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A Novel Score Function for Spherical Fuzzy Sets and Its Application to Assignment Problem

Abstract. *In our world today, keeping pace with development of technologies and the expansion of non-absolute data, modelling phenomena becomes very hard and ambiguous, which makes it difficult for deciders to take the optimal decision including the assignment problem, for instance. In this case, leadership is ahead of one serious complex problem, inexact and fuzzy. Spherical fuzzy set (SFS) is disposing the indeterminacy data by the membership, the abstinence and the non-membership functions; it is a generalisation of both Pythagorean fuzzy set (PyFS) and picture fuzzy set (PFS). In this study, some results for SFSs are established first. Then; a novel score function of spherical fuzzy numbers (SFN) is proposed to avoid the comparison problem. In addition; an existing approach to the Pythagorean fuzzy assignment problem is extended to the spherical fuzzy assignment problem (SFAP) with the new proposed score function. Finally, a numerical example and a comparative study are executed to explain the method and validate its advantages.*

Keywords: *spherical fuzzy set, spherical fuzzy number, score function, assignment problem, decision making.*

JEL Classification: C02, C11, C45, C46, C63.

1. Introduction

Today, in a world full of competition and daily development, countries, organisations and enterprises work to reduce risk, and therefore seek to be precise in their projects and economise on their resources. Thus, experts must take the correct decision in most situations, including the problem of assignment, such as assigning workers or equipment to tasks; it is a decision which has a huge effect on the performance saving time and efforts and improving the quality and productivity of production. The assignment problem is one of the applications of linear programming; it consists of establishing links between the elements of two distinct sets, so as to optimise certain cost and while respecting link uniqueness constraints

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for each element (Nemhauser and Wolsey, 1988). To express the cost of an assignment, experts take into account many factors which are generally uncertainty and inexact. So, the assignment problem becomes one serious fuzzy issue.

In decision making and in order to dispose of the vague information; Zadeh (1965) presented the fuzzy set (FS), which is signified by a membership function attaching every target one value between 0 and 1. Atanassov (1986) prolonged the FS to the intuitionist fuzzy set (IFS) that is denoted by both membership and non-membership functions such as their sum is less than or equal to 1. In the reality, there exist several cases where the last condition is not satisfied; because experts may present the values of membership and non-membership as 0.7 and 0.5 for example; it is clear that $0.7 + 0.5 = 1.2 > 1$ but $(0.7)^2 + (0.5)^2 = 0.74 \leq 1$. consequently, Yager (2014) generalised the IFS to novel fuzzy set called Pythagorean fuzzy set (PyFS) which is also denoted by the membership and non-membership functions such as the sum of their squares is less than or equal to 1. It is to highlight that PFS is more efficient to treat complex decision making problems thanks to its extensions; aggregation operators and several important algorithms (see Peng & Selvachandran, 2019). The Pythagorean fuzzy environment is applied in many domains, among them; unconventional emergency (Zhan et al., 2020), social network analysis (Wang et al., 2020), physician selection problem (Rani et al., 2020), linear programming problems (Akram et al., 2021), big data (Bechar & Benyettou, 2022), assignment problem (Kumar et al., 2023), climatic analysis (Suber et al., 2023) and recently in football analysis (Li et al., 2024). On the other hand, human nature has some type of desist and rejection subject too, in vote selection, for example, human judgments comprise more answers like yes, no, abstain, and refusal. In such situations, the concepts of IFS and PyFS fail to be applied. Thus, Cuong (2013) introduced the theory of picture fuzzy set (PFS) as an extended form of IFS by adding a third function to the membership and non-membership functions called abstain membership function with condition that their sum does not exceed 1. PFS is one of the richest research part; we find in the literature: Cuong (2014) presented the distance between picture fuzzy numbers (PFN), Cuong et al. (2015) offered the fuzzy logic operators for PFS, Wei (2017) presented the cosine similarity measure for PFSs. Garg (2017), defined the picture fuzzy aggregation, and other contributions in this environment. Sometimes, experts provide their values of membership functions whose sum is greater than 1, and then the PFS fails. Due to this situation, Ashraf et al. (2019) proposed a new structure and defined the spherical fuzzy set (SFS) which is signified by the triplet of the membership, the abstain and the non-membership functions such that the sum of their squares belongs to the interval [0;1]. It is a generalisation of both theories PFS and PyFS. So; SFS is a more effective tool for decision making to deal with uncertainty. For this reason; SFS caught the attention of many researchers; we cite, for example: Ashraf and Abdullah (2019), announced the spherical aggregation operators. Ashraf et al. (2018) presented the GRA method for a spherical fuzzy linguistic set and its applications. Zeng et al. (2019) proposed the covering based spherical fuzzy rough set hybrid model with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) approach. Further, Tehreem et al. (2021) initialised the novel Dombi aggregation operators in spherical cubic fuzzy information, and recently, Javed et al. (2023) introduced the concept of spherical fuzzy neutral aggregation operators.

