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A Novel Ranking Approach for Transportation Problem under Bipolar Fuzzy Environment

Abstract. *This paper formulates a transportation problem where all parameters are represented as bipolar fuzzy numbers (BFNs), reflecting uncertainties due to market fluctuations impacting transportation costs, supply, and demand. BFNs provide a more realistic representation by analysing positive and negative membership, which distinguishes them from traditional fuzzy numbers. This research aims to minimise the transportation cost for various commodities within a transportation problem framework using BFNs. The proposed methodology introduces a novel ranking technique for solving the bipolar fuzzy transportation problem (BFTP), in which the cost supply and demand parameters are expressed as BFNs. The efficiency of the proposed ranking function over other existing ranking functions is also exhibited. An algorithm is presented to describe the solution methodology for BFTP. A numerical example is provided to demonstrate the practical implementation and effectiveness of the proposed method in solving transportation problems under uncertainty represented by BFNs. The study concludes with a discussion of the findings and outlines future avenues for research.*

Keywords: *Bipolar fuzzy numbers, transportation problem, ranking function, satisfaction degree, dissatisfaction degree.*

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1. Introduction

Transportation problem (TP) is a specialised linear programming problem that aims to minimise total transportation cost or maximise total profit by fulfilling demand at destinations using available supply from origins. There are several methods to solve TP under crisp parameters. However, real-world transportation parameters often suffer from imprecision due to uncontrollable factors, leading to

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uncertainty in various environments. Various tools have been developed to address these uncertainties, such as fuzzy set theory and its extensions. Initially, Zadeh (1965) introduced the fuzzy set theory to handle imprecise data by assigning degrees of membership. Traditional fuzzy sets only incorporate degrees of acceptance. To counter the limitations of traditional fuzzy sets, Atanassov (1999) introduced the intuitionistic fuzzy set (IFS) theory, which includes degrees of acceptance, non-acceptance, and hesitation. IFS offers a more comprehensive framework to depict uncertainties compared to fuzzy sets. The existing fuzzy set and IFS cannot represent degrees of dissatisfaction. In practical situations, information often encompasses two polarities: satisfaction and dissatisfaction. To address this dual perspective, Zhang (1998) introduced bipolar fuzzy sets (BFS). BFS is an extension of traditional fuzzy sets incorporating positive (satisfaction) and negative (dissatisfaction) aspects. Membership degrees in BFS range from -1 to 1, where [-1, 0] denotes negative membership and [0, 1] denotes positive membership. This extension enables BFS to model and manage uncertainties effectively. Moreover, Akram and Arshad (2019) introduced various types of BFNs and established ranking functions for trapezoidal and triangular bipolar fuzzy types. The study also proposed a TOPSIS method under the bipolar fuzzy(BF) environment, utilising a ranking function for the group decision-making problem. Theoretical comparisons demonstrate the advantages of this method over existing multi-criteria decision-making methods. Khalil et al. (2022) demonstrated the effectiveness of the bipolar interval-valued neutrosophic set in optimising uncertain data within a sustainable healthcare supply chain model. By focusing on time, quality, and cost satisfaction levels, the model achieved high satisfaction rates for cost reduction, product quality, and overall time efficiency. This approach provides a robust framework for selecting the best suppliers in the medicine procurement process. Çakır et al. (2022) suggested a model for mobile COVID-19 vaccination at multi-facility locations, thus, highlighting BFS's relevance in diverse practical applications. Vidhya et al. (2024) provided a robust solution for shortest path problems under bipolar neutrosophic fuzzy numbers and demonstrated the algorithm's effectiveness in identifying optimal paths and minimising travel times.

Thus, BFS provides a powerful tool for addressing uncertainty and dualistic information complexities in various real-world applications, demonstrating their effectiveness beyond traditional fuzzy set theories.

2. Literature Review

TP holds a significant place in various real-world applications and was originally formulated by Hitchcock (1941). In the fuzzy scenario, Zimmermann (1975) introduced the concept of fuzzy linear programming, extending traditional methods to handle fuzzy parameters. Tanaka and Asai(1984) formulated a linear programming problem considering parameters as fuzzy numbers, pioneering the application of fuzzy set theory in optimisation problems. Chanas et al. (1984) proposed a model specifically designed to solve TP with fuzzy supply and demand.

Later, Chanas and Kuchta (1996) solved TP under fuzzy cost. The evolution of methods to solve TP under various fuzzy environments has been continuous. Liu and Kao (2004) presented a procedure to calculate the fuzzy objective value of the fuzzy TP by leveraging the extension principle. Through mathematical programmes and dual programming, the membership function of the objective value is derived for both inequality and equality constraints, providing a comprehensive view. Kaur and Kumar (2012) proposed a new approach for solving fuzzy TP where parameters were represented as generalised trapezoidal fuzzy numbers. After that various methods were used to solve fuzzy TP, including the MODI method by (Dhanasekar et al., 2017), a modified Vogel's approach (Pratihar et al., 2021), and the maximum modulus zero-suffix method (Roy et al., 2024).

Despite the capabilities of fuzzy set theory in handling uncertainty through membership functions, situations may arise where non-membership functions and degrees of hesitation are essential. In response, Singh and Yadav (2016) defined the accuracy function using score functions for the membership and non-membership functions of triangular intuitionistic fuzzy numbers (TrIFN). Ebrahimnejad and Verdegay (2018) developed a novel solution approach for solving fully intuitionistic fuzzy TP based on classical linear programming algorithms. Recently, Beg et al. (2023) solved a generalised Intuitionistic Fuzzy TP. They initially proposed a generalised intuitionistic fuzzy min-max product method to obtain the basic feasible solution and subsequently introduced a generalised intuitionistic fuzzy-modified distribution method to determine the optimal solution for the generalised Intuitionistic Fuzzy TP.

