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The Strategy of Constructing Elder-Friendly Communities Using Differential Game

Abstract. *This study constructs a three-party stochastic differential game model consisting of the government, project contractors, and elderly residents, based on benefit distribution and coordination equilibrium. It quantitatively solves the optimal strategy for the construction of elder-friendly communities, and qualitative solutions for three different models were computed: non-cooperative model, subsidised cooperative model, and three-party cooperative model. From our results, we can conclude that 1) government subsidies are an effective incentive to increase the optimal effort of project contractors and elderly residents and thus increase the quality and efficiency of elder-friendly retrofitting. 2) compared with the non-cooperative model and the subsidised cooperative model, the optimal benefits of each subject, the total system benefits, and the quality and efficiency of elder-friendly community construction are maximised under the three-party cooperative model when the Pareto Improvement is the greatest. 3) Higher benefits come with more significant risks, and the variance increases as the total system benefits and the quality of Elder-friendly retrofitting increase.*

Keywords: *Elder-friendly retrofit, differential game, tripartite cooperation, government subsidy, Nash Equilibrium.*

JEL Classification: C70, C72.

1. Introduction

Currently, China is facing a severe problem of aging population. At the end of 2022, the population of aged 60 and above was 280.04 million in China, accounting for approximately 19.8% of the total population. The Human Resources and Social Security Bureau predicts that the elderly population in China will exceed 300 million by 2025. The number and structure of the aging population will increase further, gradually moving from light to heavy aging. Therefore, improving service systems

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for the elderly has become a fundamental social problem that needs to be addressed. To cope with the fierce wave of "silver hair, " as early as 2011, China released the "12th Five-Year Plan for the Development of China's Aging Industry," which proposed to "establish a home-based, community-based, and institutional supported elderly service system." However, to solidify the basic position of home care, it is necessary to meet the special needs of the elderly at different levels of home care services. However, in reality, the design and construction of established communities rarely consider the actual needs and services of the elderly. Even if a few new communities consider the special needs of the elderly, it is still difficult to meet the needs of the rapidly increasing elderly population. Therefore, it has become important for China to respond to the problem of an aging population by upgrading aging-appropriate communities. In this context, in July 2020, the General Office of the State Council issued the "Guidance on Comprehensively Promoting the Transformation of Old Urban Communities," which called for further promoting the deepening of the elder-friendly transformation of built communities.

Age-appropriate retrofitting refers to the corresponding design, renovation, and facility configuration of the main activity areas of the elderly, based on the consideration of the characteristics of their physiological and psychological needs to adapt to their usage (Hu & Hu, 2013). Relevant policies, technologies, and services for age-appropriate retrofitting in China have been continuously developed and improved in recent years. However, problems pertaining to quality and efficiency have been highlighted (Yao et al., 2017). For example, in 2019, 170,000 old urban neighbourhoods were reported to require renovation across the country; however, the cumulative number of old urban neighbourhoods that started renovation was only 112,000, accounting for only 65.7%. According to the 2022 China Aging and Recreation Industry Development Report, the government is still the main investor for China's elder-friendly renovation with the purpose of underwriting protection, covering a relatively limited number of people. Elder-friendly renovation has a single function and cannot adapt to the entire life cycle of the elderly, and its quality is not high.

Academics have conducted valuable discussions on the dilemma of elder-friendly community construction and its causes (Liu, 2015; Yang & Xue, 2015; Zhou & Chen, 2022; Liu et al., 2013; Zhao et al., 2017; Li et al., 2022). But only a few scholars have discussed how to solve the problem of "low" quality and efficiency of elder-friendly renovation. Some scholars have pointed out that a multi-collaboration mechanism of the government, enterprises, academia, and people" can be established to promote the active participation of multiple parties under the government's leadership (Gu et al., 2019; Zeng et al., 2022). Scholars also believe that urban renewal projects must reasonably identify the participating parties and that a combination of government subsidies and private funding can be adopted to improve the efficiency of elder-friendly community renovations (Makigami & Pynoos, 2022; Xu, 2021). Other scholars advocate that the government should further increase financial support, encourage the participation of market forces, and promote multiparty collaboration for elder-friendly retrofitting (Zhou et al., 2023; Wang &

Zhou ,2023). Undeniably, the above research results support the idea of introducing a government-led multiparty participation model for elder-friendly community construction projects. However, they must conduct an in-depth discussion on the coordination of benefit distribution and the specific alternative strategies involved. Elder-friendly retrofitting is a systematic work involving many interested parties and an extended period, and friction or disputes among different interested parties will form a "middle block", making the retrofitting work impossible and leads to low quality and efficiency. Therefore, to solve the "double-low" dilemma of elder-friendly renovation, it is necessary to coordinate the interest demands and distribution, reveal and break the "obstacle" of interests, fully mobilise the enthusiasm, and promote performance of duties by each party, to form a multi-level participation model of elder-friendly renovation, the model of division of labour and collaboration in the transformation of aging.