To represent difference between spherical fuzzy numbers (SFN), it exists some score and accuracy functions. In this study, we present an approach to solve the assignment problem in spherical fuzzy values with new score function.

The rest of this paper is organised as the following: section 2 presents basic notions that are necessary in the other sections. In Section 3, some results for the SFSs are exposed. In Section 4, a novel score function of SFNs is proposed with its properties to overcome some limitations of the score functions defined by Ashraf et al. (2019) and Javed et al. (2023). Section 5 is an apply part in which we solve the SFAP prolonging the Pythagorean model of Kumar et al. (2023) based on the new proposed score function; the presented approach is illustrated with a numerical example before that a comparative study is examined too. The paper ends with a conclusion in Section 6.

2. Preliminaries

In this section, we present diverse basic notions of PFS, PyFS and SFS and certain proprieties that we need in this study.

Definition 1 (2014). A PyFS P in a universe of discourse X is given by

$$P = \{\langle x, u_P(x), v_P(x)\rangle | x \in X\}, \tag{1}$$

where $u_P: X \to [0,1]$ and $v_P: X \to [0,1]$ denote respectively the degree of membership and the degree of non-membership of the element $x \in X$ to the set P, and satisfy the condition that $0 \le (u_P(x))^2 + (v_P(x))^2 \le 1$ for any $x \in X$.

The degree of indeterminacy π_P is $\pi_P(x) = \sqrt{1 - \left(\left(u_P(x)\right)^2 + \left(v_P(x)\right)^2\right)}$.

For convenience, the pair $(u_P(x), v_P(x))$ is called a Pythagorean fuzzy number (PyFN) denoted by $p = (u_P, v_P)$.

Definition 2 (2013). A PFS C in a universe of discourse X is given by

$$C = \{\langle x, u_C(x), v_C(x), w_C(x)\rangle | x \in X\}, \tag{2}$$

where $u_C: X \to [0,1], v_C: X \to [0,1]$ and $w_C: X \to [0,1]$ denote respectively the degree of membership, the degree of abstinence and the degree of non-membership of the element $x \in X$ to the set C, and satisfy the condition that $0 \le u_C(x) + v_C(x) + w_C(x) \le 1$ for any $x \in X$.

The refusal degree of x is $\pi_C(x) = 1 - (u_C(x) + v_C(x) + w_C(x))$.

For convenience, the triplet $(u_C(x), v_C(x), w_C(x))$ is called a picture fuzzy number (PFN) denoted by $c = (u_c, v_c, w_c)$.

Definition 3 (2019). A SFS S in a universe of discourse X is given by

$$S = \{\langle x, u_S(x), v_S(x), w_S(x)\rangle | x \in X\}, \tag{3}$$

where $u_S: X \to [0,1], v_S: X \to [0,1]$ and $w_S: X \to [0,1]$ denote respectively the degree of membership, the degree of abstinence and the degree of non-membership of the element $x \in X$ to the set S, and satisfy the condition that $0 \leq (u_P(x))^2 + (v_P(x))^2 + (w_P(x))^2 \leq 1$ for any $x \in X$.
The refusal degree of x is $\pi_S(x) = \sqrt{1 - \left(\left(u_S(x)\right)^2 + \left(v_S(x)\right)^2 + \left(w_S(x)\right)^2\right)}$.

For convenience, the triplet $(u_S(x), v_S(x), w_S(x))$ is called a spherical fuzzy number (SFN) denoted by $s = (u_S, v_S, w_S)$.

Definition 4 (2019). For any two SFSs $\alpha = \{\langle x, u_\alpha(x), v_\alpha(x), w_\alpha(x)\rangle | x \in X\}$ and $\beta = \left\{\langle x, u_\beta(x), v_\beta(x), w_\beta(x)\rangle | x \in X\right\}$, the operations are defined as the followings:

- $\alpha^c = \{\langle x, w_\alpha(x), v_\alpha(x), u_\alpha(x)\rangle | x \in X\}$; (4)
- $\alpha \subseteq \beta \iff \begin{cases} u_\alpha(x) \leq u_\beta(x) \\ v_\alpha(x) \leq v_\beta(x) \\ w_\beta(x) \leq w_\alpha(x) \end{cases}$; (5)
- $\alpha = \beta \iff \begin{cases} u_\alpha(x) = u_\beta(x) \\ v_\beta(x) = v_\alpha(x) \\ w_\beta(x) = w_\alpha(x) \end{cases}$; (6)
- $\alpha \cap \beta = \left\{\langle x, \min(u_\alpha(x), u_\beta(x)), \min(v_\alpha(x), v_\alpha(x)), \max(w_\alpha(x), w_\alpha(x)) \rangle | x \in X\right\}$; (7)
- $\alpha \cup \beta = \left\{\langle x, \max(u_\alpha(x), u_\beta(x)), \min(v_\alpha(x), v_\alpha(x)), \min(w_\alpha(x), w_\alpha(x))\rangle | x \in X\right\}$; (8)
- $\alpha \oplus \beta = \left\{\langle x, \sqrt{(u_\alpha(x))^2 + \left(u_\beta(x)\right)^2 - \left(u_\alpha(x)\right)^2 \left(u_\beta(x)\right)^2}, v_\alpha(x)v_\beta(x), w_\beta(x)w_\alpha(x) \rangle | x \in X\right\}$; (9)
- $\alpha \otimes \beta = \left\{\langle x, u_\alpha(x)u_\beta(x), v_\alpha(x)v_\beta(x), \sqrt{(w_\alpha(x))^2 + \left(w_\beta(x)\right)^2 - \left(w_\alpha(x)\right)^2 \left(w_\beta(x)\right)^2} \rangle | x \in X\right\}$; (10)
- $m\alpha = \left\{\langle x, \sqrt{1 - (1 - (u_\alpha(x))^2)^m}, (v_\alpha(x))^m, (w_\alpha(x))^m\rangle | x \in X\right\}$, for all $m > 0$; (11)

- $\alpha^m = \left\{\langle x, (u_\alpha(x))^m, (v_\alpha(x))^m, \sqrt{1-(1-(w_\alpha(x))^2)^m} \rangle | x \in X\right\}$, for all $m > 0$. (12)

Definition 5 (2019). The score function of any SFN $s = (u_S, v_S, w_S)$ is defined as the following formula:

$$SC(s) = \frac{1}{3}(2 + u_S - v_S - w_S), \tag{13}$$

where $SC(s) \in [0, 1]$.

Definition 6 (2019). The accuracy function of any SFN $s = (u_S, v_S, w_S)$ is defined as the following formula:

$$AC(s) = u_S - w_S, \tag{14}$$

where $AC(s) \in [-1, 1]$.

For any two SFNs k and l,

1) if $(k) > SC(l)$, then $k > l$;
2) if $SC(k) = SC(l)$, then
   a) if $AC(k) > AC(l)$, then $k > l$;
   b) if $AC(k) = AC(l)$, then $k \sim l$.

3. Results for SFSs

In this section, we present some proprieties as particular cases of the SFSs with its proofs.

Theorem 1. Let $\alpha = \{\langle x, u_\alpha(x), v_\alpha(x), w_\alpha(x)\rangle | x \in X\}$ and $\beta = \left\{\langle x, u_\beta(x), v_\beta(x), w_\beta(x)\rangle | x \in X\right\}$ be two SFSs in domain X. So;

1) If $u_\alpha(x) = w_\alpha(x)$ and $u_\beta(x) = w_\beta(x)$, then $\alpha \oplus \beta = (\alpha \otimes \beta)^c$;

2) If $u_\alpha(x) = w_\alpha(x)$ and $u_\beta(x) = w_\beta(x)$, then $\alpha \otimes \beta = (\alpha \oplus \beta)^c$.

Proof: Let $\alpha = \{\langle x, u_\alpha(x), v_\alpha(x), w_\alpha(x)\rangle | x \in X\}$ and $\beta = \left\{\langle x, u_\beta(x), v_\beta(x), w_\beta(x)\rangle | x \in X\right\}$ be two SFSs, if $\begin{cases} u_\alpha(x) = w_\alpha(x) \\ u_\beta(x) = w_\beta(x) \end{cases}$, we simply have the following equalities:

$$\begin{cases} \sqrt{(u_\alpha(x))^2 + \left(u_\beta(x)\right)^2 - (u_\alpha(x))^2\left(u_\beta(x)\right)^2} = \sqrt{(w_\alpha(x))^2 + \left(w_\beta(x)\right)^2 - (w_\alpha(x))^2\left(w_\beta(x)\right)^2} \\ w_\beta(x)w_\alpha(x) = u_\alpha(x)u_\beta(x) \end{cases}$$

