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A Non-Radial Inverse Network Data Envelopment Analysis (NDEA) Model with Reversible Output and Fixed Technical and Cost Efficiency

Abstract. *Inverse Data Envelopment Analysis (DEA) models have been widely discussed in the literature and are regarded as useful decision-making tools for managers in diverse fields such as the economy and industries. However, the network-based variant of the models remains a relatively neglected area of research. In this study, a Supply Chain (SC) or, more broadly, an inverse Network DEA (NDEA) is investigated in which we estimate the input after observing increased output under a manager and where the values of technical and cost efficiencies remain unchanged. We develop an inverse NDEA model with non-radial changes to evaluate technical efficiency. Because of the model's network-based structure, new constraints are added to the model to more accurately simulate the relationships between the network's components. A significant factor to consider is the intermediate productions returned in the inverse NDEA on the condition that technical and cost efficiency either remain fixed or only increase according to the manager's judgment. Furthermore, we apply the model to a real-world industrial case and evaluate the results. To arrive at accurate estimates, the basic principles of networks should be considered when considering the intermediate productions between two components of the SC. It is also important to consider the relationships between production, supply, and consumption in the SC to describe the relationships between SC components comprehensively. A practical example is provided to investigate the different aspects of the developed model.*

Keywords: *inverse network DEA, technical efficiency, cost efficiency, supply chain, reversible intermediate productions.*

JEL Classification: C1, C3, C8.

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1. Introduction

Several inputs and outputs are involved in many real-world practical examples of performance measurement (Mocholi-Arce et al., 2022). Mathematical optimisation models and methods have been used widely in variety of applications (Dhananjay et al., 2020; Amirzadeh et al., 2023). Since DEA is a non-parametric method, it is arguably more effective in evaluating the efficiency and performance level of DMUs (Aliheidari Bioki & Khademi Zare, 2014). This is because DEA allows efficiency to increase or decrease over time and requires no presumptions about the efficient frontier (Gulati & Kumar, 2017). Thus, a variant of the DEA technique, known as the inverse DEA, was introduced. The method is useful as the fuzziness of the data may often cause the efficient frontier to shift. The central purpose of the DEA is to measure efficiency. However, in inverse DEA, the goal is to determine the best inputs and/or outputs in cases where the outputs and/or inputs are changed under the supervision of a manager (Kazemi & Galagedera, 2023). If the efficiency level remains unchanged, the manager is curious to find out how flexible the rest of the inputs or outputs can be in the face of changes. In such cases, the inverse DEA may be the most crucial tool for implementing strategies to maintain organisational efficiency. It may allow managers to make new sets of decisions in their field of work while still achieving the previous efficiency level in their respective organisations (Edalatpanah, 2020).

In this study, we investigate an inverse NDEA model in a series network in which there are reversible relations. We consider this type of modelling in a Supply Chain (SC) with the condition that the technical and cost-efficiency levels remain the same. As the output of the closed-loop SC increases according to the manager's opinion, we estimate the rest of the inputs and outputs in the other SC components. The fixed-efficiency condition can be applied to the other components, as well. The formulation obtained for the model will be tested by applying it to a real-world example.

2. Literature Review

Data Envelopment Analysis (DEA) is a popular non-parametric method of measuring efficiency of Decision-Making Units (DMUs) as a ratio of several inputs to several outputs. Traditional DEA models have the propensity to assume fixed inputs and outputs, which might not hold in instances of possible readjustments. To meet this deficiency, inverse DEA models have been conceived, indicating input and output adjustments needed to achieve a specified level of efficiency. Wei et al. (2000) conceptualised the notion of inverse DEA through the construction of models to quantify the intensity of needed input and output adjustments such that the target efficiency level is attained or maintained. They were the basis for subsequent work in this domain. Subsequently, Ghiyasi (2015) made this approach important with variable returns to scale for more practical modeling. Network DEA models that split the production process into more than one stage with interstage have been used to study compound systems such as supply chains. Kalantary and Saen (2018) proposed a network dynamic DEA model through which to study supply chain sustainability on multiple time scales. They proposed the same inverse model to make the input-output changes estimations to employ in a way that it is efficient in a dynamic situation. The introduction of undesirable products, such as waste or emissions, to DEA models has been made compulsory in estimating environmental efficiency more accurately. Krmac and Djordjević (2019) proposed a non-radial DEA model with desired and undesired inputs and outputs and provided a clearer view regarding supply chain efficiency. The technique is possible with the simultaneous reduction of undesirable factors along with the preservation or enhancement of the desirable output. The incorporation of reversibility in outputs makes DEA models more complex. Pouralizadeh (2024) developed an inverse output-oriented DEA model to evaluate the sustainability of electric supply chains. The model optimises payback of good products and reduces payback of bad products with regard to the reversibility of the outputs in evaluation. Such DEA model advancements are helpful to forecast and improve supply chain performance. Inverse DEA models, network structures, and negative output factors added allow scientists and practitioners to understand the performance of complex systems and forecast improvement measures.

3. Inverse DEA Model

In inverse DEA, we estimate the input value in view of the changes ordered by the manager under the condition that even with the increase in output value, the DMU's efficiency should remain unchanged. In this regard, let us consider the model proposed by Wei et al. (Wei et al., 2000). In this model, as expressed below, θ^* represents the technical efficiency of the DMU under evaluation. As the output increases on the order of the manager, we aim to keep the value of θ^* unchanged and then estimate the value of the inputs.

$$
\begin{aligned}
& min \sum_{j=1}^{m} \beta_i \\
& s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^* \beta_i, \quad i = 1,\dots,m \\
& \quad \sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, \quad d = 1,\dots,l \\
& \quad \sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, \quad d = 1,\dots,l \\
& \quad \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, \quad r = 1,\dots,s \\
& \quad \beta_i \geq x_{io} \geq 0, \quad i = 1,\dots,m \\
& \quad \lambda_j^1 \geq 0,\ \lambda_j^2 \geq 0, \quad j = 1,\dots,n, k = 1,2
\end{aligned}
\tag{1}
$$

4. DEA Model with Returned Output

Consider a two-stage series network. X is the input, Y is the output, and Z is the intermediate production. Output and input are this network's first and second components, respectively. Additionally, W is the output of the network's second component, which is returned to the first component. λ_j^1 and λ_j^2 ($j = 1, \ldots, n$) are the corresponding intensity factors of the network's first and second components, respectively. θ is the free-in-sign variable used to apply a radial reduction in input x. The DEA model, whose input has a network structure, can be expressed as follows:

$$\begin{aligned}
& \min \theta \\
& s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta x_{io}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j^1 w_{fj} \leq w_{fo}, \quad f = 1, \ldots, p \\
& \sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, \quad d = 1, \ldots, l \\
& \sum_{j=1}^{n} \lambda_j^2 w_{fj} \geq w_{fo}, \quad f = 1, \ldots, p \qquad (2) \\
& \sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, \quad d = 1, \ldots, l \\
& \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s \\
& \lambda_j^1 \geq 0,\ \lambda_j^2 \geq 0, \quad j = 1, \ldots, n, k = 1,2 \text{ Type equation here.}
\end{aligned}$$

By solving model (2), we obtain θ^* which denotes radial efficiency and represents the highest reduction in the input to reach the efficient frontier (You & Yan, 2011).