This study investigates the coordination mechanism and optimal strategy for elder-friendly renovation from a quantitative perspective with the help of a differential game model to break the "double-low" dilemma of elder-friendly renovation. The possible marginal contributions of this study are as follows. First, we use the stochastic differential game model to study the decision-making of elder-friendly community construction among the government, engineers, and elderly residents, which compensates for the deficiencies of the existing literature in the research method. Second, it solves the optimal strategies and system equilibrium states of the three subjects under three scenarios – noncooperative, subsidised cooperative, and tripartite cooperative modes – including the quality of retrofitting, the level of government subsidies, and system benefits, which are conducive to discussing the actual benefit distribution problems and mechanisms of the three subjects under different scenarios. Third, we further identify the driving factors of the subjects' decision-making behaviours, behaviours under multiple scenarios through numerical simulations, and provide a theoretical basis and decision-making reference for constructing and refining the aging retrofitting division of labour and collaboration models.

2. Problem Description and Model Assumptions

2.1 Problem description

There is a possibility that elderly residents are unwilling to cooperate with retrofitting due to financial difficulties, limited awareness, or conflicts of interest, which renders it difficult for engineering contractors to carry out age-appropriate retrofitting effectively (Wu et al., 2017). Additionally, although the demand for elder-friendly retrofitting is high, some engineering contractors are reluctant to conduct such businesses because of relatively low marketisation and uncertainty of returns. Therefore, local governments must act as leading parties for elder-friendly retrofitting to coordinate and promote such renovation in established communities.

Specifically, we constructed an age-appropriate retrofit ecosystem consisting of a single local government S, engineer M, and elderly resident R. It is assumed that, when promoting elder-friendly community construction, local governments can increase the willingness of elderly residents to retrofit through subsidies and education. Thus, the role of elderly residents in age-appropriate retrofitting will increase, and the process of age-appropriate retrofitting will be promoted. Additionally, local governments can subsidise commission engineers to encourage and promote elder-friendly community construction. Considering that decision makers are also affected by various stochastic factors, such as their personalities, the ability to obtain information, and external economic, policy, and cultural factors, this study adds a stochastic disturbance term to the game model.

2.2 Basic Assumptions and Descriptions

Hypothesis 1 In this study, we assume that the elder-friendly retrofit is continuous during the examined period. Therefore, the game time is an infinite interval. Game participants 1, 2, and 3 represented the elderly residents, engineers, and the local government, respectively. All three parties aimed to obtain the maximum benefit in an infinite period.

Hypothesis 2 Drawing on Qiao et al. (2022) 's hypothesis on effort cost, this study assumes that the effort cost of each party to promote age-appropriate retrofitting is a convex function of the degree of effort, and sets the effort cost of elderly residents', engineers, and local governments at time t, respectively, as:

$$C_1(t) = \frac{1}{2}\mu_1 E_1^2(t); C_2(t) = \frac{1}{2}\mu_2 E_2^2(t); C_3(t) = \frac{1}{2}\mu_3 E_3^2(t) \tag{1}$$

Among them, the μ_1, μ_2 and μ_3 represent the cost coefficients of the inputs of elderly residents, engineers, and local governments, respectively, in the process of Elder-friendly community construction (μ_1, μ_2 and μ_3 are more significant than 0). $E_1(t)$, $E_2(t)$ and $E_3(t)$ represent the efforts of elderly residents, engineers, and local governments, respectively, in the process of constructing elder-friendly communities.

Hypothesis 3 The quality of age-appropriate retrofitting is given by the variable $G(t)$, and the inverse $dG(t)/d(t)$ represents its change over timet $\in [0, +\infty)$, and this variable is determined by the joint efforts of elderly residents, engineers, and local governments. Additionally, the quality of an elder-friendly retrofit is dynamic. The analysis above shows that elder-friendly retrofit is based on the entire life cycle of the elderly; therefore, it is also time-sensitive, and the quality naturally declines after a certain period. Referring to Zhang et al. (2021), the change in elder-friendly retrofit quality over time can be expressed by the following stochastic differential equation:

$$\begin{cases} dG(t) = [\alpha_1 E_1(t) + \alpha_2 E_2(t) + \alpha_3 E_3(t) - \beta G(t)]d(t) + \sigma[G(t)]dz(t) \\ G(0) = G_0 \geq 0 \end{cases} \tag{2}$$

Among them, the α_1, α_2 and α_3 represent the degree of influence of the degree of effort put in by elderly residents, engineers, and local governments on the quality of age-appropriate retrofitting, respectively, that is, the influence coefficient of retrofitting quality (α_1, α_2 and α_3 are more significant than 0). Additionally, β Greater than 0, a constant coefficient, represents the degree of decay of the quality of age-appropriate retrofitting over time, which means $G(t)$ the growth will gradually slow down with the degree of participants' efforts, that is, the marginal utility decreases, which is usually caused by factors such as weak subsidies from local governments, engineers cutting corners, and changes in the retrofitting needs of elderly residents—$z(t)$ for the standard WINNER process.