So, and according with (9), (4) and (10) we can obtain:

$$\alpha \oplus \beta = \left\{\langle x, \sqrt{(u_\alpha(x))^2 + \left(u_\beta(x)\right)^2 - \left(u_\alpha(x)\right)^2\left(u_\beta(x)\right)^2}, v_\alpha(x)v_\beta(x), w_\beta(x)w_\alpha(x)\rangle \mid x \in X\right\} =$$

$$\left\{\langle x, \sqrt{(w_\alpha(x))^2 + \left(w_\beta(x)\right)^2 - \left(w_\alpha(x)\right)^2\left(w_\beta(x)\right)^2}, v_\alpha(x)v_\beta(x), u_\alpha(x)u_\beta(x)\rangle \mid x \in X\right\} =$$

$$\left\{\langle x, u_\alpha(x)u_\beta(x), v_\alpha(x)v_\beta(x), \sqrt{(w_\alpha(x))^2 + \left(w_\beta(x)\right)^2 - \left(w_\alpha(x)\right)^2\left(w_\beta(x)\right)^2}\rangle \mid x \in X\right\}^c = (\alpha \otimes \beta)^c.$$

2) It can be proved in a similar way.

Theorem 2. Let $\alpha = \{\langle x, u_\alpha(x), v_\alpha(x), w_\alpha(x)\rangle | x \in X\}$ and m be, respectively, a SFS in domain X and a strictly positive real number, so;

- $m\alpha = (\alpha^m)^c$, if and only if, $(u_\alpha(x) = w_\alpha(x))$.

Proof: We assume that: $m\alpha = (\alpha^m)^c,\ m > 0$, and according to (11), (12) and (4) we find that it is equivalent to:

$$\left\{\langle x, \sqrt{1-(1-(u_\alpha(x))^2)^m}, \left(v_\alpha(x)\right)^m, \left(w_\alpha(x)\right)^m\rangle | x \in X\right\} = \left\{\langle x, \sqrt{1-(1-(w_\alpha(x))^2)^m}, (v_\alpha(x))^m, (u_\alpha(x))^m\rangle | x \in X\right\},$$

and as stated in (6), it is clear that it means $\begin{cases}\sqrt{1-(1-(u_\alpha(x))^2)^m} = \sqrt{1-(1-(w_\alpha(x))^2)^m} \\ (w_\alpha(x))^m = (u_\alpha(x))^m\end{cases}$, which is equivalent to saying $u_\alpha(x) = w_\alpha(x)$.

4. Novel score function for SFNs

The goal of any decision making algorithm is to classify alternatives to select the best one. In the case where the evaluation information of the alternatives is provided in the form of SFNs, the decision maker needs a score function to rank these fuzzy numbers. In this section, we present the comparison problem that can be in practices by using some existing score functions. Before that, we propose a new score function for SFNs using two parameters as an alternate procedure with its proprieties; also, we give some examples to illustrate its efficacies and advantages.

4.1 Presentation of the comparison problem

To ranking the SFNs, Ashraf and Abdullah (2019) proposed not only their score function SC, but also an accuracy function AC as it is in definitions 2.5 and 2.6 that the largest SFN s is the one for which the value of $SC(s)$ is greater, and if the scores are equals, the greater value of $AC(s)$ refers to the greater SFN s, and denotes the extra well alternative for the decision maker. But if we take for example two SFNs k and l such as $k = (u_k = w_k,\ v_k = \frac{1}{2},\ w_k = u_k)$ and $l = (u_l = w_l,\ v_l = \frac{1}{2},\ w_l = u_l)$, and according with (13) and (14), it is clear that $SC(k) = SC(l)$ and $AC(k) = AC(l)$ and then $k \sim l$ which is not always true. So; this score function is powerless to obtain the correct ranking in these cases.

Recently, Javed et al.(2023) suggested an innovative score function SR without accuracy function such as, for any SFN $s = (u_s, v_s, w_s)$, $SR\,(s) = \frac{e^{(u_S)^2-(v_S)^2-(w_S)^2}}{2-\left((u_S(x))^2+(v_S(x))^2+(w_S(x))^2\right)}$, where $e^{-1} \leq SR(s) \leq e$, and that for all two PFNs $k = (u_k,\ v_k,\ w_k)$ and $l = (u_l,\ v_l,\ w_l)$, if $\begin{cases} SC(k) = SC(l) \\ AC(k) > Ac(l) \end{cases}$ then $SR(k) > SR(l)$, further, if $\begin{cases} SC(k) = SC(l) \\ AC(k) = AC(l) \end{cases}$ then $SR(k) = SR(l)$. Suppose that we are comparing two SFNs $k = (u_k = w_k \neq 0,\ v_k,\ w_k = u_k)$ and $l = (0,\ v_l = v_k, 0)$, and according with (13) and (14) we have $\begin{cases} SC(k) = SC(l) \\ AC(k) = AC(l) \end{cases}$ but

$$\begin{cases} SR(k) = \frac{e^{-(v_k)^2}}{2-\left(2(u_k(x))^2+(v_S(x))^2\right)} \\ SR(l) = \frac{e^{-(v_k)^2}}{2-(v_S(x))^2} \end{cases}$$

i.e. $SR(k) \neq SR(l)$, therefore, we can see that the score function SR is failed in such case. The reader may also note some unreasonable situations given later in Table 1.

