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Stochastic Directional Distance Efficiency and Scale Elasticity of Two-Stage Processes with Undesirable Outputs

Abstract. *The aim of this study is to provide approaches to analyse the performance of two-stage processes under managerial disposability and to address their scale elasticity where there are random input-output measures and undesirable outputs. For this purpose, the performance of general systems and stages is measured using the proposed chance-constrained distance directional function two-stage DEA approach under managerial disposability. Furthermore, the stochastic directional scale elasticity of two-stage processes is addressed. To accomplish this, a response function is presented, and the right and left scale elasticity of efficient general systems for the examined risk levels are taken into consideration. A dataset from the Iranian banking sector is used to illustrate the suggested models. The results show that the proposed approaches are practical for estimating the efficiency and scale elasticity of two-stage networks in the presence of random measures and undesirable outputs.*

Keywords: *stochastic data envelopment analysis, stochastic scale elasticity, two-stage processes, efficiency, undesirable outputs.*

JEL Classification: C61, C67.

1. Introduction

There are many multi-stage processes in real-world applications with undesirable outputs and random measures in which analysing their performance is essential for planning. Furthermore, describing stochastic technologies by means of

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concepts such as scale elasticity is another significant aspect for decision-makers to make decisions about enlargement or reduction.

Data envelopment analysis (DEA) is a non-parametric method to estimate the performance of decision making units (DMUs) with several inputs and outputs. In the DEA literature, some studies can be found to analyse the efficiency of networks with random measures. Zhou et al. (2017) provided a radial stochastic DEA model to evaluate the efficiency of two-stage networks under centralised control mechanisms with desirable factors. Amirteimoori et al. (2022) also provided an alternative stochastic DEA technique to estimate two-stage processes with desirable random measures. Izadikhah and Farzipoor Saen (2018) developed a stochastic two-stage DEA approach with undesirable data to assess the efficiency of sustainability of supply networks. Izadikhah et al. (2019) extended a chance-constrained enhanced Russell measure approach to determine the efficiency of humanitarian supply networks. Amirteimoori et al. (2021) presented a stochastic network DEA approach to estimate the relative efficiency of networks with reverse flows and random measures. Kord et al. (2022) rendered a network model with weight restrictions when there are stochastic data. Lin and Lu (2023) proposed a chance-constrained network DEA model based on an enhanced Russell-based directional distance measure to assess public sector efficiency. However, there is a scarcity of stochastic directional distance DEA approaches under managerial disposability to estimate the performance and scale elasticity of two-stage networks.

In the DEA literature, scale elasticity of black-box processes with external inputs and outputs has been investigated in some studies such as (Førsund and Hjalmarsson, 2004; Podinovski and Førsund, 2020; Sahoo and Tone, 2015). Zelenyuk (2013) addressed scale elasticities on the basis of a directional distance function for black-box technologies with multiple inputs and outputs. Ren et al. (2021) introduced a DEA methodology in order to tackle directional returns to scale and directional scale elasticity issues while taking into account decision-makers' management preferences. Amirmohammadi et al. (2021) estimated scale elasticities in the presence of integer-valued factors. Also, scale elasticities with undesirable outputs and non-discretionary measures were dealt with in (Amirmohammadi et al., 2021). Sahoo et al. (2014) developed network DEA models to estimate scale elasticities in two-stage processes with deterministic and desirable factors. Azizi et al. (2022) investigated the directional scale elasticities of two-stage networks with weakly disposable outputs and deterministic measures. Amirteimoori et al. (2023) provided a chance-constrained cost-efficiency technique to address a value-based measure of the scale elasticity of systems considered as black-box processes with desirable and random measures.

As the investigation shows, the majority of existing studies analysed the scale elasticity of processes with desirable and deterministic measures, although undesirable outputs and random factors are presented in many applications. There are some approaches to include undesirable outputs such as considering a strong disposability property, weak disposability assumption, and managerial disposability attribute. Under strong disposability, undesirable outputs are deemed as inputs. Under weak disposability, undesirable outputs may be reduced along with a proportional reduction of desirable outputs (Fare et al., 1989). According to managerial disposability, the optimisation of desirable outputs and concomitant mitigation of undesirable outputs is achieved through an increase in inputs (Sueyoshi and Goto, 2012). Considering the regulation change and managerial efforts, this study incorporates undesirable outputs under managerial disposability. The literature review has shown that few studies have investigated the performance and scale elasticity of two-stage processes in the presence of undesirable outputs. Furthermore, there are many situations where the performance and scale elasticity of entities should be addressed in a stochastic environment.

Thus, due to the purpose of this research and to fill the existing gap of the literature, this study develops a stochastic directional distance two-stage DEA model to assess the stochastic efficiency of network systems with managerially disposable undesirable outputs in a stochastic environment. Furthermore, stochastic directional scale elasticities of two-stage processes with undesirable outputs under managerial disposability property are addressed. A real application of the banking sector is presented to clarify the introduced approaches. In general, the contribution of this study is fourfold: first, to evaluate the efficiency of two-stage networks under managerial disposability in the presence of random measures and undesirable outputs; second, to provide a stochastic directional distance DEA model under managerial disposability to estimate stochastic efficiencies of stages and general networks; third, to calculate the scale elasticities of networks with random measures; fourth, to analyse the performance of branches of an Iranian bank.

The rest of this research is structured as follows. Stochastic DEA models are proposed in Section 2 to estimate the stochastic performance and scale elasticities of two-stage systems with undesirable outputs. An application of the banking sector is given in Section 3 to clarify the proposed approaches. Conclusions are provided in Section 4.

2. The estimation of the stochastic efficiency and scale elasticity of two-stage processes

Assume there are N two-stage processes, $DMU_j, j = 1, \dots, N$, with the frame shown in Fig. 1. As can be seen, random inputs $\tilde{x}_{ij} (i = 1, \dots, I)$ are consumed in Stage 1 and the random intermediate measures $\tilde{z}_{qj} (q = 1, \dots, Q)$are produced in Stage 1 and used in Stage 2. Furthermore, random external inputs $\widetilde{m}_{bj}$ $(b = 1, \dots, B)$ are consumed in Stage 2 in addition to $\tilde{z}_{qj}$ that lead to random desirable outputs $\tilde{y}_{tj} (t = 1, \dots, T)$ and undesirable outputs $\widetilde{w}_{fj} (f = 1, \dots, F)$. Notice that $\tilde{z}_{qj}$and $\widetilde{m}_{bj}$ are increased due to the objectives of this study. To more illustrate, $\widetilde{m}_{bj}$ are considered as random inputs with managerial disposability.

