Maryam ARBABI, PhD Candidate
E-mail: marbabi@gmail.com
Department of Mathematics, Islamic Azad University, Qazvin Branch, Qazvin, Iran

Zohreh MOGHADDAS, PhD
(corresponding author)
E-mail: zmoghaddas@qiau.ac.ir
Department of Mathematics, Islamic Azad University, Qazvin Branch, Qazvin, Iran

Alireza AMIRTEIMOORI, PhD
E-mail: aamirteimoori@riau.ac.ir
Department of Mathematics, Islamic Azad University, Rasht Branch, Rasht Iran

Mohsen KHUNSIAVASH, PhD
E-mail: msiavash@qmail.com
Department of Mathematics, Islamic Azad University, Qazvin Branch, Qazvin, Iran

INVERSE MODEL OF NETWORK DATA ENVELOPMENT ANALYSIS: REALLOCATION OF RESOURCES IN HOSPITALS

Abstract. *Data envelopment analysis (DEA) is a mathematical programming model to evaluate the relative efficiency of a set of homogeneous decision-making units considered to be 71 hospitals. Due to the importance of the sensitivity analysis in optimisation problems, a new extension of DEA models called inverse DEA (IDEA) model has been proposed. The purpose of this model is to analyse the sensitivity of some of the inputs or outputs as a result of the changes in some other inputs or outputs of the unit under evaluation, provided that the amount of efficiency remains constant or improves at the discretion of the manager. In this paper, for the first time, we intend to introduce the inverse model in data envelopment analysis with network structure. To do so, we will examine the extent to which the input parameters can change with regard to the presuppositions of the problem, for the output changes that are applied according to the manager. One of the key features of the current study is that to make the modelling more consistent with reality, the leader-follower method has been used to estimate the parameters in the network. In addition, the opinion of the manager and the decision maker in the system, who have controls over the system under their*

DOI: 10.24818/18423264/57.4.23.19

guidance and management, is considered in this modelling to estimate the values in question. Another property of this modelling is the inclusion of uncontrollable factors in the inverse model in data envelopment analysis with network structure. Finally, using some applications, the obtained results are analysed based on the proposed model in hospitals.

Keywords: *Network Data Envelopment Analysis, Inverse Data Envelopment Analysis, hospitals, technical efficiency, uncontrollable factors*

JEL Classification: G51, O16, C45, N20

1. Introduction

One of the famous techniques for the efficiency evaluation of a number of decision-making units (DMUs) with multiple inputs and outputs is Data Envelopment Analysis (DEA). DEA was first introduced by Charnes et al (1987). In 1978 based on Farrell's study (Farell, 1957). The model proposed by Charnes et al. (1978), called CCR, was based on the assumption of constant returns to scale. Later, different model was introduced assuming a variable return to scale. This technique has attracted the attention of many researchers both in theory and practice.

In traditional data envelopment analysis models in which the decision-making unit is considered irrespective of the interaction between its processes, a unit may be efficient while its components and processes are inefficient. Because of the weakness of traditional models in considering the internal structure of units, great efforts have been made by the researchers to develop the traditional models so that they can examine the internal structure of multiplier units (Kao, 2017). To troubleshoot the traditional and independent models, Fare and Grosskopf (2000) introduced network data envelopment analysis models that examine the operation of component processes in estimating the efficiency of the system under evaluation. Traditional data envelopment analysis models treat decision-making units as a "black box" and they do not consider the internal processes and the efficiency of these processes, so that the input enters the decision unit and the output is obtained. Therefore, considering the decision-making unit as a whole, they calculate its efficiency. While the inefficiency of each decision-making unit often stems from the inefficiency of the internal parts of the decision-making unit, therefore, network data envelopment analysis method was presented to evaluate the efficiency of the similar decision-making units according to their internal processes (Zhu, 2003; Kao et al., 2018; Kao et al., 2017; Kao, 2009; Chen et al., 2013; Lozano et al., 2013; Lozano, 2011; Liu et al., 2015).

In numerous studies, the researchers have examined the factors affecting the value of efficiency. Their purpose was to study and estimate the effect of factors affecting the amount of efficiency, with the assumption that the efficiency of the unit under evaluation does not change. These studies are known as inverse data envelopment analysis. Wei et al. (2000) introduced inverse data envelopment analysis models and presented the first model in this field.

They posed the following questions and sought to answer them. If all or some of the outputs of a decision-making unit increase, how much should the inputs of this unit increase (over-consumption of input) while the efficiency of the decision-making unit remains the same? Another question was that: If all or some of the inputs of a decision-making unit increase, how much should the outputs of this unit increase (overproduction) without any changes in the efficiency of the decision-making unit. They identified the necessary and sufficient conditions in order to maintain the efficiency of the unit under evaluation.