Due to this problem and to avoid it, we work to develop a new score function in the next subsection.

4.2 Proposed score function

In this subsection, we propose a novel score function for SFNs in order to managing the previous issue by considering the favourite attitudinal of SFNs. The proposed score function is characterised by dependence on two parameters m and n as a substitute technique and it has deferent properties that we discuss too.

Definition 7. The score function of any SFN $k = (u_k, v_k, w_k)$ can be defined as the following formula:

$$S_{m,n_{proposed}}(k) = (u_k)^2 + m(v_k)^2 - n(w_k)^2,\ 0 < m, n \leq 1. \tag{15}$$

Theorem 3. For any two SFNs $k = (u_k, v_k, w_k)$ and $l = (u_l, v_l, w_l)$,

i. If $u_k > u_l$ and $v_k > v_l$ and $w_k < w_l$, then $S_{m,n_{proposed}}(k) > S_{m,n_{proposed}}(l)$,
ii. If $u_k < u_l$ and $v_k < v_l$ and $w_k > w_l$, then $S_{m,n_{proposed}}(k) < S_{m,n_{proposed}}(l)$,
iii. If $u_k = u_l$ and $v_k = v_l$ and $w_k = w_l$, then $S_{m,n_{proposed}}(k) = S_{m,n_{proposed}}(l)$.

Proof:

i. By using (15), we have $S_{m,n_{proposed}}(k) = (u_k)^2 + m(v_k)^2 - n(w_k)^2$, $S_{m,n_{proposed}}(l) = (u_l)^2 + m(v_l)^2 - n(w_l)^2$. So; $S_{m,n_{proposed}}(k) - S_{m,n_{proposed}}(l) = (u_k)^2 - (u_l)^2 + m((v_k)^2 - (v_l)^2) + n((w_l)^2 - (w_k)^2)$. If $\begin{cases} u_k > u_l \\ v_k > v_l \\ w_k < w_l \end{cases}$, and since $0 < m, n \leq 1$,

we can obtain

$$\begin{cases} (u_k)^2 - (u_l)^2 > 0 \\ m((v_k)^2 - (v_l)^2) > 0 \\ n((w_l)^2 - (w_k)^2) > 0 \end{cases},$$

Consequently, $S_{m,n_{proposed}}(k) - S_{m,n_{proposed}}(l) > 0$. So: $S_{m,n_{proposed}}(k) > S_{m,n_{proposed}}(l)$.

ii. As a similar way, if $\begin{cases} u_k < u_l \\ v_k < v_l \\ w_k > w_l \end{cases}$, and since $0 < m, n \leq 1$, we can obtain

$$\begin{cases} (u_k)^2 - (u_l)^2 < 0 \\ m((v_k)^2 - (v_l)^2) < 0 \\ n((w_l)^2 - (w_k)^2) < 0 \end{cases},$$

Consequently, $S_{m,n_{proposed}}(k) - S_{m,n_{proposed}}(l) < 0$. So; $S_{m,n_{proposed}}(k) < S_{m,n_{proposed}}(l)$.

iii. It is trivial.

Theorem 4. For any SFN $k = (u_k, v_k, w_k)$, we have

$$-1 \leq S_{m,n_{proposed}}(k) \leq 2.$$

Proof: We know that $\begin{cases} 0 \le u_k \le 1 \\ 0 \le v_k \le 1 \\ 0 \le w_k \le 1 \end{cases}$, then, $\begin{cases} 0 \le (u_k)^2 \le 1 \\ 0 \le (v_k)^2 \le 1 \\ 0 \le (w_k)^2 \le 1 \end{cases}$. When $0 < m, n \le 1$, we have $\begin{cases} 0 \le (u_k)^2 \le 1 \\ 0 \le m(v_k)^2 \le 1 \\ -1 \le -n(w_k)^2 \le 0 \end{cases}$, consequently, we can obtain

$-1 \le (u_k)^2 + m(v_k)^2 - n(w_k)^2 \le 2$. So; $-1 \le S_{m,n_{proposed}}(k) \le 2$.