5. Inverse NDEA model with reversible output

The purpose of developing an inverse NDEA model is to estimate the value of the inputs when the manager orders that the outputs be increased so that the cost efficiency and non-radial technical efficiency remain unchanged (Ratner et al., 2022). We develop and implement this model on a network. Importantly, the returned intermediate productions should be considered in inverse NDEA under the condition that the technical and cost efficiencies remain unchanged (Yu, 2008). Moreover, we consider network principles on the intermediate productions between the two components of the SC. To better evaluate an SC by DEA in our model, we also consider the relationships among production, supply, and consumption in an SC where the relationship between the components is significant. In the inverse NDEA, when the technical efficiency resulting from SBM and the cost efficiency remain unchanged or increase upon the manager's order, y is increased to y_n by the manager (Pinto, 2024). Thus, we propose the model below to estimate the inputs:

$$
\begin{aligned}
& min \quad (\alpha_1, \dots, \alpha_m) - \varepsilon(\textstyle\sum_{i=1}^{m} s_i^-) \\
& s.t. \quad \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le (x_{io} + \alpha_i) - s_i^-, \quad i = 1, \dots, m, \quad (a) \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}^n, \quad r = 1, \dots, s, \quad (b) \\
& \textstyle\sum_{j=1}^{n} \lambda_j = 1 \quad (c) \\
& 1 - \frac{1}{m} \textstyle\sum_{i=1}^{m} \frac{s_i^-}{x_{io} + \alpha_i^*} = (\rho_1^* + \Delta) \quad (d) \\
& \frac{\sum_{i=1}^{m} c_i \alpha_i^*}{\sum_{i=1}^{m} c_i x_{io}} = (\rho_2^* + \nabla) \quad (e) \\
& \lambda_j \ge 0, \quad j = 1, \dots, n, \\
& \alpha_i \ge 0, \quad i = 1, \dots, m, \\
& (x_{io} + \alpha_i) \ge c(x_{io}) \quad i = 1, \dots, m. \quad c \in R^+
\end{aligned} \tag{3}
$$

Note that the model above is a two-phase one. In the second phase, given the optimal value α^* obtained in the first phase, the model tries to keep the efficiency of the DMU at its current value, which is $\rho_1^* + \Delta$. It should be noted that

$$0 \le \Delta \le 1 - (\rho_1^*), 0 < \rho_1^* \le 1 \tag{4}$$

where the desired value of Δ is selected by the manager from the specified interval. The same is true for cost efficiency.

$$0 \le \nabla \le 1 - (\rho_2^*), 0 < \rho_2^* \le 1 \tag{5}$$

As can be seen, *c* is a parameter whose value is a positive real number determined by the manager. The first phase of the model above is itself a multi-objective model. Note that in the model above, Constraint *(a)* is meant to find the minimum value that gets added to the inputs, which results in the constraint remaining unchanged. Constraint *(b)* represents a set of output constraints. Constraint *(c)* is the variables' returns to scale. Constraint *(d)* guarantees that the technical efficiency of the DMUs is maintained or improved based on the non-radial SBM framework. Lastly, constraint *(e)* ensures that the cost efficiency of the DMUs is preserved or enhanced according to the cost minimisation principles. In order to simplify the multi-objective model (3), we transform it into a single-objective optimisation model. For this purpose, instead of minimising the variables α_i one by one, we minimise the sum of variables $\sum_{i=1}^{m} \alpha_i$.

Furthermore, in constraint (*a*), we have

$$\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} = \alpha_i, \; i = 1, \dots, m \tag{6}$$

where it is possible to insert $\sum_{j=1}^{n} \lambda_j x_{ij}$ instead of α_i. Consequently, constraints *(d)* and *(e)* can also be expressed more simply. Constraint *(d)* is converted to

$$1 - \frac{1}{m} \textstyle\sum_{i=1}^{m} \frac{s_i^-}{\alpha_i^*} = (\rho_1^* + \Delta) \tag{7}$$

and constraint *(c)* is converted to

$$\frac{\sum_{i=1}^{m} c_i \alpha_i}{\sum_{i=1}^{m} c_i x_{io}} = (\rho_2^* + \nabla) \tag{8}$$

Thus, we introduce the following model:

$$
\begin{aligned}
& \min \quad \sum_{i=1}^{m} \alpha_i \\
& s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le (x_{io} + \alpha_i), \quad i = 1,\dots,m, \quad (a) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}^{n}, \quad r = 1,\dots,s, \quad (b) \\
& \sum_{j=1}^{n} \lambda_j = 1 \quad (c) \\
& \frac{\sum_{i=1}^{m} c_i \alpha_i}{\sum_{i=1}^{m} c_i x_{io}} = (\rho_2^* + \nabla) \quad (d) \\
& (x_{io} + \alpha_i) \ge c(x_{io}) \quad i = 1,\dots,m \quad c \epsilon R^+ \quad (e) \\
& \lambda_j \ge 0, \quad j = 1,\dots,n, \\
& \alpha_i \ge 0, \quad i = 1,\dots,m,
\end{aligned} \tag{9}
$$

Given the values of λ_j^* obtained in the optimal solution, the optimal value of each α_i^* can be used in the second-phase model. Note that the value of ∇ is determined by the manager based on the possible interval. It is even possible to consider the value of ∇ to be 0, meaning that the manager does not intend to make any improvements to the efficiency of the DMU.

Given the interval of changes in the value of cost efficiency, we have

$$0 \le \nabla + \rho_2^* \le 1 \tag{10}$$

By determining the optimal value of α_i^* per each i, the inputs can be estimated in accordance with the increase in the outputs, as ordered by the manager, such that the value of cost efficiency either remains unchanged or increases. We now compute the second-phase model as follows:

$$
\begin{aligned}
& \max \quad \left(\sum_{i=1}^{m} s_i^-\right) \\
& s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le (x_{io} + \alpha_i^*) - s_i^-, \quad i = 1,\dots,m, \quad (a) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}^{n}, \quad r = 1,\dots,s, \quad (b) \\
& \sum_{j=1}^{n} \lambda_j = 1 \quad (c) \\
& 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{s_i^-}{x_{io} + \alpha_i^*} = (\rho_1^* + \Delta) \quad (d) \\
& \frac{\sum_{i=1}^{m} c_i \alpha_i}{\sum_{i=1}^{m} c_i x_{io}} = (\rho_2^* + \nabla) \quad (e) \\
& \lambda_j \ge 0, \quad j = 1,\dots,n, \\
& \alpha_i \ge 0, \quad i = 1,\dots,m, \\
& (x_{io} + \alpha_i) \ge c(x_{io}) \quad i = 1,\dots,m. \quad c \epsilon R^+
\end{aligned} \tag{11}
$$

If the values of both technical and cost efficiencies do not change, then the value of allocative efficiency also remains unchanged since we have

$$Allocative\ efficiency = \frac{Cost\ efficiency}{Technical\ efficiency} \tag{12}$$

A significant advantage of the model defined above is its non-radiality, which makes it possible to estimate the value of any input component. In developing model (11), the output was not assumed to be reversible. Now, if the series network is assumed to have reversible output, the subsequent models in this study may be addressed according to the method of estimating the parameters in model (3). In order to have a clearer idea of reversible outputs in modelling, consider Figure 1 below. The diagram features a series network with three components and a reversible relationship.