Following this strand of research, Yan et al. (2002) adopted a different approach to examine the models of inverse data envelopment analysis and answered the questions posed by Wei et al. (2000). To answer the questions posed by Wei et al. (2000) to estimate the amount of inputs (outputs) when the outputs (inputs) are altered. Several of important articles about this subject are such as; Jahanshahloo et al. (2009) considered a certain percentage of efficiency improvement for the decision-making units and proposed several models, accordingly. Also, Lertworasirikul et al. (2011), Jahanshahloo et al. (2014), Mirsalehi et al. (2014), Hadi Vencheh et al. (2008), Ghiyasi (2017), Ghobadi (2018), Wegener et al. (2019), Ghiyasi and Khoshfetrat (2019), Soltanifar et al. (2022) presented a new model for inverse DEA modelling.

The main contributions of this study are as follows. The innovative idea of this study is to present IDEA models in network structural systems. This study considered the leader-follower method for considering the relations between the stages in parameter estimations. An application in healthcare systems (hospitals) is presented and the results are analysed for future strategy plannings. Non-discretionary data are considered in IDEA modelling.

The rest of the paper is organised as follows: In the next section, the prerequisites of presenting the main model will be stated. The main ideas and modelling proposed in this study are provided in section three. In section four, the features and benefits of the proposed model will be explained by giving numerical examples. Finally, section five includes conclusions and suggestions.

2. Inverse model of data envelopment analysis with network structure

In recent years, data envelopment analysis has been one of the key tools in the area of performance analysis of organisations and activities in various fields. This has resulted in the expansion of the applications of this technique with an increasing growth in its theory along with the introduction of more advanced models. One such model is the inverse model in data envelopment analysis, which was first proposed by Wei et al. (2000). They introduced this model for the estimation of input or output after making changes to the outputs or inputs, while the amount of efficiency remains intact. After being introduced in the literature, this model quickly drew the attention of researchers and has since been modified and developed from different aspects.

The main purpose of this section is to introduce an inverse model of data envelopment analysis with network structure. To do so, we first consider a two-stage series network in Figure 1. It should be mentioned that since this study is the first attempt in data envelopment analysis literature, we decided to consider a two-stage series network. In future studies, different types of network models can be considered for evaluation using inverse data envelopment analysis models.

To evaluate the technical efficiency of the network introduced in Figure 1, we consider the input-oriented radial model with constant returns to scale. Therefore, according to the changes desired by the manager or the decision maker, which is to increase the outputs, given a constant or increasing efficiency at the discretion of the manager, the purpose is to estimate input values in the network.

Consider DMU_j $(j = 1, \ldots, n)$as the jth decision-making unit, which includes input $x_{ij}(i = 1, \ldots, m)$, intermediate products, $z_{dj}(d = 1, \ldots, 1)$the output $y_{rj}(r = 1, \ldots, s)$. Further assume that DMU_oin which $o \in \{1, \ldots, n\}$is the unit under evaluation. λ_j^1 and λ_j^2 for each j = 1, ..., n represent the corresponding intensity coefficients of the first and the second components of the network, respectively. Also, θ is a free sign variable, whose function is to produce a radial reduction in the independent input of network x. Many studies have been conducted on network DEA recently (Soltanifar et al., 2022). Consider Figure 1, where a two-stage network system with a series structure is shown.

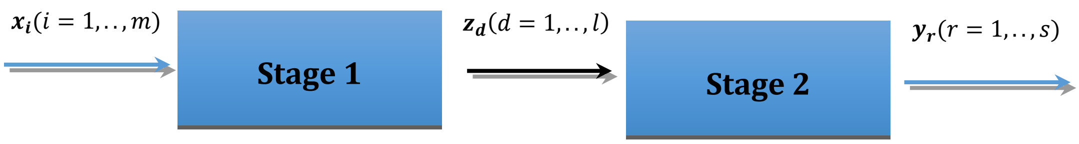

Figure 1. A two-stage network system with series structure

Consider the data envelopment analysis model with network structure in the input-oriented as follows.

$$
\begin{array}{lll}
 & \min\ \theta & \\
s.t & \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta x_{io}, & i = 1, \ldots, m \\
 & \sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, & d = 1, \ldots, l \\
 & \sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, & d = 1, \ldots, l \\
 & \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}, & r = 1, \ldots, s \\
 & \lambda_j^1 \geq 0\ ,\ \lambda_j^2 \geq 0, & j = 1, \ldots, n, k = 1,2
\end{array}
\tag{1}
$$

After solving model (1), it is concluded that θ^* represents the amount of technical efficiency that indicates the maximum radial reduction of the input until it reaches the efficiency limit. Now, if the manager or decision maker intends to increase the outputs from y to y^N, to estimate the independent network input, i.e., x, model (2) is introduced. Note that in this model θ^* is the efficiency obtained from model (1). The main purpose of model (2) is to estimate all the components of the independent input vector in the network, namely, x.