Researchers have extensively utilised BFS in various optimisation techniques. Mehmood et al. (2021) proposed a technique to solve the fully BF linear programming problem using the ranking function, which is based on the area and mean of positive and negative membership functions. Alsager and Alfahhad (2022) introduced bipolar type-2 fuzzy set and bipolar type-2 fuzzy soft set theories and provided a robust framework for handling high-order uncertainty in knowledge-based systems. Ahmed and Bashir (2022) introduced a novel technique for solving TP using bipolar single-valued neutrosophic sets (BSNS) and the BSNS-based method effectively addresses real-life uncertainties.

2.1 Research gap and motivation

In our exploration of various research studies, fuzzy sets and IFS theory have emerged as pivotal tools for addressing uncertainty in optimisation problems. Numerous methods have been developed to solve TP within fuzzy and IF environments. However, in practical scenarios where uncertainty in data is prevalent, BFS has proven to be a powerful tool for effectively managing fuzziness and uncertainty.

It has been observed that while BFS has been extensively used by researchers in various optimisation fields, there remains a notable absence of an optimisation model in the literature specifically for TP under a BF environment. The versatility and applicability of BFTP in real-life problems highlight its potential for novel applications and methodologies. This novelty has motivated us to develop a new ranking method to deal with TP in the BF environment.

2.2 Contribution and novelty to the proposed method

This research article focuses on solving the BFTP where the parameters are represented as BFNs. A novel ranking function is developed for BFNs and its properties are also presented. The efficiency of the defined ranking function is compared to the existing ones using examples. The application of the proposed method is exhibited on a TP where all the parameters are BFNs.

The organisation of the paper is as follows: Section 3 provides basic definitions and preliminaries essential for understanding the concepts discussed. Section 4 discusses the existing ranking function and its limitations through examples. Section 5 presents the proposed ranking function and compares it with the existing ranking function. Section 6 details the formulation of the model for the BFTP. Section 7 presents an algorithm designed to solve the BFTP, leveraging the transformation into a crisp TP. Section 8 demonstrates the application of the proposed approach through a numerical example. Finally, Section 9 concludes the paper, discussing its advantages and highlighting future research directions.

3. Preliminaries

In this section, basic definitions and arithmetic operations of BFNs are introduced.

Definition 1 (Akram and Arshad, 2019)**.** Let X be a nonempty set. A bipolar fuzzy system is denoted by $\tilde{A}$ in X and is defined as follows:

$$\widetilde{A} = \langle \tilde{A}^{+}, \tilde{A}^{-} \rangle = \{\langle x, \mu_{\tilde{A}}^{+}(x), \mu_{\tilde{A}}^{-}(x) \rangle : x \in X\}$$

where $\mu_{\tilde{A}}^{+}(x): X \to [0,1]$ and $\mu_{\tilde{A}}^{-}(x): X \to [-1,0]$.

The positive membership degree, denoted as $\mu_{\tilde{A}}^{+}$, signifies the degree of truth or satisfaction of an element x concerning a specific property corresponding to the bipolar fuzzy set $\tilde{A}$. Further, $\mu_{\tilde{A}}^{-}$ represents the degree of falsity or dissatisfaction of x with some counter property of $\tilde{A}$.

Definition 2 (Akram and Arshad, 2019)**.** Trapezoidal Bipolar Fuzzy Number (TpBFN). It is represented as $\langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$. Its positive membership function, $\mu_{\tilde{A}}^{+}$ and negative membership function, $\mu_{\tilde{A}}^{-}$, are represented as follows:

$$\mu_{\tilde{A}}^{+}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \le x \le a_2 \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases} \qquad \mu_{\tilde{A}}^{-}(x) = \begin{cases} \frac{b_1-x}{b_2-b_1}, & \text{if } b_1 \le x \le b_2 \\ \frac{x-b_4}{b_4-b_3}, & \text{if } b_3 \le x \le b_4 \\ 0, & \text{otherwise} \end{cases}$$

Where $\mu_{\tilde{A}}^{+}(x) \in [0,1]$ and $\mu_{\tilde{A}}^{-}(x) \in [-1,0]$.

Definition 3. A TpBFN $\tilde{A} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$ is called a non-negative TpBFN if and only if $a_1 \geq 0$ and $b_1 \geq 0$.

Definition 4. Arithmetic operations for TpBFNs.
Let $\tilde{A}_1 = \langle (a_{11}, a_{12}, a_{13}, a_{14}), (b_{11}, b_{12}, b_{13}, b_{14}) \rangle$ and $\tilde{A}_2 = \langle (a_{21}, a_{22}, a_{23}, a_{24}),$ $(b_{21}, b_{22}, b_{23}, b_{24}) \rangle$ be two TpBFNs. Then,

(1) $\tilde{A}_1 + \tilde{A}_2 = \langle (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}), (b_{11} + b_{21}, b_{12} + b_{22}, b_{13} + b_{23}, b_{14} + b_{24}) \rangle$

(2) $-\tilde{A}_1 = \langle (-a_{14}, -a_{13}, -a_{12}, -a_{11}), (-b_{14}, -b_{13}, -b_{12}, -b_{11}) \rangle$

(3) $\tilde{A}_1 - \tilde{A}_2 = \langle (a_{11} - a_{24}, a_{12} - a_{23}, a_{13} - a_{22}, a_{14} - a_{21}), (b_{11} - b_{24}, b_{12} - b_{23}, b_{13} - b_{22}, b_{14} - b_{21}) \rangle$

(4) $K\tilde{A}_1 = \begin{cases} \langle (Ka_{11}, Ka_{12}, Ka_{13}, Ka_{14}), (Kb_{11}, Kb_{12}, Kb_{13}, Kb_{14}) \rangle, & \text{if } K \geq 0 \\ \langle (Ka_{14}, Ka_{13}, Ka_{12}, Ka_{11}), (Kb_{14}, Kb_{13}, Kb_{12}, Kb_{11}) \rangle, & \text{if } K < 0 \end{cases}$

Note: In the TpBFN, $\tilde{A} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$, take $a_2 = a_3$ and $b_2 = b_3$, then, the TpBFN transforms into a triangular bipolar fuzzy number(TrBFN) as $\tilde{A} = \langle (a_1, a_2, a_4), (b_1, b_2, b_4) \rangle$.