Hypothesis 4 Referring to the research hypothesis of Yang & Mingge (2022), this study assumes that the change in the quality of age-appropriate retrofitting over time is influenced by the standard WINNER process, and the random disturbance influence coefficient is proportional to the square root of the quality of age-appropriate modifications:

$$\sigma[G(t)]dz(t) = \sigma\sqrt{G(t)}dz(t) \tag{3}$$

Hypothesis 5 Active participation of elderly residents, engineers, and local governments in age-appropriate retrofitting is a critical way to improve its quality, and the level of effort of each of the three parties to cooperate will directly affect the benefits of the ecosystem. Thus, the interactions among the three parties must be considered. Referring to the idea of the cooperative model in Xie et al. (2022), the total benefit of the collaborative ecosystem of elder-friendly community construction with government participation at time (t) can be expressed as:

$$\pi(t) = \gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t) \tag{4}$$

Among them, the γ_1, γ_2 and γ_3 represent the marginal revenue coefficients of elderly residents, engineers, and local governments, respectively, in the process of age-appropriate retrofitting (γ_1, γ_2 and γ_3 are more significant than 0). Additionally, δ Greater than 0, a constant coefficient, represents the degree of impact of the quality of elder-friendly community construction on the total revenue.

Assumption 6 The total benefits of the tripartite ecosystem are distributed among the participants according to a pre-agreed ratio, setting the coefficient of benefit distribution for the elderly resident to ϑ_1, and that for engineer is set as ϑ_2. Therefore, the coefficient of benefit distribution of the local government is $1 - \vartheta_1 - \vartheta_2$ (ϑ_1 and ϑ_2 are more significant than 0, and $\vartheta_1 + \vartheta_2$ less than 1). From the above, it is evident that the support of a local government can promote age-appropriate retrofitting. Therefore, the subsidy coefficient of the local government for elderly resident is set to φ_1, and that for the engineer is φ_2. In addition, at time $[0, +\infty)$, the three subjects have the same discount factor r ($r > 0$). To facilitate the calculation, some of the relevant parameters of the model were set as time-independent constants in this study; therefore, time is not listed below.

3. Model building and solving

3.1 Non-cooperative model

In this model, elderly residents, engineers, and local governments make simultaneous decisions independently and equally, and local governments do not subsidise elderly residents and engineers, that is, assuming $\varphi_1 = \varphi_2 = 0$. Because the game subject independently makes an optimal decision, the game strategy constitutes a Nash equilibrium solution. The specific decision functions of the elderly residents, engineers, and local governments are as follows:

$$\max J_1^F = \int_0^\infty \{e^{-r}[\vartheta_1(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)) - \frac{1}{2}\mu_1 E_1^2(t)]\} dt$$

$$\max J_2^F = \int_0^\infty \{e^{-r}[\vartheta_2(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)) - \frac{1}{2}\mu_2 E_2^2(t)]\} dt$$

$$\max J_3^F = \int_0^\infty \{e^{-r}[(1-\vartheta_1-\vartheta_2)\big(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)\big) - \frac{1}{2}\mu_3 E_3^2(t)]\} dt \tag{5}$$

Theorems 1–3 can be proven by constructing the Hamilton-Jacobi-Bellman equation for the solution.

Theorem 1. Under the non-cooperative model, the Nash equilibrium strategy of the elderly residents, engineers, and local government, that is, the optimal effort decision, can be expressed as:

$$E_1^F = \frac{\vartheta_1[\gamma_1(r+\beta) + \delta\alpha_1]}{\mu_1(r+\beta)};\ E_2^F = \frac{\vartheta_2[\gamma_2(r+\beta) + \delta\alpha_2]}{\mu_2(r+\beta)}$$

$$E_3^F = \frac{(1-\vartheta_1-\vartheta_2)[\gamma_3(r+\beta) + \delta\alpha_3]}{\mu_3(r+\beta)} \tag{6}$$

Theorem 2. Under the non-cooperative model, the Nash equilibrium benefits for elderly residents, engineers, and local governments can be expressed as:

$$V_1^F(G) = \frac{\vartheta_1\delta}{r+\beta}G + \frac{{\vartheta_1}^2[(r+\beta)\gamma_1 + \delta\alpha_1]^2}{2\mu_1 r(r+\beta)^2} + \frac{\vartheta_1\vartheta_2[(r+\beta)\gamma_2 + \delta\alpha_2]^2}{\mu_2 r(r+\beta)^2} + \frac{\vartheta_1(1-\vartheta_1-\vartheta_2)[(r+\beta)\gamma_3 + \delta\alpha_3]^2]}{\mu_3 r(r+\beta)^2}$$

$$V_2^F(G) = \frac{\vartheta_2\delta}{r+\beta}G + \frac{\vartheta_1\vartheta_2[(r+\beta)\gamma_1 + \vartheta_2\delta\alpha_1]^2}{\mu_1 r(r+\beta)^2} + \frac{{\vartheta_2}^2[(r+\beta)\gamma_2 + \delta\alpha_2]^2}{2\mu_2 r(r+\beta)^2} + \frac{\vartheta_2(1-\vartheta_1-\vartheta_2)[(r+\beta)\gamma_3 + \delta\alpha_3]^2}{\mu_3 r(r+\beta)^2}$$

$$V_3^F(G) = \frac{(1-\vartheta_1-\vartheta_2)\delta}{r+\beta}G + \frac{(1-\vartheta_1-\vartheta_2)\vartheta_1[(r+\beta)\gamma_1+\delta\alpha_1]^2}{\mu_1 r(r+\beta)^2} + \frac{(1-\vartheta_1-\vartheta_2)\vartheta_2[(r+\beta)\gamma_2+\delta\alpha_2]^2}{\mu_2 r(r+\beta)^2} + \frac{(1-\vartheta_1-\vartheta_2)^2[(r+\beta)\gamma_3+\delta\alpha_3]^2}{2\mu_3 r(r+\beta)^2} \quad (7)$$