Theorem 5. For any SFN $k = (u_k, v_k, w_k)$, the score function $S_{m,n_{proposed}}(k)$ is increasing monotonically with the parameter m and decreasing with the parameter n.

Proof: For any four parameters $0 < m_1, m_2, n_1, n_2 \le 1$, and according to (15), we have $S_{m_1,n_{1}proposed}(k) = (u_k)^2 + m_1(v_k)^2 - n_1(w_k)^2$, $S_{m_2,n_{1}proposed}(k) = (u_k)^2 + m_2(v_k)^2 - n_1(w_k)^2$ and $S_{m_1,n_{2}proposed}(k) = (u_k)^2 + m_1(v_k)^2 - n_2(w_k)^2$ Then $\begin{cases} S_{m_1,n_{1}proposed}(k) - S_{m_2,n_{1}proposed}(k) = (m_1 - m_2)(v_P)^2 \\ S_{m_1,n_{1}proposed}(k) - S_{m_1,n_{2}proposed}(k) = (n_2 - n_1)(v_P)^2 \end{cases}$. So; we can see that if $m_1 < m_2$, then $S_{m_1,n_{1}proposed}(k) < S_{m_2,n_{1}proposed}(k)$. But if $n_1 < n_2$, then $S_{m_1,n_{1}proposed}(k) > S_{m_1,n_{2}proposed}(k)$.

To illustrate the power of the proposed score function, we give the following comparison in Table 1, supposing concrete examples and taking $m = 0.5$ and $n = 0.5$ in (15). What is in bold is an unreasonable result.

Table 1. Comparison of score functions

Examples of SFNs	Score functions	Score values	Accuracy values	Ranking SFNs
$k = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $l = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.	score function by Ashraf et al.(2019)	$\begin{cases} SC(k) = \frac{1}{2} \\ SC(l) = \frac{1}{2} \end{cases}$	$\begin{cases} AC(k) = 0 \\ AC(l) = 0 \end{cases}$	$\boldsymbol{k \sim l}$
	The proposed function	$\begin{cases} S_{m,n_{proposed}}(k) = \frac{1}{4} \\ S_{m,n_{proposed}}(l) = \frac{3}{32} \end{cases}$	without	$k > l$
$k = (0,0,0)$,	score function by	$\begin{cases} SC(k) = \frac{2}{3} \\ SC(l) = \frac{2}{3} \end{cases}$	$\begin{cases} AC(k) = 0 \\ AC(l) = 0 \end{cases}$	$\boldsymbol{p \sim q}$

Examples of SFNs	Score functions	Score values	Accuracy values	Ranking SFNs
$l=\left(\frac{1}{2},0,\frac{1}{2}\right)$.	Ashraf et al.(2019)			
	The proposed function	$\begin{cases} S_{m,n_{proposed}}(k)=0 \\ S_{m,n_{proposed}}(l)=\frac{1}{8} \end{cases}$	without	$k<l$
$k=\left(0,0,\frac{1}{2}\right)$, $l=\left(0,\frac{1}{\sqrt{8}},\frac{1}{\sqrt{8}}\right)$.	score function by Javed et al. (2023)	$\begin{cases} SR(k)=\frac{4}{7}e^{-\frac{1}{4}} \\ SR(l)=\frac{4}{7}e^{-\frac{1}{4}} \end{cases}$	without	$\boldsymbol{k\sim l}$
	The proposed function	$\begin{cases} S_{m,n_{proposed}}(k)=\frac{-1}{8} \\ S_{m,n_{proposed}}(l)=0 \end{cases}$	without	$k<l$
$k=\left(0,0,\frac{1}{2}\right)$, $l=\left(0,\frac{1}{2},0\right)$.	score function by Javed et al. (2023)	$\begin{cases} SR(k)=\frac{4}{7}e^{-\frac{1}{4}} \\ SR(l)=\frac{4}{7}e^{-\frac{1}{4}} \end{cases}$	without	$\boldsymbol{k\sim l}$
	The proposed function	$\begin{cases} S_{m,n_{proposed}}(k)=\frac{-1}{8} \\ S_{m,n_{proposed}}(l)=\frac{1}{8} \end{cases}$	without	$k<l$

Source: Comparison performed based on our assumed concrete examples.

From these examples, we can see that the presented score function $S_{m,n_{proposed}}$ is able to detect the difference without any accuracy function when both Ashraf et al. (2019) and Javed et al. (2023) score functions fail. Then the novel score function is reasonable and offers an effective procedure for the process of the decision analysis.