4. Existing Ranking Functions and Their Limitations

Definition 5 (Akram and Arshad, 2019). **Ranking Function R_{f1} for $TpBFN$**
Let $\tilde{A} = \langle \tilde{A}^{+}, \tilde{A}^{-} \rangle = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$ be the TpBFN. Using Definition 2**,** the ranking function R_{f1} of $TpBFN$ is defined as:

$$R_{f1}(\tilde{A}) = (m_1(\tilde{A}^{+}) + \sigma_1(\tilde{A}^{+})) - (m_1(\tilde{A}^{-}) + \sigma_1(\tilde{A}^{-}))$$

where $m_1(\tilde{A}^{+})$ and $m_1(\tilde{A}^{-})$ denote the mean of positive and negative membership, respectively, and are defined as:

$$m_1(\tilde{A}^{+}) = \frac{a_1 + a_2 + a_3 + a_4}{4}, \quad m_1(\tilde{A}^{-}) = \frac{b_1 + b_2 + b_3 + b_4}{4},$$

and $\sigma_1(\tilde{A}^{+})$ and $\sigma_1(\tilde{A}^{-})$ denote the area of positive and negative membership, respectively, and are defined as:

$$\sigma_1(\tilde{A}^{+}) = \frac{-a_1 - a_2 + a_3 + a_4}{2}, \quad \sigma_1(\tilde{A}^{-}) = \frac{-b_1 - b_2 + b_3 + b_4}{2}.$$

Therefore,

$$R_{f1}(\tilde{A}) = \frac{(-a_1 - a_2 + 3a_3 + 3a_4) - (b_1 - b_2 + 3b_3 + 3b_4)}{4}$$
$$= \frac{-a_1 - a_2 + 3a_3 + 3a_4 + b_1 + b_2 - 3b_3 - 3b_4}{4}.$$

Definition 6 (Akram and Arshad, 2019)**. Characteristics of ranking function.** Let $K = \{k_1, k_2, k_3, \dots, k_n\}$ be the set of TpBFNs. Then, for any distinct $k_i, k_j \in K$, the ranking function R_{f1} from K to the real line $\mathbb{R}$ is a mapping, satisfying the following characteristics:

- If $R_{f1}(k_i) < R_{f1}(k_j)$, then $k_i < k_j$.
- If $R_{f1}(k_i) = R_{f1}(k_j)$, then $k_i = k_j$.
- If $R_{f1}(k_i) > R_{f1}(k_j)$, then $k_i > k_j$.

In the same way, Akram and Arshad define a ranking function for TrBFNs, $\tilde{A} = \langle \tilde{A}^+, \tilde{A}^- \rangle = \langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle$ as follows;

$$R_{f2}(\tilde{A}) = (m_2(\tilde{A}^+) + \sigma_2(\tilde{A}^+)) - (m_2(\tilde{A}^-) + \sigma_2(\tilde{A}^-))$$

Therefore,

$$R_{f2}(\tilde{A}) = \frac{(-a_1 + 2a_2 + 5a_3) - (-b_1 + 2b_2 + 5b_3)}{6}$$
$$= \frac{-a_1 + 2a_2 + 5a_3 + b_1 - 2b_2 - 5b_3}{6}.$$

4.1 Limitations of the Ranking Function

This section uses numerical examples to discuss some limitations of Akram and Arshad's ranking function by comparing different TpBFNs or TrBFNs.

Example 1: Let A=⟨(5,7,8,9),(5,7,8,9)⟩,B=⟨(90,100,110,120),(90,100,110,120)⟩, C=⟨(0,1,2,3),(0,1,2,3)⟩ and D=⟨(20,30,40,50),(20,30,40,50)⟩ be four different TpBFNs, but all have the same ranking value:

$$R_{f1}(A) = R_{f1}(B) = R_{f1}(C) = R_{f1}(D) = 0.$$

Example 2: Let $A = \langle (3,4,5,6), (2,3,4,5) \rangle$ and $B = \langle (2,3,4,5), (1,2,3,4) \rangle$ be two different TpBFNs. The ranking function R_{f1} is the same:

$$R_{f1}(A) = R_{f1}(B) = 1.$$

Example 3: Let $A = \langle (2,3,4), (1,2,3) \rangle$ and $B = \langle (4,5,6), (3,4,5) \rangle$ be two different TrBFNs. The ranking function R_{f2} is given by;

$$R_{f2}(A) = R_{f2}(B) = 1.$$

which implies that $A = B$, but A and B are two different TrBFNs.

Example 4: Let $A = \langle (3,5,7), (2,3,4) \rangle$ and $B = \langle (5,7,9), (4,5,6) \rangle$ be two different TrBFNs. The ranking function R_{f2} is the same:

$$R_{f2}(A) = R_{f2}(B) = 3.$$

Example 5: Let A=⟨(30,50,60),(30,50,60)⟩, B=⟨(40,60,70),(40,60,70)⟩, C=⟨(10, 20,30),(10,20,30)⟩, D=⟨(1,2,3),(1,2,3)⟩ and E=⟨(50,70,90),(50,70,90)⟩ be four different TpBFNs, but all have the same ranking value:

$$R_{f2}(A) = R_{f2}(B) = R_{f2}(C) = R_{f2}(D) = R_{f2}(E) = 0.$$

4.2 Limitations of the ranking function are discussed as follows:

1. Let $A_1 = \langle(a_{11}, a_{12}, a_{13}, a_{14}), (a_{11}, a_{12}, a_{13}, a_{14})\rangle$ and $A_2 = \langle(a_{21}, a_{22}, a_{23}, a_{24}), (a_{21}, a_{22}, a_{23}, a_{24})\rangle$ be any two TpBFNs. Then, $R_{f1}(A_1) = R_{f1}(A_2) = 0$ (Example 1).

2. Let $A_1 = \langle(a_{11}, a_{12}, a_{13}, a_{14}), (b_{11}, b_{12}, b_{13}, b14)\rangle$ and $A_2 = \langle(a_{11} - k, a_{12} - k, a_{13} - k, a14 - k), (b11 - k, b12 - k, b13 - k, b14 - k)\rangle$ be any two TpBFNs and $k \le a_{11}, b_{11}$. Then, $R_{f1}(A_1) = R_{f1}(A_2)$ (Example 2).