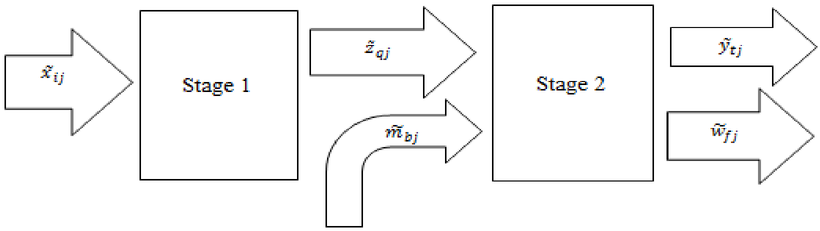

Figure 1. The frame under consideration
Source: Created by author.

In the next subsection, a chance-constrained two-stage DEA approach is proposed to measure the efficiency of systems with the frame shown in Figure 1.

2.1 Stochastic technical efficiency analysis of two-stage processes

Considering notations mentioned, we have the principles of observations inclusion, convexity, free disposability, minimum intersection. The PPS is the intersection set of all technologies satisfying the above-mentioned properties. Thus, by regarding $\tilde{X}, \tilde{Z}, \widetilde{M}, \tilde{Y}$ and $\widetilde{W}$ as matrices of performance measures, the technology is defined as follows:

$$T = \{(\tilde{x}, \tilde{z}, \widetilde{m}, \tilde{y}, \widetilde{w}) | \tilde{X}\lambda \leq \tilde{x}, \tilde{z} \leq \tilde{Z}\lambda, \tilde{z} \leq \tilde{Z}\mu, \widetilde{m} \leq \widetilde{M}\mu, \tilde{y} \leq \tilde{Y}\mu, \widetilde{W}\mu \leq \widetilde{w}, e^T\lambda = 1, \lambda \geq 0, e^T\mu = 1, \mu \geq 0\}. \quad (1)$$

In order to estimate the stochastic efficiency of two-stage processes under examination, the subsequent directional distance function is defined:

$$\vec{D}\left((\widetilde{x_{\iota J}}, \widetilde{z_{qJ}}, \widetilde{m_{bJ}}, \widetilde{y_{tJ}}, \widetilde{w_{fJ}}), g\right) = \mathrm{Sup}\left\{\beta + \alpha : \left(\widetilde{x_{\iota J}} - \beta g_{xi}, \widetilde{z_{qJ}}, \widetilde{m_{bJ}} + \alpha g_{mb}, \widetilde{y_{tJ}} + \alpha g_{yt}, \widetilde{w_{fJ}} - \alpha g_{wf}\right) \in T\right\}$$

Accordingly, we introduce the following directional distance chance-constrained DEA model:

$$\begin{aligned}
& Max \ \ \beta + \alpha \\
& s.t. \ p\left[\textstyle\sum_{j=1}^{N} \lambda_j \widetilde{x_{\iota J}} \leq \widetilde{x_{\iota o}}(1 - \beta)\right] \geq 1 - \theta, && i = 1,2, \dots, I, \quad (2) \\
& \quad p\left[\sum_{j=1}^{N} \lambda_j \widetilde{z_{qJ}} \geq \widetilde{z_{qo}}\right] \geq 1 - \theta, && q = 1,2, \dots, Q, \\
& \quad \textstyle\sum_{j=1}^{N} \lambda_j = 1, \qquad \lambda_j \geq 0,
\end{aligned}$$

$$p\left[\sum_{j=1}^{N}\mu_j\,\widetilde{z_{qj}} \geq \widetilde{z_{qo}}\right] \geq 1-\theta, \qquad q = 1,2,\dots,Q,$$

$$p\left[\sum_{j=1}^{N}\mu_j\,\widetilde{m_{bj}} \geq \widetilde{m_{bo}}\right] \geq 1-\theta, \qquad b = 1,\dots,B,$$

$$p\left[\sum_{t=1}^{N}\mu_j\,\widetilde{y_{tj}} \geq \widetilde{y_{to}}(1+\alpha)\right] \geq 1-\theta, \qquad t = 1,\dots,T,$$

$$p\left[\sum_{f=1}^{N}\mu_j\,\widetilde{w_{fj}} \leq \widetilde{w_{fo}}(1-\alpha)\right] \geq 1-\theta, \qquad f = 1,\dots,F,$$

$$\sum_{j=1}^{N}\mu_j = 1, \qquad \mu_j \geq 0,$$

that p shows probability and $\theta\epsilon[0,1]$ is a value that indicates allowable risk describing the direction of decision-maker. To evaluate the stochastic efficiency of two-stage processes and compute model (2), it can be transformed into a deterministic plan.

In this part, using the linearisation method of Cooper et al. (1998), according to the data structure, the problem with possibilistic constraints is converted into a linear deterministic form. Consider

$$\widetilde{x_{ij}} = x_{ij} + a_{ij}\epsilon_{ij},$$
$$\widetilde{z_{qj}} = z_{qj} + c_{qj}\varsigma_{qj},$$
$$\widetilde{m_{bj}} = m_{bo} + d_{bj}\tau_{bj},$$
$$\widetilde{y_{tj}} = y_{tj} + e_{tj}\eta_{tj},$$
$$\widetilde{w_{fj}} = w_{fj} + h_{fj}\psi_{fj},$$

in which a_{ij}, c_{qj}, d_{bj}, e_{tj} and h_{fj} are non-negative and ϵ_{ij}, ς_{qj}, τ_{bj}, η_{tj} and $\psi_{fj}$are independent random variables with normal distribution such that $\epsilon_{ij} \sim N(0,\bar{\sigma}^2)$, $\varsigma_{qj} \sim N(0,\bar{\sigma}^2)$, $\tau_{bj} \sim N(0,\bar{\sigma}^2)$, $\eta_{tj} \sim N(0,\bar{\sigma}^2)$ and $\psi_{fj} \sim N(0,\bar{\sigma}^2)$. The random variables ϵ_{ij}, ς_{qj}, τ_{bj}, η_{tj} and ψ_{fj} are the errors of the input, intermediate and output random variables, which are called symmetric structure due to the symmetry of the normal distribution. It follows from the above relationships:

$$\epsilon_{ij} \sim N\left(x_{ij},\bar{\sigma}^2{a_{ij}}^2\right),$$
$$\varsigma_{qj} \sim N\left(z_{qj},\bar{\sigma}^2{c_{qj}}^2\right),$$

$\tau_{pj} \sim N\left(m_{bo}, \bar{\sigma}^2 {c_{pj}}^2\right)$,
$\eta_{tj} \sim N\left(y_{tj}, \bar{\sigma}^2 {b_{tj}}^2\right)$,
$\psi_{fj} \sim N\left(w_{fj}, \bar{\sigma}^2 {b_{fj}}^2\right)$.