Therefore, the ecosystem's optimal total benefit function in the non-cooperative model can be obtained as:

$$V^F(G) = V_1^F(G) + V_2^F(G) + V_3^F(G) = \frac{\delta G}{r+\beta} + \frac{\vartheta_1(2-\vartheta_1)[(r+\beta)\gamma_1+\delta\alpha_1]^2}{2\mu_1 r(r+\beta)^2} + \frac{\vartheta_2(2-\vartheta_2)[(r+\beta)\gamma_2+\delta\alpha_2]^2}{2\mu_2 r(r+\beta)^2} + \frac{(1+\vartheta_1+\vartheta_2)(1-\vartheta_1-\vartheta_2)[(r+\beta)\gamma_3+\delta\alpha_3]^2}{2\mu_3 r(r+\beta)^2} \quad (8)$$

Theorem 3. Under the non-cooperative model, the expected and stable values of the quality of age-appropriate retrofitting are as follows:

$$E[G^F(t)] = \frac{A}{\beta} - \frac{e^{-\beta t}}{\beta}(A - G_0\beta);\ \lim_{t\to\infty} E[G^F(t)] = \frac{A}{\beta}$$
$$A = \frac{\alpha_1\vartheta_1[\gamma_1(r+\beta)+\delta\alpha_1]}{\mu_1(r+\beta)} + \frac{\alpha_2\vartheta_2[\gamma_2(r+\beta)+\delta\alpha_2]}{\mu_2(r+\beta)} + \frac{\alpha_3(1-\vartheta_1-\vartheta_2)[\gamma_3(r+\beta)+\delta\alpha_3]}{\mu_3(r+\beta)} \quad (9)$$

The variance and variance stability values for the quality of the age-appropriate retrofitting are as follows:

$$D[G^F(t)] = \frac{\sigma^2\left[A - 2(A-G_0\beta)e^{-\beta t} + (A-2G_0\beta)e^{-2\beta t}\right]}{2\beta^2}$$
$$\lim_{t\to\infty} D[G^F(t)] = \frac{A\sigma^2}{2\beta^2} \quad (10)$$

3.2 Subsidised cooperation model

In the subsidised cooperation model, this study assumes that the local government is the leader within the ecosystem, the elderly residents and engineers are followers, and that the local government provides subsidies to the elderly residents and engineers, φ_1 and φ_2, respectively. In this model, local governments first determine their efforts and subsidies for elderly residents and engineers. Further, the elderly residents and engineers make decisions based on the local government's decisions to maximise their interests. Therefore, local governments can effectively predict the behaviours of elderly residents and make optimal decisions. The objective function of a three-party game subject can be expressed as follows:

$$\max J_1^N = \int_0^{\infty} \{e^{-rt}[\vartheta_1(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)) - \frac{1}{2}\mu_1 E_1^2(t)(1 - \varphi_1)\}]\, dt$$

$$\max J_2^N = \int_0^{\infty} \{e^{-rt}[\vartheta_2(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)) - \frac{1}{2}\mu_2 E_2^2(t)(1 - \varphi_2)]\}\, dt$$

$$\max J_3^N = \int_0^{\infty} \{e^{-rt}[(1 - \vartheta_1 - \vartheta_2)\big(\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t)\big) - \frac{1}{2}\mu_3 E_3^2(t) - \frac{1}{2}\varphi_1\mu_1 E_1^2(t) - \frac{1}{2}\varphi_2\mu_2 E_2^2(t)]\} \quad (11)$$

By constructing the HJB equation to solve it, the following Theorems 4–6 can be proved.

Theorem 4. Under the subsidised cooperation model, the Nash equilibrium strategy (i.e., the optimal effort decision) of the elderly resident, engineer, and local government can be expressed as:

$$E_1^N = \frac{(2 - \vartheta_1 - 2\vartheta_2)[\gamma_1(r + \beta) + \delta\alpha_1]}{2\mu_1(r + \beta)};\ E_2^N = \frac{(2 - 2\vartheta_1 - \vartheta_2)[\gamma_2(r + \beta) + \delta\alpha_2]}{2\mu_2(r + \beta)}$$

$$E_3^N = \frac{(1 - \vartheta_1 - \vartheta_2)[\gamma_3(r + \beta) + \delta\alpha_3]}{\mu_3} \quad (12)$$

Theorem 5. Under the subsidised cooperation model, the Nash equilibrium benefits for elderly residents, engineers, and local governments can be expressed as:

$$V_1^N(G) = \frac{\vartheta_1\delta}{r + \beta}G + \frac{\vartheta_1(2 - \vartheta_1 - 2\vartheta_2)[\gamma_1(r + \beta) + \delta\alpha_1]^2}{4\mu_1 r(r + \beta)^2} + \frac{\vartheta_1(2 - 2\vartheta_1 - \vartheta_2)[\gamma_2(r + \beta) + \delta\alpha_2]^2}{2\mu_2 r(r + \beta)^2} + \frac{\vartheta_1(1 - \vartheta_1 - \vartheta_2)[\gamma_3(r + \beta) + \delta\alpha_3]^2}{\mu_3 r(r + \beta)^2}$$