3.Let $A_1 = \langle(a_{11}, a_{12}, a_{13}, a_{14}), (b_{11}, b_{12}, b_{13}, b14)\rangle$ and $A_2 = \langle(a_{11} + k, a_{12} + k, a_{13} + k, a14 + k), (b11 + k, b12 + k, b13 + k, b14 + k)\rangle$ be any two TpBFNs. Then $R_{f1}(A_1) = R_{f1}(A_2)$ (Example 2).

The limitations of Akram and Arshad's ranking functions motivate the authors to define a new ranking function. It is presented in the following section.

5. Proposed Ranking Function

In this section, a new ranking function for the class of BFNs is introduced to address the limitations of Akram and Arshad's ranking functions. In addition, the properties of the proposed ranking function are also presented.

To define a new ranking function for BFNs, we first define the value and ambiguity of positive and negative membership for BFNs as follows;

Definition 7 (Value of positive and negative membership) The value of the positive membership, $\mu^+(x)$ and the value of the negative membership, $\mu^-(x)$ for the BFN $\tilde{A}$ are defined as follows:

$$V_{\mu^+_{\tilde{A}}} = \int_0^1 \frac{(L_A^{\mu^+}(\alpha) + R_A^{\mu^+}(\alpha))}{2}\, d\alpha \tag{1}$$

$$V_{\mu^-_{\tilde{A}}} = \int_{-1}^0 \frac{(L_A^{\mu^-}(\beta) + R_A^{\mu^-}(\beta))}{2}\, d\beta \tag{2}$$

Here, $\alpha \in [0,1], \beta \in [-1,0]$ and $L_A^{\mu^+}(\alpha)$ and $R_A^{\mu^+}(\alpha)$ are the lower and upper limits of positive membership, respectively, and $L_A^{\mu^-}(\beta))$ and $R_A^{\mu^-}(\beta))$ are the lower and upper limits of negative membership, respectively.

Thus, according to Eqs. (1) and (2), the values of the positive and negative membership functions of TpBFN $\tilde{A}$ are:

$$V_{\mu^+_{\tilde{A}}} = \frac{a_1 + a_2 + a_3 + a_4}{4} \tag{3}$$

$$V_{\mu^-_{\tilde{A}}} = \frac{b_1 + b_2 + b_3 + b_4}{4} \tag{4}$$

Definition 8 (Ambiguity of the positive and negative membership) The ambiguity of the positive membership, $\mu^+(x)$ and the ambiguity of the negative membership, $\mu^-(x)$ for the BFN $\tilde{A}$ are defined as follows:

$$A_{\mu^+_{\tilde{A}}} = \int_0^1 \left(R_A^{\mu^+}(\alpha) - L_A^{\mu^+}(\alpha)\right) d\alpha \quad (5)$$

$$A_{\mu^-_{\tilde{A}}} = \int_{-1}^0 \left(R_A^{\mu^-}(\beta) - L_A^{\mu^-}(\beta)\right) d\beta \quad (6)$$

where $\alpha \in [0,1]$ and $\beta \in [-1,0]$.

Thus, according to Eqs. (5) and (6), the ambiguities of the positive and negative membership functions of TpBFN $\tilde{A}$ are:

$$A_{\mu^+_{\tilde{A}}} = \frac{-a_1-a_2+a_3+a_4}{2} \quad (7)$$

$$A_{\mu^-_{\tilde{A}}} = \frac{-b_1-b_2+b_3+b_4}{2} \quad (8)$$

5.1 Proposed Ranking Function

Using definitions 7 and 8, we proposed the following ranking function for BFNs. Let $\tilde{A}$ be a BFN. $V_{\mu^+_{\tilde{A}}}$ and $V_{\mu^-_{\tilde{A}}}$ are the values of the positive and negative membership functions of $\tilde{A}$, respectively and $A_{\mu^+_{\tilde{A}}}$ and $A_{\mu^-_{\tilde{A}}}$ are the ambiguities of the positive and negative membership functions of $\tilde{A}$, respectively. Then the ranking function $R(\tilde{A})$ of BFN $\tilde{A}$ is defined as:

$$R(\tilde{A}) = \lambda \left[V_{\mu^+_{\tilde{A}}} + A_{\mu^+_{\tilde{A}}}\right] + (1-\lambda)\left[V_{\mu^-_{\tilde{A}}} + A_{\mu^-_{\tilde{A}}}\right], \text{where } \lambda \in [0,0.5) \cup (0.5,1]. \quad (9)$$

Thus, the ranking function of a TpBFN is as follows:

$$R_{Tp}(\tilde{A}) = \lambda \left[\frac{a_1+a_2+a_3+a_4}{4} + \frac{-a_1-a_2+a_3+a_4}{2}\right] + (1-\lambda)\left[\frac{b_1+b_2+b_3+b_4}{4} + \frac{-b_1-b_2+b_3+b_4}{2}\right] \quad (10)$$

And the ranking function of a TrBFN is:

$$R_{Tr}(\tilde{A}) = \lambda \left[\frac{a_1+2a_2+a_3}{4} + \frac{a_3-a_1}{2}\right] + (1-\lambda)\left[\frac{b_1+b_2+b_3}{4} + \frac{b_3-b_1}{2}\right] \quad (11)$$

Definition 9 Characteristics of the ranking function

Let $A = \{A_1, A_2, \dots, A_n\}$ be the set of BFNs. For any distinct $A_i, A_j \in A$, the ranking function R from A to the real line $\mathbb{R}$ is a mapping satisfying the following characteristics:

- $R(A_1) > R(A_2)$ if and only if $A_1 > A_2$.
- $R(A_1) < R(A_2)$ if and only if $A_1 < A_2$.
- $R(A_1) = R(A_2)$ if and only if $A_1 = A_2$.

Theorem 1: The ranking function $R: \tilde{A} \to \mathbb{R}$ is a linear function.