Thus, we have:

$$\begin{aligned}
&\text{Max } \beta + \alpha \\
&s.t.\ \textstyle\sum_{j=1}^{N} \lambda_j x_{ij} - \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \lambda_j a_{ij} - (1-\beta)a_{io}\right| \le (1-\beta)x_{io}, i = 1,2, \dots I, \quad (3) \\
&\textstyle\sum_{j=1}^{N} \lambda_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \lambda_j c_{qj} - c_{qo}\right| \ge z_{qo}, \qquad q = 1,2, \dots Q, \\
&\textstyle\sum_{j=1}^{N} \lambda_j = 1, \quad \lambda_j \ge 0, \\
&\textstyle\sum_{j=1}^{N} \mu_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \mu_j c_{qj} - c_{qo}\right| \ge z_{qo}, \quad q = 1,2, \dots Q, \\
&\textstyle\sum_{j=1}^{N} \mu_j m_{bj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \mu_j d_{bj} - d_{bo}\right| \ge m_{bo}, \ b = 1,2, \dots B, \\
&\textstyle\sum_{j=1}^{N} \mu_j y_{tj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \mu_j e_{tj} - (1+\alpha)e_{to}\right| \ge (1+\alpha)y_{to}, \ t = 1,2, \dots T, \\
&\textstyle\sum_{j=1}^{N} \mu_j w_{fj} - \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N} \mu_j h_{fj} - (1-\alpha)h_{fo}\right| \le (1-\alpha)\mathrm{w}_{fo}, f = 1,2, \dots F, \\
&\textstyle\sum_{j=1}^{N} \mu_j = 1, \qquad \mu_j \ge 0.
\end{aligned}$$

To compute model (3), the goal programming theory introduced by Charnes and Cooper (1961, 1977) is applied, therefore

$$\begin{gathered}
\left|\sum_{j=1}^{N} \lambda_j a_{ij} - (1-\beta)a_{io}\right| = u_i^{1+} + u_i^{1-}, i = 1,2, \dots I, \\
\sum_{j=1}^{N} \lambda_j a_{ij} - (1-\beta)a_{io} = u_i^{1+} - u_i^{1-}, i = 1,2, \dots I, \\
u_i^{1+}.u_i^{1-} = 0, i = 1,2, \dots I, \\
\left|\sum_{j=1}^{N} \lambda_j c_{qj} - c_{qo}\right| = u_q^{2+} + u_q^{2-}, q = 1,2, \dots Q, \\
\sum_{j=1}^{N} \lambda_j c_{qj} - c_{qo} = u_q^{2+} - u_q^{2-}, q = 1,2, \dots Q, \\
u_q^{2+}.u_q^{2-} = 0, q = 1,2, \dots Q, \\
\left|\sum_{j=1}^{N} \mu_j c_{qj} - c_{qo}\right| = u_q^{3+} + u_q^{3-}, q = 1,2, \dots Q, \\
\sum_{j=1}^{N} \mu_j c_{qj} - c_{qo} = u_q^{3+} - u_q^{3-}, q = 1,2, \dots Q, \\
u_q^{3+}.u_q^{3-} = 0, q = 1,2, \dots Q,
\end{gathered}$$

$$\left|\sum_{j=1}^{N}\mu_j\, d_{bj} - d_{bo}\right| = u_b^{4+} + u_b^{4-}, b = 1,2, \ldots B,$$
$$\sum_{j=1}^{N}\mu_j\, d_{bj} - d_{bo} = u_b^{4+} - u_b^{4-}, b = 1,2, \ldots B,$$
$$u_b^{4+}.u_b^{4-} = 0, b = 1,2, \ldots B,$$
$$\left|\sum_{j=1}^{N}\mu_j\, e_{tj} - (1+\alpha)e_{to}\right| = u_t^{5+} + u_t^{5-}, t = 1,2, \ldots T,$$
$$\sum_{j=1}^{N}\mu_j\, e_{tj} - (1+\alpha)e_{to} = u_t^{5+} - u_t^{5-}, t = 1,2, \ldots T,$$
$$u_t^{5+}.u_t^{5-} = 0, t = 1,2, \ldots T,$$
$$\left|\sum_{j=1}^{N}\mu_j\, h_{fj} - (1-\alpha)h_{fo}\right| = u_f^{6+} + u_f^{6-}, f = 1,2, \ldots F,$$
$$\sum_{j=1}^{N}\mu_j\, h_{fj} - (1-\alpha)h_{fo} = u_f^{6+} - u_f^{6-}, f = 1,2, \ldots F,$$
$$u_f^{6+}.u_f^{6-} = 0, f = 1,2, \ldots F,$$

Accordingly, model (3) is transformed to the next one while $u_i^{1+}.u_i^{1-} = 0, i = 1,2, \ldots I, u_q^{2+}.u_q^{2-} = 0, q = 1,2, \ldots Q, u_q^{3+}.u_q^{3-} = 0, q = 1,2, \ldots Q, u_b^{4+}.u_b^{4-} = 0, b = 1,2, \ldots B, u_t^{5+}.u_t^{5-} = 0, t = 1,2, \ldots T,$ and $u_f^{6+}.u_f^{6-} = 0, f = 1,2, \ldots, F$ have been ignored:

$IE_0^* =$Max $\beta + \alpha$
$s.t.\ \sum_{j=1}^{N}\lambda_j\, x_{ij} - \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_i^{1+} + u_i^{1-}) \le (1-\beta)x_{io}, \qquad i = 1,2, \ldots I,$ (4)

$$\sum_{j=1}^{N}\lambda_j\, a_{ij} - (1-\beta)a_{io} = u_i^{1+} - u_i^{1-}, i = 1,2, \ldots I,$$

$$\sum_{j=1}^{N}\lambda_j\, z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_q^{2+} + u_q^{2-}) \ge z_{qo}, \qquad q = 1,2, \ldots Q,$$

$$\sum_{j=1}^{N}\lambda_j\, c_{qj} - c_{qo} = u_q^{2+} - u_q^{2-}, q = 1,2, \ldots Q,$$

$$\sum_{j=1}^{N}\lambda_j = 1, \qquad \lambda_j \ge 0,$$

$$\sum_{j=1}^{N}\mu_j\, z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_q^{3+} + u_q^{3-}) \ge z_{qo}, \qquad q = 1,2, \ldots Q,$$

$$\sum_{j=1}^{N}\mu_j\, c_{qj} - c_{qo} = u_q^{3+} - u_q^{3-}, q = 1,2, \ldots Q,$$

$$\sum_{j=1}^{N} \mu_j\, m_{bj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_b^{4+} + u_b^{4-}) \geq m_{bo}, \qquad b = 1,2, \dots B,$$

$$\sum_{j=1}^{N} \mu_j\, d_{bj} - d_{bo} = u_b^{4+} - u_b^{4-}, b = 1,2, \dots B,$$

$$\sum_{j=1}^{N} \mu_j\, y_{tj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_t^{5+} + u_t^{5-}) \geq (1+\alpha) y_{to}, \qquad t = 1,2, \dots T,$$

$$\sum_{j=1}^{N} \mu_j\, e_{tj} - (1+\alpha) e_{to} = u_t^{5+} - u_t^{5-}, t = 1,2, \dots T,$$

$$\sum_{j=1}^{N} \mu_j\, w_{fj} - \Phi^{-1}(\Theta)\sigma(\varepsilon)(u_f^{6+} + u_f^{6-}) \leq (1-\alpha) \mathrm{w}_{fo}, \qquad f = 1,2, \dots F,$$

$$\sum_{j=1}^{N} \mu_j\, h_{fj} - (1-\alpha) h_{fo} = u_f^{6+} - u_f^{6-}, f = 1,2, \dots F,$$

$$\sum_{j=1}^{N} \mu_j = 1,$$

$$\mu_j \geq 0, u_i^{1+}, u_i^{1-}, u_q^{2+}, u_q^{2-}, u_q^{3+}, u_q^{3-}, u_b^{4+}, u_b^{4-}, u_t^{5+}, u_t^{5-}, u_f^{6+}, u_f^{6-} \geq 0.$$

In model (4), the optimal objective function value IE_0^* shows the overall inefficiency for the risk level θ. The network under examination is called stochastic overall efficient if $E_0^* = 0$ for the risk level θ. Otherwise, it is stochastic overall inefficient for the risk level θ. α and β also indicate the stochastic inefficiency values of Stage 1 and Stage 2, respectively. For the risk level θ, the unit under examination is called stochastic efficient in Stage 1 (Stage 2) if the optimal value β^* (α^*) equals zero. Otherwise, it is inefficient.

The optimal values $1 - \beta^*$ and $1 - \alpha^*$are defined as the efficiency scores of stages 1 and 2 for the risk level θ. Furthermore, the stochastic overall efficiency is defined in the following way for the risk level θ:

$$EO^* = \frac{(1-\beta^*) + (1-\alpha^*)}{2} \tag{5}$$

Accordingly, the stochastic (overall and stage) efficiency of two-stage networks with random measures and undesirable outputs is measured.

In the next section, determining the stochastic scale elasticity of two-stage processes is addressed.

2.2 Stochastic scale elasticity analysis of two-stage processes

In this section, the maximum increase of desirable outputs of Stage 2 is estimated for expanding the inputs of Stage 1 and undesirable outputs of Stage 2. Consequently, considering the technology (1), the response function can be defined as follows:

$$
\begin{aligned}
&\beta'(\alpha') = \text{Max } \vartheta \\
&s.t.\ p\left[\sum_{j=1}^{N}\lambda_j \widetilde{x_{ij}} \le \widetilde{x_{io}}(\alpha')\right] \ge 1-\theta, && i = 1,2,\dots,I, \qquad (6)\\
&p\left[\sum_{j=1}^{N}\lambda_j \widetilde{z_{qj}} \ge \widetilde{z_{qo}}\right] \ge 1-\theta, && q = 1,2,\dots Q,\\
&\sum_{j=1}^{N}\lambda_j = 1, \qquad \lambda_j \ge 0,\\
&p\left[\sum_{j=1}^{N}\mu_j \widetilde{z_{qj}} \ge \widetilde{z_{qo}}\right] \ge 1-\theta, && q = 1,2,\dots Q,\\
&p\left[\sum_{j=1}^{N}\mu_j \widetilde{m_{bj}} \ge \widetilde{m_{bo}}\right] \ge 1-\theta, && b = 1,\dots B,\\
&p\left[\sum_{j=1}^{N}\mu_j \widetilde{y_{tj}} \ge \widetilde{y_{to}}(\vartheta)\right] \ge 1-\theta, && \mathrm{t} = 1,\dots,\mathrm{T},\\
&p\left[\sum_{j=1}^{N}\mu_j \widetilde{w_{fj}} \le \widetilde{w_{fo}}(\alpha')\right] \ge 1-\theta, && f = 1,\dots F,\\
&\sum_{j=1}^{N}\mu_j = 1, \mu_j \ge 0.
\end{aligned}
$$

In the same way illustrated in Subsection 2.1, model (6) can be transformed into the following model:

$$
\begin{aligned}
&\beta'(\alpha') = Max\ \vartheta \\
&s.t.\ \textstyle\sum_{j=1}^{N}\lambda_j x_{ij} - \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\lambda_j a_{ij} - \alpha' a_{io}\right| \le \alpha' x_{io}, && i = 1,2,\dots,I, \qquad (7)\\
&\textstyle\sum_{j=1}^{N}\lambda_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\lambda_j c_{qj} - c_{qo}\right| \ge z_{qo}, q = 1,2,\dots,Q,\\
&\textstyle\sum_{j=1}^{N}\lambda_j = 1, \qquad \lambda_j \ge 0,\\
&\textstyle\sum_{j=1}^{N}\mu_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\mu_j c_{qj} - c_{qo}\right| \ge z_{qo},\ q = 1,2,\dots,Q,\\
&\textstyle\sum_{j=1}^{N}\mu_j m_{bj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\mu_j d_{bj} - d_{bo}\right| \ge \alpha' m_{bo},\ b = 1,2,\dots,B,\\
&\textstyle\sum_{j=1}^{N}\mu_j y_{tj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\mu_j e_{tj} - \vartheta e_{to}\right| \ge \vartheta y_{to},\ \ t = 1,2,\dots,T,\\
&\textstyle\sum_{j=1}^{N}\mu_j w_{fj} - \Phi^{-1}(\Theta)\sigma(\varepsilon)\left|\sum_{j=1}^{N}\mu_j h_{fj} - \alpha' h_{fo}\right| \le \alpha' \mathrm{w}_{fo}, f = 1,2,\dots,F,\\
&\textstyle\sum_{j=1}^{N}\mu_j = 1, \mu_j \ge 0.
\end{aligned}
$$