5. Application of the Spherical Fuzzy Assignment Problem

In this section, an assignment problem in spherical fuzzy environment, called as spherical fuzzy assignment problem (SFAP), is introduced. The issue is proposed by using the SFNs in the elements of the cost matrix. Also; the existing approach to solve the Pythagorean fuzzy assignment problem of Kumar et al. (2023) extend to treat the SFAP using the proposed score function. Moreover, a numerical example is given and a comparative study is examined to validate the presented approach.

5.1 Assignment problem in spherical fuzzy environment

In optimisation, Assignment problem is an important area. It consists of optimising the allocation of p resources to p demand points. It can be mathematically represented as follows:

$$\text{Optimise } \sum_{i=1}^{p}\sum_{j=1}^{p} C_{ij}X_{ij} \text{ subject to } \begin{cases} \sum_{j=1}^{n} X_{ij} = 1,\ i = 1, \dots, p \\ \sum_{i=1}^{n} X_{ij} = 1,\ j = 1, \dots, p \\ X_{ij} \in \{0; 1\}, i = 1, \dots, p, j = 1, \dots, p \end{cases}$$

where C_{ij} is the cost of assigning the resource i to the demand j. The constraints indicate that each resource i needs to be assigned to only one demand j, and each demand j needs to be assigned to only resource i. Then the cost matrix associate $(C_{ij})_{\substack{1\le i\le p\\ 1\le j\le p}}$ may be as following:

$$\begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pp} \end{pmatrix}.$$

Assume that expert provides the evaluation information of the cost of assigning the resource i to the demand j in the form of a SFN $s_{ij} = (u_{ij}, v_{ij}, w_{ij})$, where u_{ij} , $v_{ij}$and w_{ij} are , respectively, the membership, the abstinence and the non-membership function values associate , and then, the spherical fuzzy cost matrix $M_C = (s_{ij})_{\substack{1\le i\le p\\ 1\le j\le p}}$ can be obtained as follow:

$$M_C = \begin{pmatrix} (u_{11}, v_{11}, w_{11}) & \cdots & (u_{1p}, v_{1p}, w_{1p}) \\ \vdots & \ddots & \vdots \\ (u_{p1}, v_{p1}, w_{p1}) & \cdots & (u_{pp}, v_{pp}, w_{pp}) \end{pmatrix}.$$

For finding the optimal assignment, we use the method that is summarised as below:

Step 1: Input the spherical fuzzy cost matrix $M_C = (s_{ij})_{\substack{1\le i\le p\\ 1\le j\le p}}$

Step 2: Compute the score matrix $M_R = (r_{ij})_{\substack{1\le i\le p\\ 1\le j\le p}}$ of $M_C = (s_{ij})_{\substack{1\le i\le p\\ 1\le j\le p}}$

where:

$$r_{ij} = S_{m,n_{proposed}}(s_{ij}) = (u_{s_{ij}})^2 + m\left(v_{s_{ij}}\right)^2 - n\left(w_{s_{ij}}\right)^2, 0 < m,n \le 1$$

Step 3: If the system is unbalanced, add dummy variables to convert it into a balanced one. Else, go to step 4.

Step 4: The greater score value indicate the preference of assigning the demand j to the resource i.

5.2 Numerical example

Suppose that a country in the process of implementing five different projects $\{P_1;\ P_2; P_3; P_4; P_5\}$ in five different regions $\{R_1;\ R_2; R_3; R_4; R_5\}$, where each region receives a single project and each project must be implemented in a single region too. The execution of each project P_i ; $1 \leq i \leq 5$; in relation to each region R_j ; $1 \leq j \leq 5$; has a cost that will be evaluated by decision maker. The problem is how to assign the projects to regions to get the optimal cost. Assume that this evaluation is given in Table 2.

Table 2. Expert's cost evaluation

	$\boldsymbol{R_1}$	$\boldsymbol{R_2}$	$\boldsymbol{R_3}$	$\boldsymbol{R_4}$	$\boldsymbol{R_5}$
P_1	(0, 0.5, 0)	(0.5, 0.5, 0.5)	(0.6, 0.5, 0.6)	(0.2, 0.3, 0.2)	(0.5, 0.3, 0.5)
P_2	(0.5, 0.5, 0.5)	(0.25, 0.5, 0.25)	(0, 0.5, 0)	(0, 0, 0.5)	(0.6, 0.5, 0.6)
P_3	(0.68, 0.1,0.68)	(0.3, 0.2, 0.4)	(0.3, 0.3, 0.4)	(0.7, 0.2, 0.7)	(0.3, 0.3, 0.3)
P_4	(0.68, 0.1,0.68)	(0.63, 0.1, 0.63)	(0.6, 0.1, 0.6)	(0.66,0.1,0.66)	(0.65,0.1,0.65)
P_5	(0.7, 0.1, 0.6)	(0.7, 0.3, 0.3)	(0.7, 0.2, 0.5)	(0.7, 0.1, 0.7)	(0.71, 0, 0.7)

Source: Our assuming simulation.