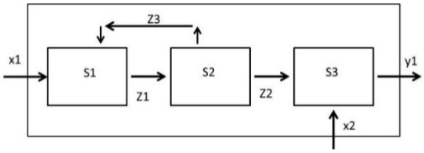

Figure 1. A three-stage series network with a reversible relationship
Source: authors' finding.

In order to expand model (12) on this network, we first develop the network's technical and cost-efficiency models. Like models (9) and (11), we propose the following to correspond to the network:

Model (9) is developed to measure the technical efficiency of the input and corresponds to the network illustrated in Figure 1. As the output increases, the target is to estimate the minimum increase in the input so that the technical and cost efficiencies either remain the same or improve according to the manager's opinion. Estimating the minimum increase in the input is important to managers because it benefits the system by helping prevent resource wastage. Since intermediate productions are part of intra-network relationships in network-based models, no restriction is placed on their changes to establish a balance between the network's two components. However, the main input of the network, i.e., SC, is assumed to increase.

$$
\begin{aligned}
min \quad & \rho_3 = \left(\tfrac{1}{3}\right)\Bigg(\left(1-\tfrac{1}{|f_3+i_1|}\left(\textstyle\sum_{f_3}\tfrac{s^-_{f_3}}{z^3_{f_3o}}+\sum_{i_1}\tfrac{s^-_{i1}}{x^1_{i_1o}}\right)\right)+ \\
& \left(1-\tfrac{1}{|f_1|}\textstyle\sum_{f_1}\tfrac{s^-_{f_1}}{z^1_{f_1o}}\right)+\left(1-\tfrac{1}{|i_2+f_2|}\left(\textstyle\sum_{f_3}\tfrac{s^-_{i_2}}{x^2_{i_2o}}+\sum_{f_2}\tfrac{s^-_{f_2}}{z^2_{f_2o}}\right)\right)\Bigg) \\
s.t. \quad & \textstyle\sum_{j=1}^{n}\lambda^1_j\, z^3_{f_3j} \le z^3_{f_3o}-s^-_{f_3}, && \forall f_3 \\
& \textstyle\sum_{j=1}^{n}\lambda^1_j\, x^1_{i_1j} \le x^1_{i_1o}-s^-_{i1}, && \forall i_1 \\
& \textstyle\sum_{j=1}^{n}\lambda^1_j\, z^1_{f_1j} \ge z^1_{f_1o}, && \forall f_1, \\
& \textstyle\sum_{j=1}^{n}\lambda^1_j = 1, \lambda^1_j \ge 0, s^-_{f_3} \ge 0, s^-_{i1} \ge 0, s^-_{f_1} \ge 0, && \forall j, \forall f_3, \forall i_1, \\
& \textstyle\sum_{j=1}^{n}\lambda^2_j\, z^1_{f_1j} \le z^1_{f_1o}-s^-_{f_1}, && \forall f_1, \\
& \textstyle\sum_{j=1}^{n}\lambda^2_j\, z^2_{f_2j} \ge z^2_{f_2o}, && \forall f_2, \\
& \textstyle\sum_{j=1}^{n}\lambda^2_j\, z^3_{f_3j} \ge z^3_{f_3o}, && \forall f_3, \\
& \textstyle\sum_{j=1}^{n}\lambda^2_j = 1, \lambda^2_j \ge 0, s^-_{f_1} \ge 0,, && \forall j, \forall f_1,, \\
& \textstyle\sum_{j=1}^{n}\lambda^3_j\, z^2_{f_2j} \le z^2_{f_2o}-s^+_{f_2}, && \forall f_2, \\
& \textstyle\sum_{j=1}^{n}\lambda^3_j\, x^2_{i_2j} \le x^2_{i_2o}-s^-_{i_2}, && \forall i_2 \\
& \textstyle\sum_{j=1}^{n}\lambda^3_j y_{rj} \ge y_{ro}, && \forall r \\
& \textstyle\sum_{j=1}^{n}\lambda^3_j = 1, \lambda^3_j \ge 0, s^-_{i_2} \ge 0, s^+_{f_2} \ge 0, && \forall j, \forall f_2, \forall i_2,,
\end{aligned}
\tag{13}
$$

Note that the network's total technical efficiency is ρ^*_3. Efficiency can be obtained for each component using the following formula: in view of the optimal solution previously obtained, the technical efficiency of the network's first component is calculated as follows:

$$e^{*1} = 1 - \frac{1}{|f_3 + i_1|}\left(\sum_{f_3} \frac{s_{f_3}^{*-}}{z_{f_3 o}^3} + \sum_{i_1} \frac{s_{i1}^{*-}}{x_{i_1 o}^1}\right) \quad (14)$$

And the technical efficiency of the network's second component is obtained as follows:

$$e^{*2} = 1 - \frac{1}{|f_1|}\sum_{f_1} \frac{s_{f_1}^{*-}}{z_{f_1 o}^1} \quad (15)$$

$$e^{*3} = 1 - \frac{1}{|i_2 + f_2|}\left(\sum_{f_3} \frac{s_{i_2}^{-}}{x_{i_2 o}^2} + \sum_{f_2} \frac{s_{f_2}^{-}}{z_{f_2 o}^2}\right) \quad (16)$$

Since $e^{*1} < 1$, $e^{*2} < 1$, and $e^{*3} < 1$, then $\rho_3^* < 1$.

The model to measure the corresponding cost efficiency of the network, based on Figure 1, is as follows:

$$\begin{aligned}
\min \quad & \rho_4 = \left(\sum_{f_3} c_{f_3}^1 z_{f_3}^3 + \sum_{i_1} c_{i1}^2 x_{i_1}^1\right) + \left(\sum_{f_1} c_{f_1}^3 z_{f_1}^1\right) \\
& +\left(\sum_{f_3} c_{i_2}^4 z_{f_2}^2 + \sum_{i_2} c_{i_2}^5 x_{i_2}^2\right) \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^n \lambda_j^1 z_{f_3 j}^3 = z_{f_3}^3, && \forall f_3 \\
& \textstyle\sum_{j=1}^n \lambda_j^1 x_{i_1 j}^1 = x_{i_1}^1, && \forall i_1 \\
& \textstyle\sum_{j=1}^n \lambda_j^1 z_{f_1 j}^1 \geq z_{f_1 o}^1, && \forall f_1, \\
& \textstyle\sum_{j=1}^n \lambda_j^1 = 1, \lambda_j^1 \geq 0,\ z_{f_3}^3 \geq 0, x_{i_1}^1 \geq 0, && \forall j, \forall f_3, \forall i_1, \\
& \textstyle\sum_{j=1}^n \lambda_j^2 z_{f_1 j}^1 = z_{f_1}^1, && \forall f_1, \\
& \textstyle\sum_{j=1}^n \lambda_j^2 z_{f_2 j}^2 \geq z_{f_2 o}^2, && \forall f_2, \\
& \textstyle\sum_{j=1}^n \lambda_j^2 z_{f_3 j}^3 \geq z_{f_3 o}^3, && \forall f_3, \\
& \textstyle\sum_{j=1}^n \lambda_j^2 = 1, \lambda_j^2 \geq 0,\ z_{f_1}^1 \geq 0, && \forall j, \forall f_1, \\
& \textstyle\sum_{j=1}^n \lambda_j^3 z_{f_2 j}^2 = z_{f_2}^2, && \forall f_2, \\
& \textstyle\sum_{j=1}^n \lambda_j^3 x_{i_2 j}^2 = x_{i_2}^2, && \forall i_2 \\
& \textstyle\sum_{j=1}^n \lambda_j^3 y_{rj} \geq y_{ro}, && \forall r \\
& \textstyle\sum_{j=1}^n \lambda_j^3 = 1, \lambda_j^3 \geq 0,\ z_{f_2}^2 \geq 0,\ x_{i_2}^2 \geq 0, && \forall j, \forall f_2, \forall i_2,
\end{aligned} \quad (17)$$

