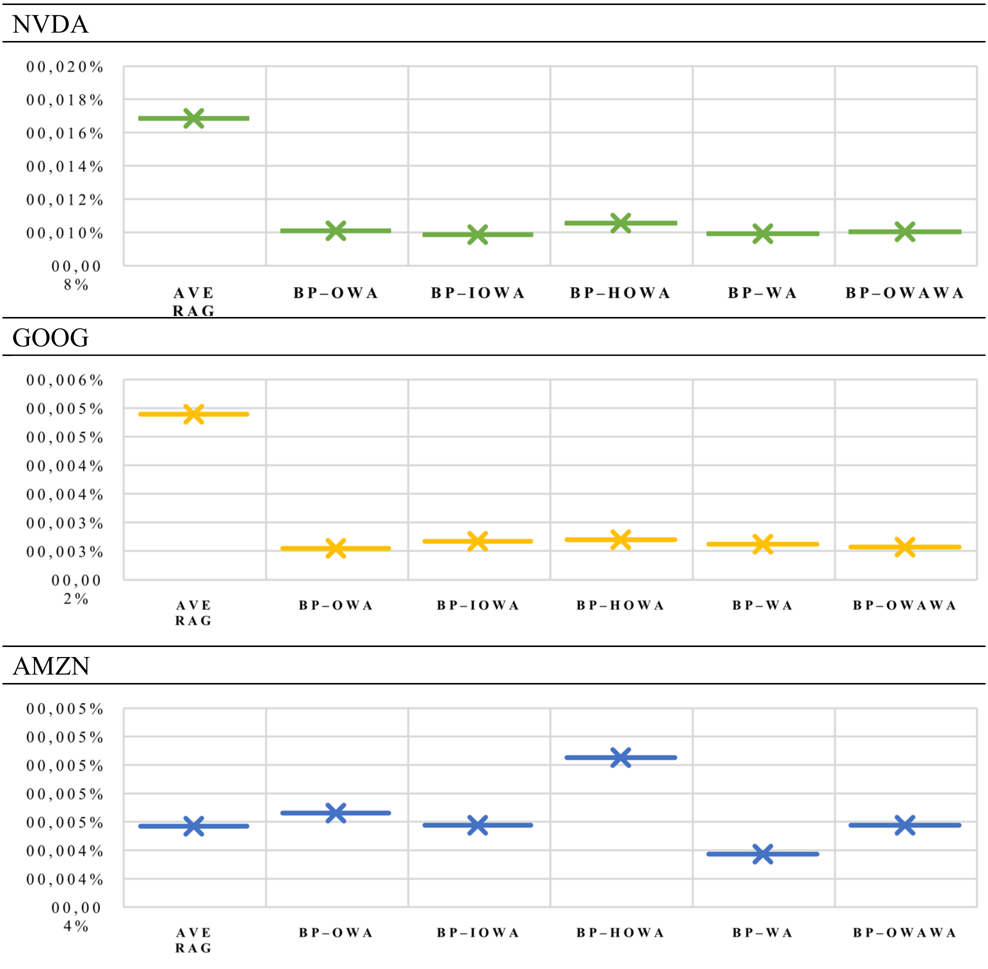


Figure 2. Results of each return of stocks by Penalty-based B-OWA's operators

Source: Authors' own creation.

Consequently, the decision-maker may consider several parameters in the decision-making process, including factors such as importance (what it highlights), preference (what it desires), expectations (what it projects), attitude (what it feels), and penalty (what it disagrees with), all of which are considered simultaneously. Each parameter may yield a different result, depending on the circumstances. In this context, the term "parameter" refers to a method of approximate reasoning in decision-making processes that seeks to maximise returns. Moreover, integrating penalty functions with each parameter, specifically with each OWA family operator, provides a novel perspective that incorporates disagreement and nuanced ideal positions in decision-making. In this instance, the focus is on selecting financial assets.