Now, the goal programming theory introduced by Charnes and Cooper (1961, 1977) is applied, thus, we have:

$\beta'(\alpha') = Max\ \vartheta$

$$s.t.\ \sum_{j=1}^{N} \lambda_j x_{ij} - \Phi^{-1}(\Theta)\sigma(\varepsilon)(v_i^{+} + v_i^{-}) \le \alpha' x_{io}, \qquad i = 1,2, \dots I, \tag{8}$$

$$\sum_{j=1}^{N} \lambda_j a_{ij} - \alpha' a_{io} = v_i^{+} - v_i^{-}, i = 1,2, \dots I,$$

$$\sum_{j=1}^{N} \lambda_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left(v_q^{1+} + v_q^{1-}\right) \ge z_{qo}, \qquad q = 1,2, \dots Q,$$

$$\sum_{j=1}^{N} \lambda_j c_{qj} - c_{qo} = v_q^{1+} - v_q^{1-}, q = 1,2, \dots, Q,$$

$$\sum_{j=1}^{N} \lambda_j = 1, \qquad \lambda_j \ge 0,$$

$$\sum_{j=1}^{N} \mu_j z_{qj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)\left(v_q^{2+} + v_q^{2-}\right) \ge z_{qo}, \qquad q = 1,2, \dots Q,$$

$$\sum_{j=1}^{N} \mu_j c_{qj} - c_{qo} = v_q^{2+} - v_q^{2-}, q = 1,2, \dots, Q,$$

$$\sum_{j=1}^{N} \mu_j m_{bj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(v_b^{3+} + v_b^{3-}) \ge m_{bo}, \qquad b = 1,2, \dots B,$$

$$\sum_{j=1}^{N} \mu_j d_{bj} - d_{bo} = v_b^{3+} - v_b^{3-}, b = 1,2, \dots, B,$$

$$\sum_{j=1}^{N} \mu_j y_{tj} + \Phi^{-1}(\Theta)\sigma(\varepsilon)(v_t^{4+} + v_t^{4-}) \ge \vartheta y_{to}, \qquad t = 1,2, \dots T,$$

$$\sum_{j=1}^{N} \mu_j e_{tj} - \vartheta e_{to} = v_t^{4+} - v_t^{4-}, t = 1,2, \dots, T,$$

$$\sum_{j=1}^{N} \mu_j w_{fj} - \Phi^{-1}(\Theta)\sigma(\varepsilon)(v_f^{5+} + v_f^{5-}) \le \alpha' \mathrm{w}_{fo}, \qquad f = 1,2, \dots F,$$

$$\sum_{j=1}^{N} \mu_j h_{fj} - \alpha' h_{fo} = v_f^{5+} - v_f^{5-}, f = 1,2, \dots, F,$$

$$\sum_{j=1}^{N} \mu_j = 1,$$

$$\mu_j \ge 0, v_i^{+}, v_i^{-}, v_q^{1+}, v_q^{1-}, v_q^{2+}, v_q^{2-}, v_b^{3+}, v_b^{3-}, v_t^{4+}, v_t^{4-}, v_f^{5+}, v_f^{5-} \ge 0.$$

The dual of model (8) is as follows:

$$\beta'(\alpha') = Max\ \ \alpha'\left(\sum_{i=1}^{m}(\bar{a}_i\, x_{io} + \bar{a}_i^1 a_{io})\right) - \sum_{q=1}^{Q}\bar{a}_q^2\, z_{qo} + \sum_{q=1}^{Q}\bar{a}_q^3 c_{qo} + \gamma_1 - \sum_{q=1}^{Q}\bar{a}_q^4\, z_{qo}$$
$$+ \sum_{q=1}^{Q}\bar{a}_q^5\, c_{qo} + \left(\sum_{b=1}^{B}(-\bar{a}_b^6\, m_{bo} + \bar{a}_b^7 d_{bo})\right)$$
$$+ \alpha'\left(\sum_{f=1}^{F}(\bar{a}_f^{10}\, w_{fo} + \bar{a}_f^{11} h_{fo})\right) + \gamma_2$$

$$s.t. \sum_{i=1}^{m}(\bar{a}_i\, x_{ij} + \bar{a}_i^1 a_{ij}) + \sum_{q=1}^{Q}(-\bar{a}_q^2\, z_{qj} + \bar{a}_q^3 c_{qj}) + \gamma_1 \geq 0, j = 1, \dots, N, \tag{9}$$

$$\sum_{q=1}^{Q}(-\bar{a}_q^4\, z_{qj} + \bar{a}_q^5 c_{qj}) + \sum_{b=1}^{B}(-\bar{a}_b^6\, m_{bj} + \bar{a}_b^7 d_{bj}) + \sum_{t=1}^{T}(-\bar{a}_t^8\, y_{tj} + \bar{a}_t^9 e_{tj}) + \sum_{f=1}^{F}(\bar{a}_f^{10}\, w_{fj}$$
$$+ \bar{a}_f^{11} h_{fj}) + \gamma_2 \geq 0, j = 1, \dots, N,$$

$$\sum_{t=1}^{T}(\bar{a}_t^8\, y_{tj} - \bar{a}_t^9 e_{tj}) = 1,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_i - \bar{a}_i^1 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_i + \bar{a}_i^1 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^2 - \bar{a}_q^3 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^2 + \bar{a}_q^3 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^4 - \bar{a}_q^5 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^4 + \bar{a}_q^5 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_b^6 - \bar{a}_b^7 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_b^6 + \bar{a}_b^7 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_t^8 - \bar{a}_t^9 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_t^8 + \bar{a}_t^9 \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_f^{10} - \bar{a}_f^{11} \leq 0,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_f^{10} + \bar{a}_f^{11} \leq 0,$$
$$\bar{a}_i, \bar{a}_q^2, \bar{a}_q^4, \bar{a}_b^6, \bar{a}_t^8, \bar{a}_f^{10} \geq 0,$$
$$\bar{a}_i^1, \bar{a}_q^3, \bar{a}_q^5, \bar{a}_b^7, \bar{a}_t^9, \bar{a}_f^{11}, \gamma_1, \gamma_2 : free.$$