$$V_2^N(G) = \frac{\vartheta_2\delta}{r + \beta}G + \frac{\vartheta_2(2 - \vartheta_1 - 2\vartheta_2)[\gamma_1(r + \beta) + \delta\alpha_1]^2}{2\mu_1 r(r + \beta)^2} + \frac{\vartheta_2(2 - 2\vartheta_1 - \vartheta_2)[\gamma_2(r + \beta) + \delta\alpha_2]^2}{4\mu_2 r(r + \beta)^2} + \frac{\vartheta_2(1 - \vartheta_1 - \vartheta_2)[\gamma_3(r + \beta) + \delta\alpha_3]^2}{\mu_3 r(r + \beta)^2}$$

$$V_3^N(G) = \frac{(1-\vartheta_1-\vartheta_2)\delta}{r+\beta}G + \frac{(2-\vartheta_1-2\vartheta_2)^2[\gamma_1(r+\beta)+\delta\alpha_1]^2}{8\mu_1 r(r+\beta)^2} + \frac{(2-2\vartheta_1-\vartheta_2)^2[\gamma_2(r+\beta)+\delta\alpha_2]^2}{8\mu_2 r(r+\beta)^2} + \frac{(1-\vartheta_1-\vartheta_2)^2[\gamma_3(r+\beta)+\delta\alpha_3]^2}{2\mu_3 r(r+\beta)^2} \tag{13}$$

Therefore, in this model, the optimal total benefit of the tripartite ecosystem can be obtained as:

$$V^N(G) = V_1^N(G) + V_2^N(G) + V_3^N(G) = \frac{\delta}{r+\beta}G + \frac{(2+\vartheta_1+2\vartheta_2)(2-\vartheta_1-2\vartheta_2)[\gamma_1(r+\beta)+\delta\alpha_1]^2}{8\mu_1 r(r+\beta)^2} + \frac{(2+2\vartheta_1+\vartheta_2)(2-2\vartheta_1-\vartheta_2)[\gamma_2(r+\beta)+\delta\alpha_2]^2}{8\mu_2 r(r+\beta)^2} + \frac{(1+\vartheta_1+\vartheta_2)^2[\gamma_3(r+\beta)+\delta\alpha_3]^2}{2\mu_3 r(r+\beta)^2} \tag{14}$$

Theorem 6. Under the subsidised cooperation model, the expected value and expected stable value of the quality of the age-appropriate retrofit are as follows:

$$E[G^N(t)] = \frac{B}{\beta} - \frac{e^{-\beta t}}{\beta}(B - G_0\beta);\ \lim_{t\to\infty} E[G^N(t)] = \frac{B}{\beta}$$

$$B = \frac{\alpha_1(2-\vartheta_1-2\vartheta_2)[\gamma_1(r+\beta)+\delta\alpha_1]}{2\mu_1(r+\beta)} + \frac{\alpha_2(2-2\vartheta_1-2\vartheta_2)[\gamma_2(r+\beta)+\delta\alpha_2]}{2\mu_2(r+\beta)} + \frac{\alpha_3(1-\vartheta_1-\vartheta_2)[\gamma_3(r+\beta)+\delta\alpha_3]}{\mu_3} \tag{15}$$

The variance and variance stability values for the quality of the age-appropriate retrofitting are as follows:

$$D[G^N(t)] = \frac{\sigma^2\left[B - 2(B-G_0\beta)e^{-\beta t} + (B-2G_0\beta)e^{-2\beta t}\right]}{2\beta^2}$$

$$\lim_{t\to\infty} D[G^N(t)] = \frac{B\sigma^2}{2\beta^2} \tag{16}$$

3.3 Tripartite cooperation model

To further promote elder-friendly retrofitting and improve its quality in built communities, the relationship among elderly residents, engineers, and local government is transformed from the existing government subsidy guidance to a unified, organic system. The three parties prioritise the overall benefit of the ecosystem to maximise the quality of elder-friendly retrofitting in built communities and jointly determine the best-effort strategy of the three parties. Therefore, the objective function of the three-party game subjects can be set as follows:

$$\max J^{M} = J_1^{M} + J_2^{M} + J_3^{M}$$
$$= \int_0^{+\infty} e^{-rt}[\gamma_1 E_1(t) + \gamma_2 E_2(t) + \gamma_3 E_3(t) + \delta G(t) - \frac{1}{2}\mu_1 E_1^2(t) - \frac{1}{2}\mu_2 E_2^2(t) - \frac{1}{2}\mu_3 E_3^2(t)]dt \quad (17)$$

By constructing the HJB equation to solve it, Theorems 7–9 can be proved.