Proof: Let $\tilde{A}_1 = \langle(a_{11}, a_{12}, a_{13}, a_{14}), (b_{11}, b_{12}, b_{13}, b_{14})\rangle$ and $\tilde{A}_2 = \langle(a_{21}, a_{22}, a_{23}, a_{24}), (b_{21}, b_{22}, b_{23}, b_{24})\rangle$ be two TpBFNs. Then for $k > 0$, we have:

$$R(\tilde{A}_1 + k\tilde{A}_2) = R(\langle(a_{11}, a_{12}, a_{13}, a_{14}), (b_{11}, b_{12}, b_{13}, b_{14})\rangle + \langle(ka_{21}, ka_{22}, ka_{23}, ka_{24}), (kb_{21}, kb_{22}, kb_{23}, kb_{24})\rangle)$$

$$= R(\langle(a_{11} + ka_{21}, a_{12} + ka_{22}, a_{13} + ka_{23}, a_{14} + ka_{24}), (b_{11} + kb_{21}, b_{12} + kb_{22}, b_{13} + kb_{23}, b_{14} + kb_{24})\rangle)$$

$$= \lambda\left[\frac{a_{11} + a_{12} + a_{13} + a_{14} + k(a_{21} + a_{22} + a_{23} + a_{24})}{4} + \frac{-a_{11} - a_{12} + a_{13} + a_{14} + k(-a_{21} - a_{22} + a_{23} + a_{24})}{2}\right] +$$

$$(1-\lambda)\left[\frac{b_{11} + b_{12} + b_{13} + b_{14} + k(b_{21} + b_{22} + b_{23} + b_{24})}{4} + \frac{-b_{11} - b_{12} + b_{13} + b_{14} + k(-b_{21} - b_{22} + b_{23} + b_{24})}{2}\right]$$

$$= R(\tilde{A}_1) + kR(\tilde{A}_2).$$

Similarly, it can be proved for $k < 0$. This implies that R is a linear function.

5.2 Comparison with Akram's Ranking Function

The comparative analysis of the proposed ranking function to Akram and Arshid's ranking function is illustrated in Table 1.

Table 1. Comparative analysis of the proposed ranking function

Examples	**Akram Ranking Principle**	**Proposed Ranking Principle**
		Let $\lambda = 0.6$
Let $\tilde{A} = \langle(5,7,8,9), (5,7,8,9)\rangle$, $\tilde{B} = \langle$ (90,100,110,120), (90,100,110,120)$\rangle$, $\tilde{C} = \langle(0,1,2,3), (0,1,2,3)\rangle$, *and* $\tilde{D} = \langle(20,30,40,50)$, (20,30,40,50)$\rangle$ be four different TpBFNs.	$R_{f1}(\tilde{A}) = R_{f1}(\tilde{B}) = R_{f1}(\tilde{C}) = R_{f1}(\tilde{D}) = 0$, $\tilde{A} = \tilde{B} = \tilde{C} = \tilde{D}$	$R_{Tp}(\tilde{A}) = 9.75, R_{Tp}(\tilde{B}) = 125$, $R_{Tp}(\tilde{C}) = 3.5, R_{Tp}(\tilde{D}) = 55$ $R_{Tp}(\tilde{C}) < R_{Tp}(\tilde{A}) < R_{Tp}(\tilde{D}) < R_{Tp}(\tilde{B})$ Then, $\tilde{C} < \tilde{A} < \tilde{D} < \tilde{B}$
Let $\tilde{A} = \langle(3,4,5,6), (2,3,4,5)\rangle$ and $\tilde{B} = \langle(2,3,4,5), (1,2,3,4)\rangle$ be two different TpBFNs.	$R_{f1}(\tilde{A}) = R_{f1}(\tilde{B}) = 1$ $\tilde{A} = \tilde{B}$	$R_{Tp}(\tilde{A}) = 6.1, R_{Tp}(\tilde{B}) = 5.1$, $R_{Tp}(\tilde{A}) > R_{Tp}(\tilde{B})$ Then, $\tilde{A} > \tilde{B}$
Let $\tilde{A} = \langle(2,3,4), (1,2,3)\rangle$, $\tilde{B} = \langle(4,5,6), (3,4,5)\rangle$ be two different TrBFNs	$R_{f2}(\tilde{A}) = R_{f2}(\tilde{B}) = 1$ $\tilde{A} = \tilde{B}$	$R_{Tr}(\tilde{A}) = 3.6, R_{Tr}(\tilde{B}) = 5.6$ $R_{Tr}(\tilde{A}) < R_{Tr}(\tilde{B})$ Then, $\tilde{A} < \tilde{B}$
Let $\tilde{A} = \langle(3,5,7), (2,3,4)\rangle$ $\tilde{B} = \langle(5,7,9), (4,5,6)\rangle$ be two different TrBFNs.	$R_{f2}(\tilde{A}) = R_{f2}(\tilde{B}) = 3$ $\tilde{A} = \tilde{B}$	$R_{Tr}(\tilde{A}) = 5.8, R_{Tr}(\tilde{B}) = 7.8$ $R_{Tr}(\tilde{A}) < R_{Tr}(\tilde{B})$ Then, $\tilde{A} < \tilde{B}$
Let$\tilde{A} = \langle(30,50,60), (30,50,60)\rangle$ $\tilde{B} = \langle(40,60,70), (40,60,70)\rangle$ $\tilde{C} = \langle(10,20,30), (10,20,30)\rangle$ $\tilde{D} = \langle(1,2,3), (1,2,3)\rangle$ $\tilde{E} = \langle(50,70,90), (50,70,90)\rangle$ be five different TrBFNs.	$R_{f2}(\tilde{A}) = R_{f2}(\tilde{B}) = R_{f2}(\tilde{C}) = R_{f2}(\tilde{D}) = R_{f2}(\tilde{E}) = 0$, $\tilde{A} = \tilde{B} = \tilde{C} = \tilde{D} = \tilde{E}$	$R_{Tr}(\tilde{A}) = 65, R_{Tr}(\tilde{B}) = 75$, $R_{Tr}(\tilde{C}) = 30, R_{Tr}(\tilde{D}) = 3, R_{Tr}(\tilde{E}) = 90$, $R_{Tr}(\tilde{D}) < R_{Tr}(\tilde{C}) < R_{Tr}(\tilde{A}) < R_{Tr}(\tilde{B}) < R_{Tr}(\tilde{E})$ Then, $\tilde{D} < \tilde{C} < \tilde{A} < \tilde{B} < \tilde{E}$

Source: The data used in the numerical illustrations is hypothetical.