The subsequent transformation function is provided for each two-stage network $0 \in \{1, \dots, N\}$.

$$\psi(\alpha' x_o, \alpha' a_o, z_o, c_o, M_o, d_o, \beta'(\alpha') Y_o, \beta'(\alpha') e_o, \alpha' W_o, \alpha' h_o) =$$
$$\beta'(\alpha') \sum_{t=1}^{T}(\bar{a}_t^8\, y_{tj} - \bar{a}_t^9 e_{tj}) - \alpha'\left(\sum_{i=1}^{m}(\bar{a}_i\, x_{io} + \bar{a}_i^1 a_{io})\right) - \gamma_1 -$$
$$\alpha'\left(\sum_{f=1}^{F}(\bar{a}_f^{10}\, w_{fo} + \bar{a}_f^{11} h_{fo})\right) - \gamma_2 = 0, \tag{10}$$

We have the following statement with differentiating (10) respect to α':

$$\frac{\partial\psi}{\partial\alpha'}=\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\alpha' x_{io})}x_{io}+\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\alpha' a_{io})}a_{io}+\sum_{f=1}^{F}\frac{\partial\psi(.)}{\partial(\alpha' w_{fo})}w_{fo}+\sum_{f=1}^{F}\frac{\partial\psi(.)}{\partial(\alpha' h_{fo})}h_{fo}$$
$$+\sum_{t=1}^{T}\frac{\partial\psi(.)}{\partial(\beta' y_{to})}y_{to}\frac{\partial\beta'}{\partial\alpha'}+\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\beta' e_{to})}e_{to}\frac{\partial\beta'}{\partial\alpha'}=0$$

$$\frac{\partial\beta'}{\partial\alpha'}=-\frac{\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\alpha' x_{io})}x_{io}+\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\alpha' a_{io})}a_{io}+\sum_{f=1}^{F}\frac{\partial\psi(.)}{\partial(\alpha' w_{fo})}w_{fo}+\sum_{f=1}^{F}\frac{\partial\psi(.)}{\partial(\alpha' h_{fo})}h_{fo}}{\sum_{t=1}^{T}\frac{\partial\psi(.)}{\partial(\beta' y_{to})}y_{to}+\sum_{i=1}^{m}\frac{\partial\psi(.)}{\partial(\beta' e_{to})}e_{to}}$$
$$=\frac{\sum_{i=1}^{m}(\bar{a}_i\,x_{io}+\bar{a}_i^1 a_{io})+\sum_{f=1}^{F}(\bar{a}_f^{10}\,w_{fo}+\bar{a}_f^{11}h_{fo})}{\sum_{t=1}^{T}(\bar{a}_t^8\,y_{tj}-\bar{a}_t^9 e_{tj})}$$
$$=\frac{\beta'(\alpha')\sum_{t=1}^{T}(\bar{a}_t^8\,y_{tj}-\bar{a}_t^9 e_{tj})-\gamma_1-\gamma_2}{\alpha'}=\frac{\beta'(\alpha')-\gamma_1-\gamma_2}{\alpha'}$$

Thus, the measure of stochastic scale elasticity of the network 0 can be determined in the following way:

$$\varepsilon(\alpha' x_o,\alpha' a_o,z_o,c_o,m_o,d_o,\beta'(\alpha')Y_o,\beta'(\alpha')e_o,\alpha' W_o,\alpha' h_o)=\frac{\partial\beta'(\alpha')}{\partial\alpha'}.\frac{\alpha'}{\beta'(\alpha')}$$
$$=\frac{\beta'(\alpha')-\gamma_1-\gamma_2}{\alpha'}.\frac{\alpha'}{\beta'(\alpha')}=1-\frac{\gamma_1+\gamma_2}{\beta'(\alpha')}$$

For stochastic efficient points, $\beta'(\alpha')=\alpha'=1$, thus, we have:

$$\varepsilon(\alpha' x_o,\alpha' a_o,z_o,c_o,m_o,d_o,\beta'(\alpha')Y_o,\beta'(\alpha')e_o,\alpha' W_o,\alpha' h_o)=1-(\gamma_1+\gamma_2).$$

Notice that the DEA technologies are not smooth at vertices according to (Amirteimoori et al., 2022). Therefore, the maximum (minimum) amount of $\gamma_1+\gamma_2$ is estimated in the following way:

$$\rho'_{max}(\rho'_{min})=Max\ \ (Min)\,\gamma_1+\gamma_2$$
$$s.t.\sum_{i=1}^{m}(\bar{a}_i\,x_{ij}+\bar{a}_i^1 a_{ij})+\sum_{q=1}^{Q}(-\bar{a}_q^2\,z_{qj}+\bar{a}_q^3 c_{qj})+\gamma_1\geq 0, j=1,\dots,N, \tag{11}$$
$$\sum_{q=1}^{Q}(-\bar{a}_q^4\,z_{qj}+\bar{a}_q^5 c_{qj})+\sum_{b=1}^{B}(-\bar{a}_b^6\,m_{bj}+\bar{a}_b^7 d_{bj})+\sum_{t=1}^{T}(-\bar{a}_t^8\,y_{tj}+\bar{a}_t^9 e_{tj})+\sum_{f=1}^{F}(\bar{a}_f^{10}\,w_{fj}$$
$$+\bar{a}_f^{11}h_{fj})+\gamma_2\geq 0, j=1,\dots,N,$$
$$\sum_{t=1}^{T}(\bar{a}_t^8\,y_{to}-\bar{a}_t^9 e_{to})=1,$$
$$\alpha'(\sum_{i=1}^{m}(\bar{a}_i\,x_{io}+\bar{a}_i^1 a_{io}))-\sum_{q=1}^{Q}\bar{a}_q^2\,z_{qo}+\sum_{q=1}^{Q}\bar{a}_q^3 c_{qo}+\gamma_1-\sum_{q=1}^{Q}\bar{a}_q^4\,z_{qo}+$$
$$\sum_{q=1}^{Q}\bar{a}_q^5\,c_{qo}+(\sum_{b=1}^{B}(-\bar{a}_b^6\,m_{bo}+\bar{a}_b^7 d_{bo}))+\alpha'\left(\sum_{f=1}^{F}(\bar{a}_f^{10}\,w_{fo}+\bar{a}_f^{11}h_{fo})\right)+\gamma_2=\beta'(\alpha')$$