In the model above c^1 and c^2 are the cost vectors of z^3 and x^1, the inputs of the network's first component. c^3 is the cost vector of z^1, the input of the network's second component. c^4 and c^5 are the cost vectors of z^2 and x^2, the inputs of the network's first component. The purpose of solving the model above is to obtain the optimal values of the inputs of the two network components. Note that the input variables of each component are assumed to be non-negative.

Given the optimal solution obtained from the model above by solving the following equation:

$$\rho_4^* = \left(\sum_{f_3} c_{f_3}^1 z_{f_3}^{*3} + \sum_{i_1} c_{i1}^2 x_{i_1}^{*1}\right) + \left(\sum_{f_1} c_{f_1}^3 z_{f_1}^{*1}\right) + \left(\sum_{f_3} c_{i_2}^4 z_{f_2}^{*2} + \sum_{i_2} c_{i_2}^5 x_{i_2}^{*2}\right) \quad (18)$$

The value of the network's total cost efficiency is obtained from the following relation:

$$Cost\ Efficiency = C.E. = \frac{\left(\sum_{f_3} c_{f_3}^1 z_{f_3}^{*3} + \sum_{i_1} c_{i1}^2 x_{i_1}^{*1}\right) + \left(\sum_{f_1} c_{f_1}^3 z_{f_1}^{*1}\right) + \left(\sum_{f_3} c_{i_2}^4 z_{f_2}^{*2} + \sum_{i_2} c_{i_2}^5 x_{i_2}^{*2}\right)}{\left(\sum_{f_3} c_{f_3}^1 z_{f_3 o}^3 + \sum_{i_1} c_{i1}^2 x_{i_1 o}^1\right) + \left(\sum_{f_1} c_{f_1}^3 z_{f_1 o}^1\right) + \left(\sum_{f_3} c_{i_2}^4 z_{f_2 o}^2 + \sum_{i_2} c_{i_2}^5 x_{i_2 o}^2\right)} \quad (19)$$

Note that we are addressing a minimisation model. Hence:

$$C.E. \leq 1 \tag{20}$$

The cost efficiencies of the network's first, second, and third components are obtained by relations (25), (26), and (27), respectively, as follows:

$$Cost\ Efficiency\ Stage\ 1 = CE^1 = \frac{\left(\sum_{f_3} c_{f_3}^1\, z_{f_3}^{*3} + \sum_{i_1} c_{i1}^2\, x_{i_1}^{*1}\right)}{\left(\sum_{f_3} c_{f_3}^1\, z_{f_3o}^3 + \sum_{i_1} c_{i1}^2\, x_{i_1o}^1\right)} \tag{21}$$

$$Cost\ Efficiency\ Stage\ 2 = CE^2 = \frac{\left(\sum_{f_1} c_{f_1}^3\, z_{f_1}^{*1}\right)}{\left(\sum_{f_1} c_{f_1}^3\, z_{f_1o}^1\right)} \tag{22}$$

$$Cost\ Efficiency\ Stage\ 3 = CE^3 = \frac{\left(\sum_{f_3} c_{i_2}^4\, z_{f_2}^{*2} + \sum_{i_2} c_{i_2}^5\, x_{i_2}^{*2}\right)}{\left(\sum_{f_3} c_{i_2}^4\, z_{f_2o}^2 + \sum_{i_2} c_{i_2}^5\, x_{i_2o}^2\right)} \tag{23}$$

Clearly, the cost efficiency of each component is equal to or less than 1.

Similar to model (21), in case the manager decides to increase the outputs and under the condition that cost and technical efficiencies either remain the same or improve, the inputs of the network's components can be estimated by the following two-phase model. As the outputs increase, the goal is to estimate the minimum increase in the value of the inputs, such that cost and technical efficiencies remain the same or improve upon the manager's order. In network-based models, since intermediate productions are part of intra-network relationships, no restriction is placed on their changes to establish a balance between the network's two components, but the SC/network's independent input is assumed to increase.

$$\begin{aligned}
&Min\ \rho_7 = \sum_{i_1} \alpha_{i_1}^{x^1} + \sum_{f_3} \alpha_{f_3}^{z^3} + \sum_{f_1} \dot{\alpha}_{f_1}^{z^1} + \sum_{f_1} \alpha_{f_1}^{z^1} + \\
&\sum_{f_2} \dot{\alpha}_{f_2}^{z^2} + \sum_{f_3} \dot{\alpha}_{f_3}^{z^3} + \sum_{i_2} \alpha_2^{x^2} + \sum_{y} \alpha_{f_2}^{z^2} \\
&s.t. \sum_{j=1}^{n} \lambda_j^1\, \mathrm{z}_{f_3 j}^3 \leq \mathrm{z}_{f_3 o}^3 + \alpha_{f_3}^{z^3},\ \forall f_3 \\
&\quad \sum_{j=1}^{n} \lambda_j^1\, \mathrm{x}_{i_1 j}^1 \leq \mathrm{x}_{i_1 o}^1 + \alpha_{i_1}^{x^1}, \forall i_1 \\
&\quad \sum_{j=1}^{n} \lambda_j^1\, \mathrm{z}_{f_1 j}^1 \geq \mathrm{z}_{f_1 o}^1 + \dot{\alpha}_{f_1}^{z^1}, \forall f_1, \\
&\quad \sum_{j=1}^{n} \lambda_j^1 = 1, \lambda_j^1 \geq 0, \forall j, \alpha_{i_1}^{x^1} \geq 0, \forall i_1 \\
&\quad \sum_{j=1}^{n} \lambda_j^2\, \mathrm{z}_{f_1 j}^1 \leq \mathrm{z}_{f_1 o}^1 + \alpha_{f_1}^{z^1}, \forall f_1, \\
&\quad \sum_{j=1}^{n} \lambda_j^2\, \mathrm{z}_{f_2 j}^2 \geq \mathrm{z}_{f_2 o}^2 + \dot{\alpha}_{f_2}^{z^2}, \forall f_2, \\
&\quad \sum_{j=1}^{n} \lambda_j^2\, \mathrm{z}_{f_3 j}^3 \geq \mathrm{z}_{oj}^3 + \dot{\alpha}_{f_3}^{z^3}, \forall f_3, \\
&\quad \sum_{j=1}^{n} \lambda_j^2 = 1, \lambda_j^2 \geq 0, \forall j, \\
&\quad \sum_{j=1}^{n} \lambda_j^3\, \mathrm{z}_{f_2 j}^2 \leq \mathrm{z}_{f_2 o}^2 + \alpha_{f_2}^{z^2}, \forall f_2, \\
&\quad \sum_{j=1}^{n} \lambda_j^3\, \mathrm{x}_{i_2 j}^2 \leq \mathrm{x}_{i_2 o}^2 + \alpha_{i_2 o}^{x^2}, \forall i_2, \\
&\quad \sum_{j=1}^{n} \lambda_j^3 y_{rj} \geq \mathrm{y}_{ro}^N, \forall r, \\
&\quad \sum_{j=1}^{n} \lambda_j^3 = 1, \lambda_j^3 \geq 0, \forall j, \alpha_{i_2 o}^{x^2} \geq 0, \forall i_2,
\end{aligned} \tag{24}$$