6. Model Formulation for BFTP

A BFTP is a TP where all the parameters involved are represented by BFNs. Consider a scenario where there are m sources of commodities capable of supplying n destination points. Each transportation route from a source i to a destination j is associated with a BF cost coefficient $\tilde{c}_{ij}$, where $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$. Let $\tilde{A}_i$ denote the bipolar fuzzy amount of commodities available at source i, and $\tilde{B}_j$ denote the BF amount of commodities required at destinations j. The decision variable x_{ij} represents the amount of commodities transported from source i to destination j.

Mathematically, the BFTP can be formulated as follows:

$$\text{Minimise} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}$$

subject to the constraints:

$$\sum_{j=1}^{n} x_{ij} \le \tilde{A}_i, \qquad (i = 1,2,\dots,m)$$
$$\sum_{i=1}^{m} x_{ij} \ge \tilde{B}_j, \qquad (j = 1,2,\dots,n),$$
$$x_{ij} \ge 0, \qquad \forall i, j,$$
$$\tilde{c}_{ij} \ge 0, \quad \tilde{A}_i \ge 0, \quad \tilde{B}_j \ge 0, \qquad \forall i, j.$$

The objective is to minimise the total transportation cost considering the uncertainty represented by the BFNs, while ensuring that the supply from each source meets or exceeds the demand at each destination.

Here, $\tilde{c}_{ij} = \langle (c_{ija1}, c_{ija2}, c_{ija3}, c_{ija4}), (c_{ijb1}, c_{ijb2}, c_{ijb3}, c_{ijb4}) \rangle$. The positive membership function $\mu^{+}_{\tilde{c}_{ij}}$ and negative membership function $\mu^{+}_{\tilde{c}_{ij}}$ of transportation cost $\tilde{c}_{ij}$ ($\forall$ i and j) are given as follows;

$$\mu^{+}_{\tilde{c}_{ij}}(x) = \begin{cases} \frac{x - c_{ija1}}{c_{ija2} - c_{ija1}}, & \text{if } \mathrm{c_{ija1}} \le \mathrm{x} \le \mathrm{c_{ija2}} \\ \frac{c_{ija4} - x}{c_{ija4} - c_{ija3}}, & \text{if } \mathrm{c_{ija3}} \le \mathrm{x} \le \mathrm{c_{ija4}} \\ 0, & \text{otherwise} \end{cases}$$

$$\mu^{-}_{\tilde{c}_{ij}}(x) = \begin{cases} \dfrac{c_{ijb1} - x}{c_{ijb2} - c_{ijb1}}, & \text{if } \mathrm{c_{ijb1}} \le \mathrm{x} \le \mathrm{c_{ijb2}} \\ \dfrac{x - c_{ijb4}}{c_{ijb4} - c_{ijb3}}, & \text{if } \mathrm{c_{ijb3}} \le \mathrm{x} \le \mathrm{c_{ijb4}} \\ 0, & \text{otherwise} \end{cases}$$

Similarly, for supply $\tilde{A}_i = \langle (A_{ia1}, A_{ia2}, A_{ia3}, A_{ia4}), (A_{ib1}, A_{ib2}, A_{ib3}, A_{ib4}) \rangle$, the positive and negative membership functions are given as follow

$$\mu^{+}_{\tilde{A}_i}(x) = \begin{cases} \frac{x - A_{ia1}}{A_{ia2} - A_{ia1}}, & \text{if } A_{ia1} \le x \le A_{ia2} \\ \frac{A_{ia4} - x}{A_{ia4} - A_{ia3}}, & \text{if } A_{ia3} \le x \le A_{ia4} \\ 0, & \text{otherwise} \end{cases}$$

$$\mu^{-}_{\tilde{A}_i}(x) = \begin{cases} \dfrac{A_{ib1} - x}{A_{ib2} - A_{ib1}}, & \text{if } A_{ib1} \leq x \leq A_{ib2} \\ \dfrac{x - A_{ib4}}{A_{ib4} - A_{ib3}}, & \text{if } A_{ib3} \leq x \leq A_{ib4} \\ 0, & \text{otherwise} \end{cases}$$

Similarly, for demand $\tilde{B}_j = \langle (B_{ja1}, B_{ja2}, B_{ja3}, B_{ja4}), (B_{jb1}, B_{jb2}, B_{jb3}, B_{jb4}) \rangle$, the positive and negative membership functions are given as follows:

$$\mu^{+}_{\tilde{B}_j}(x) = \begin{cases} \frac{x - B_{ja1}}{B_{ja2} - B_{ja1}}, & \text{if } B_{ja1} \leq x \leq B_{ja2} \\ \frac{B_{ja4} - x}{B_{ja4} - B_{ja3}}, & \text{if } B_{ja3} \leq x \leq B_{ja4} \\ 0, & \text{otherwise} \end{cases}$$

$$\mu^{-}_{\tilde{B}_j}(x) = \begin{cases} \dfrac{B_{jb1} - x}{B_{jb2} - B_{jb1}}, & \text{if } B_{jb1} \leq x \leq B_{jb2} \\ \dfrac{x - B_{jb4}}{B_{jb4} - B_{jb3}}, & \text{if } B_{jb3} \leq x \leq B_{jb4} \\ 0, & \text{otherwise} \end{cases}$$

In the mathematical model of BFTP, the ranking function is applied. Since the ranking function is linear, the model is transformed into the following form:

$$\text{Minimise} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} R(\tilde{c}_{ij}) x_{ij}$$

subject to the constraints:

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} \leq R(\tilde{A}_i), && (i = 1,2, \dots, m) \\ &\sum_{i=1}^{m} x_{ij} \geq R(\tilde{B}_j), && (j = 1,2, \dots, n) \\ &x_{ij} \geq 0, && \forall i, j \end{aligned}$$

Solve the BFTP using software like LINGO 21.0 to find the optimal solution.