$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_i - \bar{a}_i^1 \le 0, \forall i,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_i + \bar{a}_i^1 \le 0, \forall i,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^2 - \bar{a}_q^3 \le 0, \forall q,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^2 + \bar{a}_q^3 \le 0, \forall q,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^4 - \bar{a}_q^5 \le 0, \forall q,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_q^4 + \bar{a}_q^5 \le 0, \forall q,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_b^6 - \bar{a}_b^7 \le 0, \forall b,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_b^6 + \bar{a}_b^7 \le 0, \forall b,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_t^8 - \bar{a}_t^9 \le 0, \forall t,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_t^8 + \bar{a}_t^9 \le 0, \forall t,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_f^{10} - \bar{a}_f^{11} \le 0, \forall f,$$
$$-\Phi^{-1}(\Theta)\sigma(\varepsilon)\bar{a}_f^{10} + \bar{a}_f^{11} \le 0, \forall f,$$
$$\bar{a}_i, \bar{a}_q^2, \bar{a}_q^4, \bar{a}_b^6, \bar{a}_t^8, \bar{a}_f^{10} \ge 0, \forall i, q, b, t, f,$$
$$\bar{a}_i^1, \bar{a}_q^3, \bar{a}_q^5, \bar{a}_b^7, \bar{a}_t^9, \bar{a}_f^{11}, \forall i, q, b, t, f, \gamma_1, \gamma_2 : free.$$

Accordingly, the right- and left-hand scale elasticity of the network 0 can be computed as follows:

$$\varepsilon_o^+ = 1 - \rho'_{max}, \ \varepsilon_o^- = 1 - \rho'_{min.} \tag{12}$$

For stages 1 and 2, they can be estimated in the following ways:

$$\varepsilon_1^+ = 1 - \gamma^*_{1\,max}, \ \varepsilon_1^- = 1 - \gamma^*_{1\,min,} \tag{13}$$

$$\varepsilon_2^+ = 1 - \gamma^*_{2\,max}, \ \varepsilon_2^- = 1 - \gamma^*_{2min.} \tag{14}$$

Notice that for inefficient networks, the scale elasticity should be determined for the target points.

In the next section, the proposed approaches are applied to analyse the performance of the Iranian banking sector.

3. An application to Iranian banking sector

This section endeavours to assess the efficacy of the presented methodologies in examining the performance of 21 branches belonging to an Iranian financial institution. In the context of Iran as a developing nation, the banking industry assumes a crucial function. The banking system is perceived as a fundamental pillar of an economy. Thus, it is imperative to undertake a performance assessment of the banking sector with the intent of securing financial stability in an economy. Figure 2 shows the structure of the two-stage networks under examination. Deposits are intermediate measures that increase in both stages. Furthermore, operational costs are treated as inputs of the loan system that, according to managerial disposability, are increased. They relate to the years 2021-2022. It is postulated that all stochastic variables follow a normal distribution. To analyse the performance of branches, model (4) and the expression (5) are applied. The overall and stage efficiencies are estimated so that the results are shown in Table 1. Notice that $1-\beta$ and $1-\alpha$ are shown the efficiency values related to stages 1 and 2, respectively, in which β and α are achieved from model (4). Furthermore, stochastic efficiencies are estimated for the risk levels $\theta = 0.01, 0.3$ and 0.5. As can be seen in Table 3, 9 branches are determined as efficient in Stage 1 for the risk levels 0.01 and 0.3. But, this amount reaches 5 for the risk level 0.5. Also, in Stage 2, 16, 15 and 7 branches are determined as stochastic efficient for the risk levels 0.01, 0.3 and 0.5, respectively. As a general network system, also, 7, 7 and 3 branches are obtained as efficient for the risk levels 0.01, 0.3 and 0.5. Actually, the number of branches does not increase as the risk level increases. Furthermore, branch 11 is the most inefficient branch in Stage 1 for all the risk levels examined. At Stage 2 and general network, branch 14 is ascertained as the most inefficient one for the risk levels under consideration.

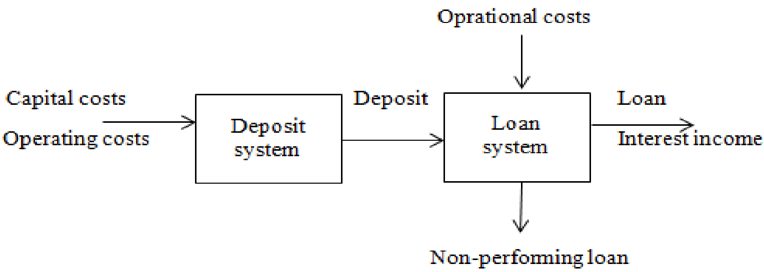

Figure 2. Banking structure

Source: Created by author.

Table 1. Efficiency values

	Efficiency								
	$\theta = 0.01$			$\theta = 0.3$			$\theta = 0.5$		
	$1-\beta$	$1-\alpha$	**Overall**	$1-\beta$	$1-\alpha$	**Overall**	$1-\beta$	$1-\alpha$	**Overall**
1	0.695	1	0.8475	0.688	1	0.844	0.682	0.519	0.6005
2	0.805	1	0.9025	0.803	1	0.9015	0.792	1	0.896
3	1	1	1	1	1	1	1	1	1
4	0.921	1	0.9605	0.863	1	0.9315	0.699	1	0.8495
5	0.925	0.799	0.862	0.88	0.402	0.641	0.771	0.166	0.4685
6	0.724	1	0.862	0.702	1	0.851	0.685	0.605	0.645
7	1	0.883	0.9415	1	0.85	0.925	1	0.436	0.718
8	0.73	1	0.865	0.705	1	0.8525	0.695	0.706	0.7005
9	0.945	1	0.9725	0.897	1	0.9485	0.831	0.621	0.726
10	0.855	1	0.9275	0.82	1	0.91	0.737	0.938	0.8375
11	0.638	1	0.819	0.63	1	0.815	0.61	1	0.805
12	1	1	1	1	1	1	0.832	1	0.916
13	1	1	1	1	1	1	1	1	1
14	0.789	0.751	0.77	0.779	0.194	0.4865	0.767	0.053	0.41

	Efficiency								
	$\theta = 0.01$			$\theta = 0.3$			$\theta = 0.5$		
	$1-\beta$	$1-\alpha$	**Overall**	$1-\beta$	$1-\alpha$	**Overall**	$1-\beta$	$1-\alpha$	**Overall**
15	1	1	1	1	1	1	0.912	0.462	0.687
16	0.699	0.927	0.813	0.691	0.731	0.711	0.646	0.376	0.511
17	0.872	1	0.936	0.863	0.644	0.7535	0.857	0.249	0.553
18	1	1	1	1	1	1	0.709	0.779	0.744
19	1	1	1	1	1	1	0.793	0.952	0.8725
20	1	0.767	0.8835	1	0.407	0.7035	1	0.097	0.5485
21	1	1	1	1	1	1	1	1	1

Source: Created by author.