$$\mathrm{z}_{f_1 o}^1 + \dot{\alpha}_{f_1}^{z^1} \geq \mathrm{z}_{f_1 o}^1 + \alpha_{f_1}^{z^1}, \forall f_1, \qquad (e)$$

$$\mathrm{z}_{f_2 o}^2 + \dot{\alpha}_{f_2}^{z^2} \geq \mathrm{z}_{f_2 o}^2 + \alpha_{f_2}^{z^2}, \forall f_2, \qquad (f)$$

$$\mathrm{z}_{oj}^3 + \dot{\alpha}_{f_3}^{z^3} \geq \mathrm{z}_{f_3 o}^3 + \alpha_{f_3}^{z^3}, \qquad \forall f_3, \qquad (g)$$

Note that in the model above, the inputs and outputs of each network component (intermediate relationships between the components) are free to change. Constraints *(e)*, *(f)*, and *(g)* express the principles of the network for intermediate productions. On this basis, the input of the second component cannot exceed the output of the first component because the second component cannot consume more than what the first input has produced. Additionally, the input of the third component cannot exceed the output of the second component. By the same token, the input of the first component, which is supplied by the output returned by the second component, cannot exceed the output of the second component. It is imperative to pay sufficient attention to the intermediate productions of a network, as misunderstanding the relationship between the network's components leads to erroneous results. Considering the relationships between the components, the intermediate productions, and the principle that a component's outputs serve as the next component's inputs, form the basis of network modelling in DEA. Relations *(c)* and *(d)* are given to summarise the relationship between the network's components and its intermediate productions. Based on the explanations provided thus far, the three final constraints *(e)*, *(f)*, and *(g)* are proposed to describe the intermediate input-output relationships between the network's components. The phase two model is given as follows:

$$
\begin{aligned}
&Max\ \rho_8 = (\textstyle\sum_{i_1} \vartheta_{i_1}^{x^1} + \sum_{i_2} \vartheta_{i_2}^{x^2} + \sum_{f_1} \vartheta_{f_1}^{z^1} + \sum_{f_2} \vartheta_{f_2}^{z^2} + \sum_{f_3} \vartheta_{f_3}^{z^3} + \\
&\textstyle\sum_{f_1} \dot{\vartheta}_{f_1}^{z^1} + \sum_{f_2} \dot{\vartheta}_{f_2}^{z^2} + \sum_{f_3} \dot{\vartheta}_{f_3}^{z^3}) \\
&s.t. \textstyle\sum_{j=1}^{n} \lambda_j^1\, z_{f_3 j}^3 \leq (z_{f_3 o}^3 + \alpha_{f_3}^{*z^3}) - \vartheta_{f_3}^{z^3},\ \forall f_3 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1\, x_{i_1 j}^1 \leq (x_{i_1 o}^1 + \alpha_{i_1}^{*x^1}) - \vartheta_{i_1}^{x^1}, \forall i_1 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1\, z_{f_1 j}^1 \geq (z_{f_1 o}^1 + \dot{\alpha}_{f_1}^{*z^1}) + \dot{\vartheta}_{f_1}^{z^1}, \forall f_1, \qquad (25) \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1 = 1, \lambda_j^1 \geq 0, \forall j, \vartheta_{i_1}^{x^1} \geq 0, \vartheta_{f_3}^{z^3} \geq 0, \forall i_1, \forall f_3 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2\, z_{f_1 j}^1 \leq (z_{f_1 o}^1 + \alpha_{f_1}^{*z^1}) - \vartheta_{f_1}^{z^1}, \forall f_1, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2\, z_{f_2 j}^2 \geq (z_{f_2 o}^2 + \dot{\alpha}_{f_2}^{*z^2}) + \dot{\vartheta}_{f_2}^{z^2}, \forall f_2, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2\, z_{f_3 j}^3 \geq (z_{oj}^3 + \dot{\alpha}_{f_3}^{*z^3}) + \dot{\vartheta}_{f_3}^{z^3}, \forall f_3, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2 = 1, \lambda_j^2 \geq 0, \forall j,\ \vartheta_{f_1}^{z^1}, \forall f_1 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^3\, z_{f_2 j}^2 \leq (z_{f_2 o}^2 + \alpha_{f_2}^{*z^2}) - \vartheta_{f_2}^{z^2}, \forall f_2, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^3\, x_{i_2 j}^2 \leq (x_{i_2 o}^2 + \alpha_{i_2 o}^{*x^2}) - \vartheta_{i_2}^{x^2}, \forall i_2, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^3 y_{rj} \geq y_{ro}^N, \forall r, \vartheta_{f_2}^{z^2} \geq 0, \vartheta_{i_2}^{x^2} \geq 0, \forall f_2, , \forall i_2, \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^3 = 1, \lambda_j^3 \geq 0, \forall j, \forall i_2,
\end{aligned}
$$

$$
\left(\tfrac{1}{3}\right)\left(\left(1 - \frac{1}{|f_3 + i_1|}\left(\sum_{f_3} \frac{\vartheta_{f_3}^{z^3}}{z_{f_3 o}^3 + \alpha_{f_3}^{*z^3}} + \sum_{i_1} \frac{\vartheta_{i_2}^{x^2}}{x_{i_1 o}^1 + \alpha_{i_1}^{*x^1}}\right)\right) + \right. \qquad (a)
$$

$$
\left(1 - \frac{1}{|f_1|}\sum_{f_1} \frac{\vartheta_{f_1}^{z^1}}{z_{f_1 o}^1 + \alpha_{f_1}^{*z^1}}\right) + \left(1 - \frac{1}{|i_2 + f_2|}\left(\sum_{f_3} \frac{\vartheta_{i_2}^{x^2}}{x_{i_2 o}^2 + \alpha_{i_2 o}^{*x^2}} + \sum_{f_2} \frac{\dot{\vartheta}_{f_2}^{z^2}}{z_{f_2 o}^2 + \dot{\alpha}_{f_2}^{*z^2}}\right)\right)) = \rho_3^*
$$