7. Methodology and algorithm

The proposed method, in algorithmic form, to find the optimal basic feasible solution in a bipolar fuzzy environment is outlined below:

1. **Step 1: Construct the Transportation Table**.

From the given data, construct the transportation table where the cost matrix, supplies, and demands are represented as BFNs.

2. **Step 2: Calculate Ranking Values**.

Calculate the ranking value for each cell in the transportation table. This involves assessing the BFNs and assigning a ranking value based on their characteristics.

3. **Step 3: Replace with Ranking Values**.

Replace each BFN in the transportation table with its corresponding ranking value.

4. **Step 4: Check Balance TP**.

Check whether the TP is balanced. If so, proceed to the next step. If it is unbalanced, first convert it into a balanced TP, then proceed to the next step.

5. **Step 5: Solve the Crisp TP**.

Solve the transformed TP using software such as LINGO 21.0 to find the optimal solution.

6. **Step 6: Convert to BF Solution**.

Convert the crisp solution into a BF solution with the help of BF cost coefficients.

This method ensures that the uncertainty and ambiguity inherent in BFNs are addressed by converting them into ranking values, facilitating the application of traditional TP-solving techniques. The detailed step-by-step procedure for a numerical example is explained in the subsequent section.

8. Application for BFTP

8.1 Example 6: TP with TpBFNs

Consider a TP for a company that collects dairy products from three sources: Jaipur, Punjab, and Lucknow, delivering them to three destination centres: Delhi, Uttar Pradesh, and Haryana.The availability, demand, and transportation costs are not known precisely due to factors such as machine failures, road conditions, market fluctuations, and more. These uncertainties are represented using TpBFNs in the transportation Table 2.

Table 2. Transportation cost, supply and demand of BFTP

Supplier /Demander	**Delhi**	**Uttar Pradesh**	**Haryana**	**Supply ($\tilde{A}_i$)**
Jaipur	$\langle(3,4,5,6),(1,2,3,4)\rangle$	$\langle(4,5,6,7),(3,4,5,6)\rangle$	$\langle(2,3,4,5),(1,2,3,4)\rangle$	$\langle(110,115,120,125),(105,110,115,120)\rangle$
Punjab	$\langle(4,5,6,7),(3,4,5,6)\rangle$	$\langle(3,4,5,6),(2,3,4,5)\rangle$	$\langle(7,8,9,10),(6,7,8,9)\rangle$	$\langle(120,125,130,135),(115,120,125,130)\rangle$
Lucknow	$\langle(2,3,4,5),(1,2,3,4)\rangle$	$\langle(3,4,5,6),(2,3,4,5)\rangle$	$\langle(4,5,6,7),(3,4,5,6)\rangle$	$\langle(90,95,100,105),(85,90,95,100)\rangle$
Demand ($\tilde{B}_j$)	$\langle(120,125,130,135),(115,120,125,130)\rangle$	$\langle(110,115,120,125),(105,110,115,120)\rangle$	$\langle(90,95,100,105),(85,90,95,100)\rangle$	

Source: The data used in the numerical illustrations is hypothetical.

The objective function $\tilde{Z}$ of the TP aims to minimise the total transportation cost, considering the uncertainty encapsulated by the BFN in the cost matrix, supplies and demands.

Solution. Step 2: Calculate Ranking Value.

The ranking values of BFTP are shown in Table 3 where $\lambda = 0.6$.

Table 3. Ranking Values of BFTP

Supplier/Demander	Delhi	Uttar Pradesh	Haryana	Supply ($\widetilde{A}_i$)
Jaipur	5.7	7.1	5.1	125.5
Punjab	7.1	6.1	10.1	135.5
Lucknow	5.1	6.1	7.1	105.5
Demand ($\widetilde{B}_j$)	135.5	125.5	105.5	-

Source: The data used in the numerical illustrations is hypothetical.

Step 5: Solve the crisp TP.

Minimise $\tilde{Z} = 5.7x_{11} + 7.1x_{12} + 5.1x_{13} + 7.1x_{21} + 6.1x_{22} + 10.1x_{23} + 5.1x_{31}$

$$+6.1x_{32} + 7.1x_{33}$$

Subject to constraints:

$$x_{11} + x_{12} + x_{13} \leq 125.5$$
$$x_{21} + x_{22} + x_{23} \leq 135.5$$
$$x_{31} + x_{32} + x_{33} \leq 105.5$$
$$x_{11} + x_{21} + x_{31} \geq 135.5$$
$$x_{12} + x_{22} + x_{32} \geq 125.5$$
$$x_{13} + x_{23} + x_{33} \geq 105.5$$
$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0$$

Solve the crisp TP with the help of LINGO 21.0.

Step 6: Convert to BF solution.

$\tilde{Z} = \langle(898.5, 1265, 1631.5, 1998), (512, 878.5, 1245, 1611.5)\rangle$.