Theorem 7. In the case where three parties choose to cooperate, the feedback Nash equilibrium strategy (i.e., the optimal effort decision) of the elderly resident, engineer, and local government can be expressed as follows:

$$E_1^{M} = \frac{\gamma_1(r+\beta)+\delta\alpha_1}{\mu_1(r+\beta)}; E_2^{M} = \frac{\gamma_2(r+\beta)+\delta\alpha_2}{\mu_2(r+\beta)}; E_3^{M} = \frac{\gamma_3(r+\beta)+\delta\alpha_3}{\mu_3(r+\beta)} \quad (18)$$

Theorem 8. In the tripartite cooperation model, the overall system Nash equilibrium gain can be expressed as:

$$V^{M}(G) = \frac{\delta}{r+\beta}G + \frac{[\gamma_1(r+\beta)+\delta\alpha_1]^2}{2\mu_1 r(r+\beta)^2} + \frac{[\gamma_2(r+\beta)+\delta\alpha_2]^2}{2\mu_2 r(r+\beta)^2} + \frac{[\gamma_3(r+\beta)+\delta\alpha_3]^2}{2\mu_3 r(r+\beta)^2} \quad (19)$$

Theorem 9. In the tripartite cooperation model, the expected value and the expected stable value of the quality of age-appropriate retrofitting are as follows:

$$E[G^{M}(t)] = \frac{C}{\beta} - \frac{e^{-\beta t}}{\beta}(C - G_0\beta); \ \lim_{t\to\infty} E[G^{M}] = \frac{C}{\beta}$$
$$C = \frac{\alpha_1[\gamma_1(r+\beta)+\delta\alpha_1]}{\mu_1(r+\beta)} + \frac{\alpha_2[\gamma_2(r+\beta)+\delta\alpha_2]}{\mu_2(r+\beta)} + \frac{\alpha_3[\gamma_3(r+\beta)+\delta\alpha_3]}{\mu_3(r+\beta)} \quad (20)$$

The variance and variance stability values for the quality of the age-appropriate retrofitting are as follows:

$$D[G^{M}(t)] = \frac{\sigma^2\left[C - 2(C - G_0\beta)e^{-\beta t} + (C - 2G_0\beta)e^{-2\beta t}\right]}{2\beta^2}$$
$$\lim_{t\to\infty} D[G^{M}(t)] = \frac{C\sigma^2}{2\beta^2} \quad (21)$$

3.4 Comparative analysis

By comparing the non-cooperative model, subsidised cooperation model and tripartite cooperation model, the optimal decision-making and optimal benefits of elderly residents, contractors, and local governments, as well as the quality of aging renovation of established communities and the overall benefits of the tripartite ecosystem, the following conclusions can be drawn:

Theorem 10. The optimal strategies of elderly residents, engineers, and local governments in the three models are compared and analysed as follows: (i) the optimal strategy of the elderly resident, that is, the optimal level of effort $E_1^F < E_1^N \leq E_1^M$ (ii) the optimal strategy of the engineer, that is, the optimal level of effort $E_2^F < E_2^N \leq E_2^M$ (iii) the optimal strategy of the local government, that is, the optimal level of effort $E_3^F = E_3^N \leq E_3^M$ (iv) the subsidy coefficient provided by the local government for elderly residents $\varphi_1 = 1 - E_1^F/E_1^N = (2 - 3\vartheta_1 - 2\vartheta_2)/(2 - \vartheta_1 - 2\vartheta_2)$ (v) The subsidy coefficient provided by the local government for engineers $\varphi_2 = 1 - E_2^F/E_2^N = (2 - 2\vartheta_1 - 3\vartheta_2)/(2 - \vartheta_1 - 2\vartheta_2)$.

Corollary 1. The non-cooperative model without local government subsidies is compared with the subsidised cooperation model with local government subsidies. However, the efforts of the local government for elder-friendly community construction remain the same, and local government subsidies increase the motivation of elderly residents and engineers to invest in retrofitting; thus, the optimal strategy of elderly residents and engineers, that is, the optimal effort of elder-friendly community construction increases. This increase was equal to the value of the local government for the optimal subsidy coefficient for elderly residents and engineers.

Corollary 2. Compared with the subsidised cooperation model with local government subsidies, under the tripartite cooperation model, elderly residents, engineers, and the local government as a whole aim to maximise system benefits; therefore, all three parties are willing to make more efforts for the improvement of system benefits, and the optimal strategy of the three parties, that is, the optimal degree of effort, reaches the maximum. It can be seen that the cooperative development model, which adheres to the development concept of win-win cooperation and is supported by the participation of multiple forces and the sharing of development results, is an effective mechanism to promote the sustainable implementation of elder-friendly retrofitting.

Theorem 11. Under the non-cooperative model, subsidised cooperation model, and tripartite cooperation model, the expected value and stable value of elder-friendly retrofit quality are compared and analysed as follows: (i) Expected value of elder-friendly retrofit quality $E[G^F(t)] < E[G^N(t)] < E[G^M(t)]$; (ii)Stable value of elder-friendly retrofit quality $\lim_{t\to\infty} E[G^F(t)] < \lim_{t\to\infty} E[G^N(t)] < \lim_{t\to\infty} E[G^M(t)]$.

Corollary 3. On the one hand, appropriate government subsidies can increase the efforts of elderly residents and engineers and, therefore, promote the quality of elder-friendly retrofitting. On the other hand, the elderly residents, engineers, and the local government improve the benefits of the system through cooperative efforts; that is, the three parties play their roles to the maximum, avoiding frictions and interest disputes in the process of elder-friendly retrofitting, creating a synergistic benefit of 1+1>2 cooperation, and bringing additional benefits to the system so that the quality of elder-friendly retrofitting can reach the highest value under the tripartite cooperation model.