In order to maintain or improve technical efficiency, we only focus on optimising the non-radial changes in the inputs of the network's components. To expand the model, it is also possible to replace constraints that maintain or improve the network's total cost and technical efficiencies with constraints that maintain or improve the cost and technical efficiencies of one, two, or all three network components. In classic DEA models, only the inputs or outputs of one DMU were estimated to achieve this goal. However, we expand the model proposed in this study to multiple DMUs in a network structure. To this end, we can replace constraints *(a)* and *(b)* with any one of the relations (26), (27), or (28), which represent the technical efficiency of the first, second, and third components, respectively.

$$e^{*1} = \left(1 - \frac{1}{|f_3 + i_1|}\left(\sum_{f_3} \frac{\alpha_{f_3}^{z^3}}{z_{f_3 o}^{3}} + \sum_{i_1} \frac{\alpha_{i_1}^{x^1}}{x_{i_1 o}^{1}}\right)\right) \tag{26}$$

$$e^{*2} = \left(1 - \frac{1}{|f_1|}\sum_{f_1} \frac{\alpha_{f_1}^{z^1}}{z_{f_1 o}^{1}}\right) \tag{27}$$

$$e^{*3} = \left(1 - \frac{1}{|i_2 + f_2|}\left(\sum_{f_3} \frac{\alpha_{f_2}^{z^2}}{x_{i_2 o}^{2}} + \sum_{f_2} \frac{\alpha_{2}^{x^2}}{z_{f_2 o}^{2}}\right)\right) \tag{28}$$

Moreover, constraint *(b)* can be replaced by any one of relations (29), (30), or (31), which represent the cost efficiency of the first, second, and third components, respectively.

$$\mathrm{CE}^1 = \frac{\left(\sum_{f_3} c_{f_3}^{1}(z_{f_3 o}^{3} - \alpha_{f_3}^{z^3}) \ + \sum_{i_1} c_{i1}^{2}(x_{i_1 o}^{1} - \alpha_{i_1}^{x^1})\right)}{\left(\sum_{f_3} c_{f_3}^{1} z_{f_3 o}^{3} + \sum_{i_1} c_{i1}^{2} x_{i_1 o}^{1}\right)} \tag{29}$$

$$\mathrm{CE}^2 = \frac{\left(\sum_{f_1} c_{f_1}^{3}(z_{f_1 o}^{1} - \alpha_{f_1}^{z^1})\right)}{\left(\sum_{f_1} c_{f_1}^{3} z_{f_1 o}^{1}\right)} \tag{30}$$

$$\mathrm{CE}^3 = \frac{\left(\sum_{f_3} c_{i_2}^{4}(z_{f_2 o}^{2} - \alpha_{f_2}^{z^2}) + \sum_{i_2} c_{i_2}^{5}(x_{i_2 o}^{2} - \alpha_{2}^{x^2})\right)}{(\sum_{f_3} c_{i_2}^{4} z_{f_2 o}^{2} + \sum_{i_2} c_{i_2}^{5} x_{i_2 o}^{2})} \tag{31}$$

where the intermediate inputs and outputs of all the network components are considered as variables except for x^1 and x^2 . This is because the intermediate inputs and outputs of the entire network should be capable of taking different values. Once the manager defines new outputs, the main inputs of the network can be estimated. To achieve this goal, we allow intermediate productions to change freely (no matter whether to increase or decrease). However, we estimate the changes in the problem's independent inputs assuming an upcoming increase. As a result, the impact of changes in the entire network can be estimated per each new output defined by the manager. That is, the changes in intermediate productions, returned output, and inputs can be estimated and evaluated under the condition that the cost and technical efficiencies remain unchanged. An important point regarding the network evaluation in this study is that the changes made to the productions in real-world examples should be acceptable to the system's managers and decision-makers. Therefore, the constraints are used in the modelling process to ensure that the estimated solutions are neither unexpected nor far from what the managers can logically implement in the system. In general, consider the constraints as follows:

$$p_2(x^2_{i_2o}) \leq x^2_{i_2o} - \alpha_2^{x^2} \leq p_1(x^2_{i_2o}), \qquad p_1, p_2 \epsilon R$$

Another significant matter is describing the relationship between the network's components in mathematical modelling. The network's model can be expressed as follows:

$$\begin{aligned}
&Min\ \rho_7 = \textstyle\sum_{i_1} \alpha_{i_1}^{x^1} + \sum_{i_2} \alpha_2^{x^2} \\
&s.t. \textstyle\sum_{j=1}^n \lambda_j^1 z^3_{f_3j} \leq \sum_{j=1}^n \lambda_j^2 z^3_{f_3j},\ \forall f_3 \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^1 x^1_{i_1j} \leq x^1_{i_1o} + \alpha_{i_1}^{x^1}, \forall i_1 \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^1 z^1_{f_1j} \geq \sum_{j=1}^n \lambda_j^2 z^1_{f_1j}, \forall f_1, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^1 = 1, \lambda_j^1 \geq 0, \forall j, \alpha_{i_1}^{x^1} \geq 0, \forall i_1 \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^2 z^2_{f_2j} \geq \sum_{j=1}^n \lambda_j^3 z^2_{f_2j}, \forall f_2, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^2 = 1, \lambda_j^2 \geq 0, \forall j, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^3 x^2_{i_2j} \leq x^2_{i_2o} + \alpha_{i_2o}^{x^2}, \forall i_2, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^3 y_{rj} \geq y^N_{ro}, \forall r, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j^3 = 1, \lambda_j^3 \geq 0, \forall j, \alpha_{i_2o}^{x^2} \geq 0, \forall i_2,
\end{aligned} \tag{32}$$

Although intermediate relationships are considered in the model above, their values cannot be estimated because of their particular formulation. Thus, we only estimate the impact of increasing the output on the network's independent inputs. Indeed, this is another type of network modelling; however, in this research, we are focused on the previous models and will return to this type of network modelling in our recommendations for future research.

6. Real-world example

Consider the following real-world example. Table 1 contains data on 10 SCs classified into three stages. In this example from the concrete industry, it is important to consider the conditions of production, supply, and consumption in the SC as follows. In the first stage, the network cannot store more raw materials than the permitted limit. As a result, in each SC under evaluation, a condition is in place that determines the permitted sum of inputs in the first stage. The values of π_1 and π_2 are determined by the decision-maker of the system as follows:

$$\pi_1(\sum_i x_{io} + \sum_{f3} z^3_{f3o}) \leq \sum_i (x_i - \alpha_{i_1}^{x^1}) + \sum_{f3} (z^3_{f_3o} - \alpha_{f_3}^{z^3}) \leq \pi_2 (\sum_i x_{io} + \sum_{f3} z^3_{f3o})$$

To estimate the second-stage outputs in each of the evaluated SCs, the condition that the sum of a given component's outputs should be within the decision-maker's desired interval should be met. The values of τ_1 and τ_2 are determined by the decision-maker of the system.