Based on the last paragraph, when the investor is going to make its final decision on stock selection, each investor usually has different attributes and does different analyses, for example, risk aversion, investment horizon, and capital to invest. In this sense and as has been write at the end of Section 3, the main idea of using different aggregation operators is to obtain a range of different results using the same information and that the investor can visualise the scenarios of the different stocks and with that it can visualise if the returns that are presented for each one of them are well according to the risk perception that the same has. Also, with the use of the weighting vector, induced values, or a heavy weighting vector, the investor can include their knowledge and expectation of the future, not just of the general market, but also based on the company or sector, generating different values for each analysis. As a result, it can yield different results depending on their expectations.

Finally, by proposing new extensions based on penalty functions, it is possible to improve the literature on this topic. Since the introduction of the concept, it has been used in different ways to improve the decision-making process and voting methods (Bernau, 1990; Bustince et al., 2013; Elkano et al., 2018), and because of its importance in that topic, different approaches have been developed, like the use of fuzzy logic and interval values (Souza Oliveira et al., 2022), linguistic and granulation models (Zhang et al., 2022), and with the use of OWA operators (Rathod et al., 2025). In this paper, we propose new propositions for applying penalty functions in more complex situations, using Bonferroni means, induced values, and heavy-weighting vectors.

5. Conclusions

One of the main problems in asset selection is that traditional models assume returns follow a normal distribution, which is not always true in practice (Sheikh & Qiao 2009). Given the above, the decision-maker seeks new techniques to adapt to market changes for improved results (Jarrett et al., 2019). To solve the above, there are aggregation operators. This work presents the Bonferroni OWA-based Penalty function and its application to selecting listed assets. Special cases are also studied, and some extensions for the operator are developed. In this way, the proposed model offers a systematic process and is adaptable to different contexts, highlighting the relevance of including subjective factors, such as penalties, in selecting financial assets.

This study's investment asset selection process highlights the importance of incorporating customised criteria, such as penalties based on the perception of future risk, to adjust investment decisions in line with the investor's expectations and profile. The results show that, even with the application of penalties, NVIDIA remains the most attractive option, followed by Apple, highlighting the resilience of certain assets despite negative adjustments. In addition, it is evident that aggregation methodologies such as BP–OWA, BP–IOWA, BP–HOWA, and BP–OWAWA enable a more robust analysis by combining different weights and parameters. Likewise, these operators can capture importance, preference, expectations, and attitudes associated with the penalty gradation of a decision and approximate reasoning in decision-making processes that expect to yield the maximum benefit.

Besides, these flexible approaches, which integrate the investor's vision and expectations, can be extended to more diversified portfolios, including non-technological sectors, and serve as an effective decision-making tool in complex environments.

Likewise, this new operator can be applied across multiple disciplines, such as finance, the social sciences, and medicine, where data can be treated probabilistically, and different criteria can be managed, enabling evaluation. This study can continue generating new proposals and operator extensions, allowing the decision-maker various options. For future research, the BP–OWA can be applied in fields such as engineering, politics, management, innovation, and related areas. Additionally, the inclusion of various mathematical methods, such as distance measures (Szmidt & Kacprzyk, 2000), group decision-making (Leyva-Lopez & Fernandez-Gonzalez, 2003), and heuristic algorithms (Singh et al., 2010), among others, will be considered in future work.

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Annex 1

Proof 1**:** Commutativity: Let alternatives $(\boldsymbol{x_i}, \mathrm{y})$ has the same argument products $p(\boldsymbol{x_i}, \mathrm{y})$, then:

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}}$$

$$= \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} = P - OWA(x_i, \mathrm{y})$$

Proof 2: Monotonicity: Let $\boldsymbol{x_i} \geq \boldsymbol{c_i}$ for all i_i and $\mathrm{y} \in [0,1]$. Then: $p(\boldsymbol{x_i} \cdot \mathrm{y}) \geq p(\boldsymbol{c_i} \cdot \mathrm{y})$

After that, applying non-negative weights W, we have

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}}$$

$$\geq \left(\frac{1}{n}\sum_i p(c_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} = P - OWA(c_i, \mathrm{y})$$

Hence, monotonicity is proved

Proof 3: Bounded: Since alternatives $(\boldsymbol{x_i}, \mathrm{y}) \in [0,1]$, the product $p(\boldsymbol{x_i}, \mathrm{y}) \in [0,1]$. Let $p(\boldsymbol{x_i}, \mathrm{y})^{min} = \min p(\boldsymbol{x_i}, \mathrm{y})$ and $p(\boldsymbol{x_i}, \mathrm{y})^{max} = \max p(\boldsymbol{x_i}, \mathrm{y})$, then:

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} \in [p(\boldsymbol{x_i}, \mathrm{y})^{min}, p(\boldsymbol{x_i}, \mathrm{y})^{max}]$$

Proof 4: Idempotency. Since alternatives $(\boldsymbol{x_i}, \mathrm{y}) = a$ for all i, the product $p(\boldsymbol{x_i}, \mathrm{y}) = a$. Let $p(\boldsymbol{x_i}, \mathrm{y}) = a$ for all i, then

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} = a$$

Proof 5: Non-negativity. Since alternatives $f(\boldsymbol{x_i}, \mathrm{y}) \geq 0$, the product $p(\boldsymbol{x_i}, \mathrm{y}) \geq 0$. Let $\mathrm{BP} - \mathrm{OWA} \geq 0$, then

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} \geq 0$$

Proof 6: Reflexivity. Since alternatives $f(\boldsymbol{x_i}, \mathrm{y}) = 0$, the product $p(\boldsymbol{x_i}, \mathrm{y}) = 0$. Let $\mathrm{BP} - \mathrm{OWA} = 0$, then

$$BP - OWA(x_i, \mathrm{y}) = \left(\frac{1}{n}\sum_i p(x_i, \mathrm{y})_i^r pOWA_W(V^i)\right)^{\frac{1}{r+q}} = 0$$

Likewise, it is considered the different measures to characterise the weighting vector such as the entropy of dispersion, the balance operator, the divergence of W and the degree of orness (Blanco-Mesa et al., 2016; Merigó & Casanovas 2011; Yager, 1988), which are defined as follows:

The entropy of dispersion:

$$\mathrm{H(W)} = -\left(\frac{1}{\mathrm{n}}\sum_{\mathrm{i}} \ln \mathrm{p}(\mathrm{w_i})\left(\sum_{\substack{\mathrm{j}=1\\ \mathrm{j}\neq\mathrm{i}}}^{\mathrm{n}} \mathrm{pw_i} \ln \mathrm{p}(\mathrm{w_i})\right)\right)^{\frac{1}{\mathrm{r+q}}}. \quad (19)$$

For the balance operator:

$$Bal(W) = \left(\frac{1}{n}\sum_{i=1}^{n} p\left(\frac{n+1-2_i}{n-1}\right)\left(\sum_{\substack{j=1\\ j\neq i}}^{n} p\left(\frac{n+1-2_j}{n-1}\right) w_i\right)\right)^{\frac{1}{r+q}}. \quad (20)$$

For the divergence of W:

$$Div(W) = \left(\frac{1}{n}\sum_{i=1}^{n} p\left(\frac{n-i}{n-1} - \alpha(W)\right)^2 \left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_i\, p\left(\frac{n-j}{n-1} - \alpha(W)\right)^2\right)\right)^{\frac{1}{r+q}}. \quad (21)$$

For the degree of orness:

$$\alpha(W) = \left(\frac{1}{n}\sum_{i=1}^{n} p\left(\frac{n-i}{n-1}\right)\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_i\, p\left(\frac{n-j}{n-1}\right)\right)\right)^{\frac{1}{r+q}}. \quad (22)$$

Annex 2. R code to download the information

````
```{r}
tickers <- c("AAPL","MSFT","NVDA","GOOG","AMZN")
start <- "2024-01-01"
end <- "2024-06-30"
n <- length(tickers)
p <- getSymbols(Symbols = tickers[1], src = "yahoo",
                from = start, to = end,
                auto.assign = F)[, 6]
```
```{r}
for (i in 2:n){
  p = merge(p, getSymbols(Symbols = tickers[i], src = "yahoo",
                          from = start, to = end,
                          auto.assign=F)[, 6])
}
names(p) = gsub(".Adjusted", "", names(p))
head(p)
```
```{r}
ret = p / lag(p) - 1
ret = ret[-1, ]
tail(ret)
```
```{r}
apply(is.na(ret), 2, sum)
```
```{r}
ret2 = na.omit(ret)
```
````