Theorem 12. The results of the comparative analysis of the optimal benefits of the elderly resident, engineer, local government, and the system under the three models are as follows: (i) the optimal benefits of the elderly resident $V_1^F(G) < V_1^N(G)$; (ii) the optimal benefit of the engineer $V_2^F(G) < V_2^N(G)$; (iii)The optimal benefit of the local government $V_3^F(G) < V_3^N(G)$; (iv)the optimal benefit of the system $V^F(G) < V^N(G) < V^M(G)$.

Corollary 4. In the subsidised cooperation model with local government subsidies, the optimal benefits of senior residents, engineers, local government, and the system are more significant than the relevant values in the non-cooperative model without local government subsidies. This indicates that government subsidies can adjust the relative relationship between the three parties of interest and improve the roles of senior residents and engineers in elder-friendly retrofitting, and therefore can promote the Pareto optimal improvement of the three parties. In the tripartite cooperation model, the direct connection of the three parties can form an effective resource-sharing system and a vast promotion body for elder-friendly community construction, which allows the tripartite cooperation system to reach the maximum level of total benefits, that is, the Pareto-optimal level.

Theorem 13. Under the non-cooperative model, subsidised cooperation model, and tripartite cooperation model, the results of the comparative analysis of Variance and variance stability value of elder-friendly retrofit quality are as follows: (i)Variance of elder-friendly retrofit quality $D[G^F(t)] < D[G^N(t)] < D[G^M(t)]$; (ii)Stable value of the variance of elder-friendly retrofit quality $\lim_{t\to\infty} D[G^F(t)] < \lim_{t\to\infty} D[G^N(t)] < \lim_{t\to\infty} D[G^M(t)]$.

Corollary 5. On the one hand, excessive use of government subsidies can cause engineers and elderly residents to depend on the government to the extent that it increases the formation of other types of risks in elder-friendly retrofits, such as improper use of government subsidies and engineers cutting corners, which leads to higher variance and variance stability values of elder-friendly retrofit quality. On the other hand, connecting the three parties to form a huge body of elder-friendly community construction promotion can maximise the benefits of the system. However, high risks accompany high benefits, and the three parties must bear the risk that any one of them may deviate from the main body; therefore, the variance and variance stability of the tripartite cooperation system are also the largest.

Theorem 14. The results of the comparative analysis of the efficiency expectation of the optimal benefit change of the tripartite ecosystem under the non-cooperative, subsidised, and tripartite cooperation models are as follows: $E[dV^F(G)/dt] < E[dV^N(G)/dt] < E[dV^N(G)/dt]$.

Corollary 6. Government subsidies increase communication and contact between the government and elderly residents and engineers, which can effectively reduce the friction between them during the retrofitting process. This can accelerate the efficiency of elder-friendly community construction. On the other hand, under the tripartite cooperation model, the three parties are connected as a whole, and the degree of connection between each participant is further deepened so that the possibility of friction is minimised, and the efficiency of elderly friendly community construction can be further improved.

4. Numerical simulation

Under the basic assumptions, the parameters were first set such that $r = 0.3$, $G_0 = 6$, $\mu_1 = \mu_2 = \mu_3 = 2$, $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.2$, $\gamma_1 = \gamma_2 = 5$, $\gamma_3 = 2$, $\delta = 2$, $\vartheta_1 = \vartheta_2 = 0.3$. Referring to Prasad and Sethi (2004), the analytical equations for the quality of age-appropriate retrofitting under the non-cooperative, subsidised cooperation, and tripartite cooperation models can be obtained by combining equations (6) and (10), equations (12) and (15), and equations (18) and (20) into equation (2) for discretisation as follows:

$$G^F(t+\Delta t) = \frac{A}{\beta} - \frac{e^{-\beta(t++\Delta t)}}{\beta}(A - G_0\beta) + [A - G_0\beta)]\Delta t + \sigma\sqrt{G^F(t)}\sqrt{\Delta t}\zeta(t)$$
$$G^N(t+\Delta t) = \frac{B}{\beta} - \frac{e^{-\beta(t++\Delta t)}}{\beta}(B - G_0\beta) + [B - G_0\beta)]\Delta t + \sigma\sqrt{G^N(t)}\sqrt{\Delta t}\zeta(t)$$
$$G^M(t+\Delta t) = \frac{C}{\beta} - \frac{e^{-\beta(t++\Delta t)}}{\beta}(C - G_0\beta) + [C - G_0\beta)]\Delta t + \sigma\sqrt{G^M(t)}\sqrt{\Delta t}\zeta(t) \quad (22)$$

where$\zeta(t)$ is the standard normally distributed quantity with identical independent distribution, and Δt is the step size. Further, the parameters $\Delta t = 0.09$, $\sigma = 10$ and $t \in [0,18]$; MATLAB's numerical simulation leads to the following Figures 1-2.

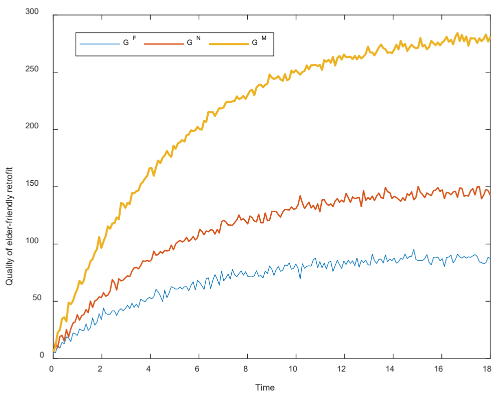

Figure 1. Comparison of the quality of elder-friendly retrofit
Source: This chart was drawn by the author.