$$\tau_1(\sum_i z^2_{f_2o} + \sum_{f3} z^3_{oj}) \leq \sum_i (z^2_{f_2o} - \dot{\alpha}_{f_2}^{z^2}) + \sum_{f3} (z^3_{oj} - \dot{\alpha}_{f_3}^{z^3}) \leq \tau_2 (\sum_i z^2_{f_2o} + \sum_{f3} z^3_{oj})$$

Table 1. Data

DMUs	x-1	x-2	x-3	z-1	z-2	Z-interm	w-1	w-2	Y	y-new
1	3	21	124	213	564	11	325	567	1512	1590
2	5	27	231	316	435	12	453	673	1414	1480
3	4	23	271	218	365	10	247	487	1315	1374
4	7	26	341	419	453	11	267	582	1216	1278
5	12	29	298	211	463	12	298	643	1613	1692
6	6	26	351	317	345	14	341	985	1125	1163
7	9	28	317	311	452	18	287	598	1664	1700
8	7	23	283	212	632	17	286	453	1271	1304
9	15	24	256	217	354	13	283	468	1585	1632
10	7	20	251	319	265	15	186	498	1489	1572

Source: authors' finding.

As seen in the table above, the outputs have indeed increased. The two stages have a single general input, a single general output, an intermediate production parameter, and a production returned from the second to the first.

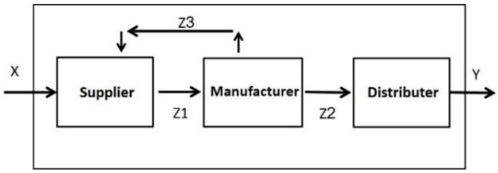

Figure 2. A three-stage SC with reversible output (source: author finding)
Source: authors' finding.

Table 2. Technical and cost efficiencies of the entire SC

DMUs	Cost Effi.	SBM Effi.	SBM.Effi	Cost.Effi
1	0.8	0.91	1	1
2	0.74	0.85	1	0.96
3	0.82	0.75	0.64	0.63
4	0.68	0.76	1	0.97
5	0.8	0.72	0.39	0.52
6	0.55	0.65	0.55	0.71
7	0.9	0.74	0.39	0.7
8	0.82	0.92	1	0.61
9	0.97	0.76	0.27	0.59
10	1	1	1	0.92

Source: authors' finding.

Table 2 presents the technical and cost efficiencies of both the entire supply chain (SC) and the supplier. The results show the efficiency scores for each DMU, allowing for a direct comparison between the overall SC performance and the supplier's performance. The graph in Figure 3 compares the technical efficiency of the entire SC and that of the supplier, and the graph in Figure 4 compares their cost efficiencies.

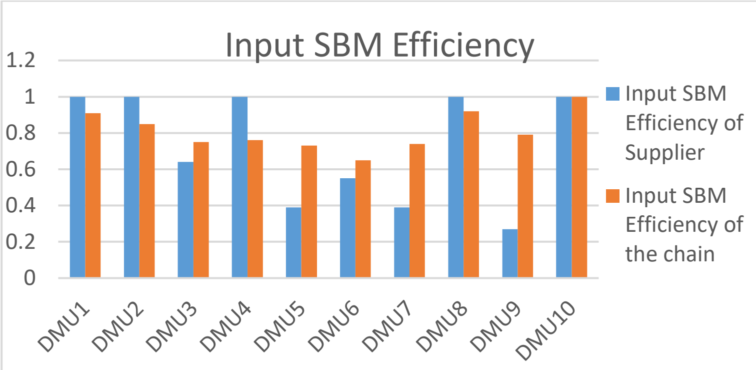

Figure 3. Technical efficiency of the entire SC and supplier
Source: authors' finding.

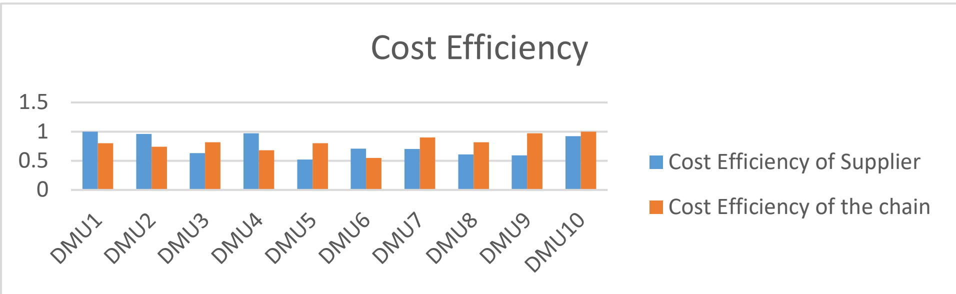

Figure 4. Cost efficiency of the entire SC and supplier
Source: authors' finding.

Table 3 presents the results of executing the inverse NDEA, including the estimated values for the inputs and outputs of the SC's various components. As previously mentioned regarding modelling, changes in output are dictated by managers and SC decision-makers. If the value of the non-radial input-based efficiency and cost efficiency of the SC improve or remain unchanged, the goal is to estimate the independent inputs of each of the network's various components. In other words, we seek to ensure a minimum input increase under the aforementioned conditions. Therefore, the intercomponent inputs and outputs of the network are allowed to change freely. However, it should be noted that changes should be made so that the manager can implement them. During the production process, the managers and decision-makers of the systems face several constraints, such as warehouse capacity, intermediate production, etc. Consequently, to estimate the changes in the model, we first define the range of changes based on the manager's needs. The inputs and outputs of the SC's various stages can increase or decrease as needed to ensure that the input-based non-radial efficiency and cost efficiency of the proposed SC remain unchanged or improve according to the manager's opinion under the condition that the SC's output increases. Table 3 presents the estimated values of the SC's independent inputs and the intermediate productions.

Table 3. Estimation results

DMUs	x-1	x-2	x-3	z-inter-inp-1	z-inter-out-1	z-out-1	z-out-2	z-inp-1
1	4.8	33.6	198.4	0.11	0.11	42.6	112.8	42.6
2	8	43.2	369.6	0.12	0.12	63.2	87	63.2
3	6.4	36.8	433.6	0.1	0.1	43.6	73	43.6
4	11.2	41.6	545.6	0.11	0.11	83.8	90.6	83.8
5	19.2	46.4	476.8	0.12	0.12	42.2	92.6	42.2
6	9.6	41.6	561.6	0.14	0.14	63.4	69	63.4
7	14.4	44.8	507.2	0.18	0.18	62.2	90.4	62.2
8	11.2	36.8	452.8	0.17	0.17	42.4	126.4	42.4
9	24	38.4	409.6	0.13	0.13	43.4	70.8	43.4
10	11.2	32	401.6	0.15	0.15	63.8	53	63.8
DMUs	**z-inp-2**	**w-inp-1**	**w-inp-2**	**w-out-1**	**w-out-2**	**x.final.1**	**x.final.2**	**x.final 3**
1	112.8	32.5	56.7	32.5	56.7	4.32	30.24	178.56
2	87	45.3	67.3	45.3	67.3	7.2	38.88	332.64
3	73	24.7	48.7	24.7	48.7	5.76	33.12	390.24
4	90.6	26.7	58.2	26.7	58.2	10.08	37.44	491.04
5	92.6	29.8	64.3	29.8	64.3	17.28	41.76	429.12
6	69	34.1	98.5	34.1	98.5	8.64	37.44	505.44
7	90.4	28.7	59.8	28.7	59.8	12.96	40.32	456.48
8	126.4	28.6	45.3	28.6	45.3	10.08	33.12	407.52
9	70.8	28.3	46.8	28.3	46.8	21.6	34.56	368.64
10	53	18.6	49.8	18.6	49.8	10.08	28.8	361.44

Source: authors' finding.