As can be found, the efficiency values decrease or are without change when the risk levels increase. Also, the response function $\beta'(\alpha')$ is estimated for values $\alpha' = 0.9$ and $\alpha' = 1.02$ and the risk levels 0.01, 0.3 and 0.5. The results can be found in Table 2. As can be seen in column 7 of Table 2, all response functions for $\alpha' = 1.02$ and $\theta = 0.5$ are feasible, which means changing the inputs leads to feasible points in the technology. However, we confront with infeasibility for some branches in other cases. The right- and left-hand scale elasticities of overall stochastic efficient branches are calculated using (11) and (12). Table 3 indicates the right- and left-hand scale elasticity of overall stochastic efficient branches for the risk levels 0.01, 0.3 and 0.5. The right- and left-hand scale elasticity of overall stochastic efficient branches are infeasible for the risk levels 0.01 and 0.3 except branch 21. For instance, in stage 1, we have $\varepsilon^- = 1$ and $\varepsilon^+ = 0$ for the risk level 0.3. It is clear that no change is allowable for desirable outputs of Stage 2 with $\varepsilon^+ = 0$. For the risk level 0.5, model (4) is unbounded. Likewise, an analysis can be conducted on the outcomes of all branches.

Table 2. Results related to the response function

Branches	$\theta = 0.01$, $\alpha' = 0.9$	$\theta = 0.01$, $\alpha' = 1.02$	$\theta = 0.3$, $\alpha' = 0.9$	$\theta = 0.3$, $\alpha' = 1.02$	$\theta = 0.5$, $\alpha' = 0.9$	$\theta = 0.5$, $\alpha' = 1.02$
1	In	1.000	In	1.000	1.964	2.088
2	In	1.000	0.524	1.000	0.941	1.002
3	In	1.000	In	1.000	In	1.058
4	In	1.000	In	1.001	In	1.005
5	In	1.201	1.598	1.598	3.635	3.930
6	In	1.000	In	1.000	1.666	1.742
7	In	1.117	In	1.150	In	1.661
8	In	1.000	In	1.000	1.563	1.729
9	In	1.000	In	1.000	1.728	1.780

Branches	$\theta=0.01$, $\alpha'=0.9$	$\theta=0.01$, $\alpha'=1.02$	$\theta=0.3$, $\alpha'=0.9$	$\theta=0.3$, $\alpha'=1.02$	$\theta=0.5$, $\alpha'=0.9$	$\theta=0.5$, $\alpha'=1.02$
10	In	1.000	0.773	1.000	1.055	1.077
11	In	1.000	In	1.000	0.864	1.001
12	In	1.000	In	1.000	In	1.003
13	In	1.002	In	1.005	In	1.013
14	1.249	1.249	1.986	1.986	5.300	5.449
15	In	In	In	1.000	In	1.983
16	In	1.216	1.618	1.618	2.205	2.223
17	In	In	2.266	2.266	3.246	3.275
18	In	In	In	In	1.412	1.444
19	In	In	In	In	0.968	1.152
20	In	1.233	In	1.593	In	3.245
21	In	In	In	In	In	1.000

In: Infeasible

Source: Created by author

Table 3. The right- and left-hand scale elasticity of efficient branches

Branch	Stage 1					
	0.01	0.01	0.3	0.3	0.5	0.5
	ε^-	ε^+	ε^-	ε^+	ε^-	ε^+
3	In	In	In	In	Un	Un
12	In	In	In	In	-	-
13	In	In	In	In	Un	Un
15	In	In	In	In	-	-
18	In	In	In	In	-	-
19	In	In	In	In	-	-
21	In	0	1	0	Un	Un
Branch	Stag 2					
	0.01	0.01	0.3	0.3	0.5	0.5
	ε^-	ε^+	ε^-	ε^+	ε^-	ε^+
3	In	In	In	In	Un	Un
12	In	In	In	In	-	-
13	In	In	In	In	Un	Un
15	In	In	In	In	-	-
18	In	In	In	In	-	-
19	In	In	In	In	-	-
21	In	1	0.768	1	Un	Un

Branch	Overall					
	0.01	0.01	0.3	0.3	0.5	0.5
	ε^{-}	ε^{+}	ε^{-}	ε^{+}	ε^{-}	ε^{+}
3	In	In	In	In	Un	Un
12	In	In	In	In	-	-
13	In	In	In	In	Un	Un
15	In	In	In	In	-	-
18	In	In	In	In	-	-
19	In	In	In	In	-	-
21	In	0	0.768	0	Un	Un

In: Infeasible, Un: Unbounded

Source: Created by author.

4. Conclusions

The consideration of performance evaluation for two-stage processes that involve undesirable outputs is of paramount significance in a stochastic setting across various practical scenarios. Accordingly, this research aimed to investigate the operational efficiency of two-stage networks, incorporating stochastic input-output measures and undesirable outputs while taking into consideration managerial disposability. Actually, stochastic directional distance DEA models under managerial disposability were proposed to estimate the efficiency of two-stage systems and scale elasticities. Applying the presented approach, stochastic overall and stages efficiencies are measured. A real-world application of Iranian banking was provided to analyse the performance of the suggested approaches.

The findings show that the efficiency scores related to general networks and stages decrease or are without change when the risk levels increase. Furthermore, the directional scale elasticity of two-stage systems with random measures can be evaluated using the presented models.

The proposed techniques can be extended to other network structures. Furthermore, estimating the directional scale elasticities of two-stage processes in a fuzzy stochastic environment is an interesting topic for future investigation.

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