The trends in elder-friendly community construction in the tripartite ecosystem under the three models over time are shown in Figure 1. It can be found that the quality of age-appropriate retrofitting improves and eventually tends to flattening out of fluctuations over time, and the trend of expected value is shown as $E[G^F(t)] < E[G^N(t)] < E[G^M(t)]$. This expectation trend is consistent with that described in Theorem 11: This indicates that the expectation of the quality of elder-friendly community construction is the lowest in the case of a non-cooperative model and increases when the local government provides subsidies to elderly reside through relevant data nts and engineers. Additionally, the expected value of the elder-friendly retrofit quality in the case of the tripartite cooperation model is much higher than that of the other two models, indicating that a cooperative mechanism with a unified goal can effectively mobilise resources in the ecosystem, improve the efficiency and effectiveness of elder-friendly retrofit quality, and enable the system to reach the maximum Pareto improvement.

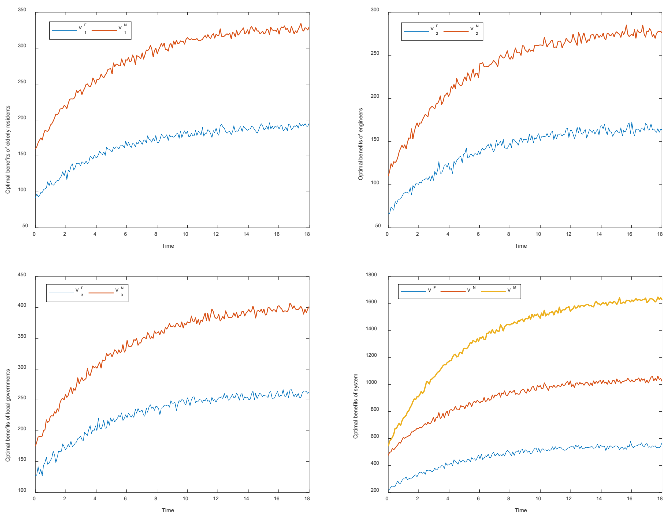

Figure 2. Comparison of optimal benefits

Source: This chart was drawn by the author through relevant data.

As shown in Figures 2, elderly residents, engineers, and local governments obtained Pareto improvements from the non-cooperative model to the subsidised cooperation model, consistent with that described in Theorem 12. Obviously, in the case of the subsidised cooperation model, the local government, as the leader, subsidises the elderly residents and engineers and shares the costs for the elderly residents and engineers, which can lead to an increase in the optimal benefits of the tripartite ecosystem. When the system tripartite redistributes the increase in the optimal benefits of the system, it contributes to the Pareto improvement of the ecosystem tripartite. Further, the optimal benefits of the tripartite ecosystem, from the non-cooperative model to the subsidised cooperation model to the tripartite cooperation model, reflect Pareto improvement, as described consistently in

Theorem 12. This indicates that as the degree of tripartite cooperation increases, the overall benefit to the ecosystem also increases. In the tripartite cooperation model case, each ecosystem member takes the overall benefit of the system as the starting point for decision-making, making all three parties work with the highest level of effort among the three model cases when the Pareto improvement of the system benefit reaches its maximum value. Additionally, Figure 2 shows that the optimal gain of the tripartite ecosystem is positively correlated with time, and the change in the earlier period is greater than that in the later period. The ranking of the change efficiency of the three models in high and low order is: the tripartite cooperation model, the subsidised cooperation model, and the non-cooperative model, as described consistently in Theorem 14. This indicates that under the three models, the system operation efficiency is effectively improved as the degree of connection among the three parties of the system is continuously strengthened.

5. Conclusions

Based on differential game theory, this study explores the optimisation of an elder-friendly retrofit ecosystem consisting of elderly residents, engineers, and local governments from a quantitative perspective. Using the HJB equation, we investigate the optimal effort, optimal benefit, expectation, and variance of the quality of elder-friendly community construction of elderly residents, engineers, and local governments in three scenarios: non-cooperative, subsidised cooperation, and tripartite cooperation models by constructing a three-way dynamic stochastic differential game model. The following conclusions are drawn: First, the local government can effectively increase the optimal effort of elderly residents and engineers by sharing the costs of elderly residents and engineers through subsidies, which is equal to the optimal cost-sharing ratio of the local government. Therefore, government subsidies can serve as an effective regulatory mechanism to significantly improve the efforts of elderly residents and contractors, thus increasing the quality and efficiency of elder-friendly community construction, which has a Pareto improvement effect. Under the tripartite cooperation model, the optimal strategy of the three parties and the optimal degree of effort reach a maximum, and the system's Pareto improvement also reaches a maximum. Second, in terms of the quality of the aging retrofit and the total benefits of the ecosystem, government subsidies as an incentive can effectively improve the total benefits of the ecosystem. However, its variance simultaneously increases, indicating that an increase in benefits requires a certain degree of risk. The tri-partnership model maximises the Pareto improvement of the system compared to the subsidised cooperation and non-cooperative models; however, the risk is also maximised.

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