As indicated in Table 3, the pattern obtained for intermediate productions is compatible with the network structure, which means that the second-stage input does not exceed the first-stage output. The same condition is also true for the second-and third-stage intermediate production, and the results show that the values are equal in the final solution. Consider the final estimated values for x as *x-final-1* and *x-final-*

2. With these values, when the outputs increase, the technical and cost efficiencies of the DMUs under evaluation improve and reach $\frac{1-effi.sbm}{2}$.

7. Conclusions

In this study, we investigated an SC or, in broader terms, a network through an inverse NDEA. The basic goal was to estimate the input(s) after the manager increased the output(s) so that the values of technical and cost efficiencies remained unchanged. The inverse NDEA was applied to a network while considering non-radial changes to evaluate technical efficiency meticulously. Because the system was network-based, many constraints were added to the model to accurately assess the relationships between the network's components and the intermediate inputs and outputs in the model. We consider the intermediate production returned from one component to another in the inverse NDEA. The proposed model was then tested in practice by being applied to a real-world example from the concrete industry. It should be noted that considering the conditions of production, supply, and consumption in a SC is particularly important when applying the model to real-world cases. In future research, other modes and circumstances in the network can be investigated. Developing a similar model on a parallel or general network may be another promising choice. Finally, it is possible to redesign the model in this research under different conditions. For instance, the condition that the values of technical and cost efficiencies remain unchanged can be dictated to the network's components.

This study presents a new approach using an inverse Network Data Envelopment Analysis (NDEA) model to maintain constant technical and cost efficiencies for an expanded output network system. The model is novel since it is capable of approximating the inputs that should be changed when the quantity of outputs varies by management, with the guarantee of increasing or maintaining the system efficiency at a constant value. The theory has integrated the concept of reversible output and complicated interdependence between factors in a supply chain (SC). The reason why it works when applied to some industries is that there are interconnected production stages and that any alteration in one stage can cause a shift to the others. The application in the building industry in real life demonstrated real-world relevance of the model in real life by measuring the input change required to sustain levels of efficiency under a situation of increase in output. One of the main benefits of this technique is that it focuses on network structures, in which one can create more accurate technical and cost effectiveness estimates. Including constraints that take into account intermediate productions and output flows between components, the model creates a better estimation of real systems. It is particularly valuable for decision-makers who must optimise resource usage while not sacrificing overall performance. The model is a supply chain management tool at different levels of granularity without compromising technical and cost efficiencies. In future research, other modes and circumstances in the network can be investigated. Developing a similar model on a parallel or general network may be another promising choice. Finally, it is possible to redesign the model in this research under different conditions. For instance, the condition that the values of technical and cost efficiencies remain unchanged can be dictated to the network's components.

References

[1] Aliheidari Bioki, T., Khademi Zare, H. (2014), *Adaptive DEA for clustering of credit clients. Journal of Applied Research on Industrial Engineering*, 1(1), 35-49.

[2] Amirzadeh, R., Nazari, A., Thiruvady, D., Ee, M.S. (2023), *Modelling determinants of cryptocurrency prices: A bayesian network approach*. arXiv preprint arXiv:2303.16148, 2023.

[3] Chen, L., Wang, Y., Jin, S. (2022), *How Green Credit Guidelines Policy Affect the Green Innovation In China? Environmental Engineering and Management Journal,* 21(3), 469-481.

[4] Dhananjay, T., Nazari, A., Elmi, A. (2021), *An ant colony optimisation based heuristic for mixed-model assembly line balancing with setups*. 2020 IEEE CEC, 1-8.

[5] Edalatpanah, S.A. (2020), *Data envelopment analysis based on triangular neutrosophic numbers. CAAI Transactions on Intelligence Technology*, 5(2), 94-98.

[6] Ghiyasi, M. (2015), *On inverse DEA model: The case of variable returns to scale. Computers & Industrial Engineering*, 87, 407-409.

[7] Gulati, R., Kumar, S. (2017), *Analysing banks' intermediation and operating efficiencies using the two-stage network DEA model: The case of India. International Journal of Productivity and Performance Management*, 66(4), 500-516.

[8] Kalantary, M., Saen, R.F. (2019), *Assessing sustainability of supply chains: An inverse network dynamic DEA model. Computers & Industrial Engineering*, 135, 1224-1238.

[9] Kao, C., Hwang, S.N. (2010), *Efficiency measurement for network systems: IT impact on firm performance. Decision Support Systems*, 48(3), 437-446.

[10] Kao, C., Hwang, S.-N. (2008), *Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan. European Journal of Operational Research*, 185(1), 418-429.

[11] Kazemi, A., Galagedera, D.U. (2023), *An inverse DEA model for intermediate and output target setting in serially linked general two-stage processes. IMA Journal of Management Mathematics*, 34(3), 511-539.

[12] Krmac, E., Djordjević, B. (2019), *A new DEA model for evaluation of supply chains: a case of selection and evaluation of environmental efficiency of suppliers. Symmetry*, 11(4), 565.

[13] Mocholi-Arce, M., Sala-Garrido, R., Molinos-Senante, M., Maziotis, A. (2022), *Measuring the eco-efficiency of the provision of drinking water by two-stage network data envelopment analysis. Environment, Development and Sustainability*, 1-17.

[14] Pinto, C. (2024). *An axiomatic production model for circular directional indirect network technologies with undesirable output production and relational NDEA models. Research Square*, https://doi.org/10.21203/rs.3.rs-4788363/v1.

[15] Pouralizadeh, M. (2024), *Sustainability evaluation of supply chain by inverse data envelopment analysis model: A case study of the power industry. International Journal of Operational Research,* 12(1), 43-61.

[16] Ratner, S.V., Shaposhnikov, A.M., Lychev, A.V. (2023), *Network DEA and its applications (2017–2022): A systematic literature review. Mathematics*, 11(9), 2141.

[17] Tone, K. (2001), *A slacks-based measure of efficiency in data envelopment analysis. European Journal of Operational Research*, 130(3), 498-509.

[18] Wei, Q., Zhang, J., Zhang, X. (2000), *An inverse DEA model for inputs/outputs estimate. European Journal of Operational Research*, 121(1), 151-163.

[19] You, S., Yan, H. (2011), *A new approach in modelling undesirable output in DEA model. Journal of the Operational Research Society*, 62(12), 2146-2156.

[20] Yu, M.M. (2008), *Assessing the technical efficiency, service effectiveness, and technical effectiveness of the world's railways through NDEA analysis. Transportation Research Part A: Policy and Practice*, 42(10), 1283-1294.