Now, we apply the algorithm to select the optimal assignment.

Step 1: According to Table 2, the spherical fuzzy cost matrix is the following:

$$M_C = (s_{ij})_{\substack{1\leq i\leq 5\\ 1\leq j\leq 5}}$$
$$= \begin{pmatrix} (0,0.5,0) & (0.5,0.5,0.5) & (0.6,0.5,0.6) & (0.2,0.3,0.2) & (0.5,0.3,0.5) \\ (0.5,0.5,0.5) & (0.25,0.5,0.25) & (0,0.5,0) & (0,0,0.5) & (0.6,0.5,0.6) \\ (0.68,0.1,0.68) & (0.3,0.2,0.4) & (0.3,0.3,0.4) & (0.7,0.2,0.7) & (0.3,0.3,0.3) \\ (0.68,0.1,0.68) & (0.63,0.1,0.63) & (0.6,0.1,0.6) & (0.66,0.1,0.66) & (0.65,0.1,0.65) \\ (0.7,0.1,0.6) & (0.7,0.3,0.3) & (0.7,0.2,0.5) & (0.7,0.1,0.7) & (0.71,0,0.7) \end{pmatrix}$$

Step 2: Compute the score matrix $M_R = (r_{ij})_{\substack{1\leq i\leq 5\\ 1\leq j\leq 5}}$ of $M_C = (s_{ij})_{\substack{1\leq i\leq 5\\ 1\leq j\leq 5}}$ using (15) with $m = 0.5$ and $n = 0.5$ as follows: (the results are given in rounded to 10^{-3})

$$M_R = \begin{pmatrix} 0.125 & 0.25 & 0.305 & 0.065 & 0.17 \\ 0.25 & 0.156 & 0.125 & -0.125 & 0.305 \\ 0.236 & 0.03 & 0.055 & 0.265 & 0.09 \\ 0.236 & 0.203 & 0.185 & 0.222 & 0.216 \\ 0.315 & 0.49 & 0.385 & 0.25 & 0.259 \end{pmatrix}$$

Step 3: The problem is balanced.
Step 4: The optimal assignment is: project1→ region3, project2 → region5, project3 → region4, project4 → region1, and project5 → region2.

5.3 Comparative study

In this subsection, we conduct a comparison applying the same algorithm for the same last numerical example with changing the score function each time. The corresponding results are summarised in Table 3.

Table 3. Comparative analysis

Score matrix						The optimal assignment
using score function by Ashraf et al.(2019)	0.5	0.5	0.5	0.566	0.656	The score function cannot determine the optimal assignment.
	0.5	0.5	0.5	0.5	0.5	
	0.633	0.566	0.533	0.6	0.566	
	0.633	0.633	0.633	0.633	0.633	
	0.666	0.7	0.666	0.633	0.67	
using score function by Javed et al. (2023)	0.445	0.623	0.756	0.499	0.648	project1→ region3,
	0.623	0.519	0.445	0.445	0.756	project2→ region5,
	0.929	0.523	0.513	0.945	0.528	project3→ region4,
	0.929	0.827	0.779	0.884	0.864	project4→ region1,
	0.989	1.025	1.001	0.980	1.008	project5→ region2.

Source: Our calculation results.

Through this comparison, it becomes clear that the results of the presented approach with the proposed score function are the same as the optimal assignment based on Javed et al. (2023) score function, while Ashraf et al. (2019) score function may be unable to determine the optimal assignment due to the equality in several cases (score values and accuracy are the same) and thus not distinguishing between differences in such situations despite its presence. Therefore, the method presented in this paper has strong advantages.

6. Conclusions

The key contributions of this work can be planned in the following:

- Some particular cases of SFSs as necessary conditions and necessary and sufficient conditions are demonstrated.
- A novel score function for ranking SFNs is suggested with its properties to avoid the comparison problem in practice that can be by using the score functions defined by Ashraf et al. (2019) and by Javed et al. (2023); some examples are given to illustrate its effectiveness when these existing score functions are limited (Table 1).
- The existing approach to the Pythagorean fuzzy assignment problem of Kumar et al. (2023) is extended to resolve the SFAP based on the presented score function of the SFNs. The method is illustrated with a numerical example. Also, a comparative analysis is given to prove the rationality and efficacy of the method presented in this paper, it can determine the optimal assignment without unreasonable situations problem that the decision maker may face (Table 3).

As a future work, we may extend this study into different fuzzy environment and to deal with other problems.

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