The positive membership $\mu^{+}_{\tilde{Z}}(x)$ and negative membership $\mu^{-}_{\tilde{Z}}(x)$ functions of the TpBFN cost are given as follows:

$$\mu^{+}_{\tilde{Z}}(x) = \begin{cases} \dfrac{x - 898.5}{1265 - 898.5}, & \text{if } 898.5 \leq x \leq 1265 \\ \dfrac{1998 - x}{1998 - 1631.5}, & \text{if } 1631.5 \leq x \leq 1998 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu^{-}_{\tilde{Z}}(x) = \begin{cases} \dfrac{512 - x}{878.5 - 512}, & \text{if } 512 \leq x \leq 878.5 \\ \dfrac{x - 1611.5}{1611.5 - 1245}, & \text{if } 1245 \leq x \leq 1611.5 \\ 0, & \text{otherwise} \end{cases}$$

8.2 Example 7: TP with TrBFNs

Consider a TP for a company that collects food products from four production centres and delivers them to four destination centres. The transportation costs, supplies, and demands are represented as TrBFNs. The objective function $\tilde{Z}$, which represents the transportation cost, along with the cost matrix, supply, and demand, are specified in Table 4:

Table 4. Transportation cost, supply and demand of BFTP

Supplier /Demander	**Destination 1**	**Destination 2**	**Destination 3**	**Destination 4**	**supply ($\tilde{A}_i$)**
Production Center 1	$\langle(3,5,7),(2,3,4)\rangle$	$\langle(5,7,9),(4,5,6)\rangle$	$\langle(7,9,11),(7,8,9)\rangle$	$\langle(6,8,10),(5,6,7)\rangle$	$\langle(170,180,190),(165,170,175)\rangle$
Production Center 2	$\langle(5,7,9),(3,4,5)\rangle$	$\langle(6,8,10),(5,6,7)\rangle$	$\langle(8,10,12),(11,12,13)\rangle$	$\langle(15,16,17),(14,15,16)\rangle$	$\langle(140,150,160),(135,140,145)\rangle$
Production Center 3	$\langle(8,10,12),(7,8,9)\rangle$	$\langle(9,11,13),(8,9,10)\rangle$	$\langle(3,5,7),(3,4,5)\rangle$	$\langle(10,12,14),(9,10,11)\rangle$	$\langle(150,160,170),(145,150,155)\rangle$
Production Center 4	$\langle(8,10,12),(7,8,9)\rangle$	$\langle(5,7,9),(4,5,6)\rangle$	$\langle(4,6,8),(2,3,4)\rangle$	$\langle(3,5,7),(2,3,4)\rangle$	$\langle(160,170,180),(155,160,165)\rangle$
Demand ($\tilde{B}_j$)	$\langle(160,170,180),(155,160,165)\rangle$	$\langle(180,190,200),(175,180,185)\rangle$	$\langle(160,170,180),(155,160,165)\rangle$	$\langle(150,160,170),(145,150,155)\rangle$	

Source: The data used in the numerical illustrations is hypothetical.

Solution. Step 2: Calculate ranking value.

Ranking values of BFTP are shown in Table 5 where $\lambda = 0.6$.

Table 5. Ranking Values of BFTP

Supplier/Demander	**Destination 1**	**Destination 2**	**Destination 3**	**Destination 4**	**Supply($\tilde{A}_i$)**
Production Center 1	5.8	7.8	10.2	8.8	184
Production Center 2	7.4	8.8	12.4	16.6	154
Production Center 3	10.8	11.8	6.2	12.8	164
Production Center 4	10.8	7.8	6.4	5.8	174
Demand ($\tilde{B}_j$)	174	194	174	164	-

Source: The data used in the numerical illustrations is hypothetical.

Step 4: Check balance of the crisp TP.

The balanced problem is shown in Table 6.

Table 6. Modifying costs matrix of BFTP

Supplier/Demander	**Destination 1**	**Destination 2**	**Destination 3**	**Destination 4**	**Supply ($\tilde{A}_i$)**
Production Center 1	5.8	7.8	10.2	8.8	184
Production Center 2	7.4	8.8	12.4	16.6	154
Production Center 3	10.8	11.8	6.2	12.8	164
Production Center 4	10.8	7.8	6.4	5.8	174
Dummy Center	0	0	0	0	80
Demand ($\tilde{B}_j$)	174	194	174	164	-

Source: The data used in the numerical illustrations is hypothetical.

Step 5: Solve the TP. Use LINGO 21.0 to solve the TP.

$$\tilde{Z} = \langle(2220,3472,4724),(1748,2374,3000)\rangle$$

The positive membership $\mu^{+}_{\tilde{Z}}(x)$ and negative membership $\mu^{-}_{\tilde{Z}}(x)$ functions of the Bipolar Fuzzy Cost are given as follows:

$$\mu_{\tilde{Z}}^{+}(x) = \begin{cases} \frac{x-2220}{3472-2220}, & \text{if } 2220 \le x \le 3472 \\ \frac{4724-x}{4724-3472}, & \text{if } 3472 \le x \le 4724 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{Z}}^{-}(x) = \begin{cases} \frac{1748-x}{2374-1748}, & \text{if } 1748 \le x \le 2374 \\ \frac{x-3000}{3000-2374}, & \text{if } 2374 \le x \le 3000 \\ 0, & \text{otherwise} \end{cases}$$

9. Conclusions

In this paper, the shortcomings of the ranking function introduced by Akram and Arshad (2019) are discussed using various cases. Accordingly, a new ranking function is proposed and its properties are analysed. The efficiency of our proposed ranking function is demonstrated by comparing arbitrary BFNs. The proposed ranking principle shows promising results for decision-making problems in the BF environment. The conclusion is that the proposed approach provides a better ranking function as compared to the existing ones.

Furthermore, the TP is analysed under the BF environment. Parameters such as transportation costs, supply, and demand are treated as BFNs. With the help of the proposed ranking function approach, BFTP is converted into crisp TP, which is solved using LINGO 21.0 software. Example 6 illustrates the application of TP in a trapezoidal BF environment, while Example 7 demonstrates the solution approach for a triangular BFTP and the solution is also BFN.

9.1 Advantages of Our Proposed Method

This section highlights the main advantages of our proposed method over existing approaches:

- The proposed method considers all transportation parameters as BFNs, which are not considered in existing methods.
- The method effectively handles both triangular and trapezoidal BFNs, expanding its applicability.
- The proposed approach provides a better ranking function compared to existing ranking functions.

9.2 Suggestions for Future Research Directions

Based on our findings, several future research directions can be considered. First, the proposed method could be applied to non-linear membership functions, allowing for more complex problem-solving. Furthermore, utilising the ranking function in models with greater complexity, such as Multi-Objective Transportation Problems within the BF environment, would enhance its practical significance. Finally, extending the application of the ranking function to other methodologies, such as LPP, would further expand its scope and usefulness in decision-making processes.

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