But because minimising each of the components $x_i (i = 1, \ldots, m)$ yields a model with m objective functions, model (2) is a multi-objective optimisation model. The basic idea of Model (2) is based on the first idea introduced by Wei et al. [21]. In model (2), the corresponding variable vector of the input vector $x = (x_1, \ldots, x_m)$is vector β which has m components as $\beta = (\beta_1, \ldots, \beta_m)$. According to Wei et al. [21], to estimate inputs, for each m input component the condition $\beta_i \geq x_{io}$ is considered. The linear model (2) is introduced as follows:

$$
\begin{aligned}
\min \quad & \textstyle\sum_{j=1}^{m} \beta_i \\
s.t \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^* \beta_i, && i = 1, \ldots, m \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, && d = 1, \ldots, l \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, && d = 1, \ldots, l \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, && r = 1, \ldots, s \\
& \beta_i \geq x_{io} \geq 0, && i = 1, \ldots, m \\
& \lambda_j^1 \geq 0\ ,\ \lambda_j^2 \geq 0, && j = 1, \ldots, n, k = 1,2
\end{aligned}
\tag{2}
$$

For the efficiency improvement range that can be considered by managers is:

$$0 \leq \theta \leq 1 - \theta^*, 0 < \theta^* \leq 1 \tag{3}$$

Considering relation (3) the improved model (4) is as follows.

$$
\begin{aligned}
\min \quad & \textstyle\sum_{j=1}^{m} \beta_i \\
s.t \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq (\theta^* + \theta) \beta_i, && i = 1, \ldots, m, (a) \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, && d = 1, \ldots, l,\ (b) \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, && d = 1, \ldots, l,\ (c) \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, && r = 1, \ldots, s, (d) \\
& \beta_i \geq x_{io} \geq 0, && i = 1, \ldots, m\ (e) \\
& \lambda_j^1 \geq 0\ ,\ \lambda_j^2 \geq 0, && j = 1, \ldots, n, k = 1,2
\end{aligned}
\tag{4}
$$

Note that in model (4), the intermediate products are not assumed to be variable. In this case, a new value may not be derived to estimate x because if y increases to y^N then λ_2 can take different values depending on the new output value. But since λ_2 should also help the constraint (c) to hold, λ2 cannot take a different value. In this case, no estimate can be obtained for x.

According to model (4), model (5) is then presented in which the intermediate products are also considered to be variable. After increasing y to y^N by the decision maker, in model (5), the goal is to estimate the values corresponding to z and x provided that the efficiency value remains constant or improves. Thus, the corresponding vector of the input variable $x = (x_1, \ldots, x_m)$ is the vector $\beta = (\beta_1, \ldots, \beta_m)$ and the corresponding variable vector of the intermediate product $z = (z_1, \ldots, z_l)$ is the vector $\gamma = (\gamma_1, \ldots, \gamma_l)$. Borrowing from Wei et al. [21], to estimate inputs and intermediate products, the conditions $\beta_i \geq x_{io}$and $\gamma_d \geq z_{do}$are assumed to hold for each input component i = 1, …, m and each intermediate product component $d = 1, \ldots, l$,. Note that minimising each of the components $x_i (i = 1, \ldots, m)$and $z_d (d = 1, \ldots, l)$results in a model with $m + l$ objective functions. The linear model (5) is then introduced as follows:

$$
\begin{array}{lll}
\min \sum_{j=1}^{m} \beta_i + \sum_{d=1}^{l} \gamma_d & & \\
s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq (\theta^* + \theta)\beta_i, & i = 1, \ldots, m & \\
\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq \gamma_d, & d = 1, \ldots, l & \\
\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq \gamma_d, & d = 1, \ldots, l & \\
\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, & r = 1, \ldots, s & (5) \\
\beta_i \geq x_{io} \geq 0, & i = 1, \ldots, m & \\
\gamma_d \geq z_{do} \geq 0, & d = 1, \ldots, l & \\
\lambda_j^1 \geq 0 \, , \, \lambda_j^2 \geq 0, & j = 1, \ldots, n, k = 1,2 &
\end{array}
$$

For more comprehensive evaluation of inverse analysis using DEA technique according to the network structure shown in Figure 1, models (6) and (7) are presented to evaluate the technical efficiency of the second and the first components of the network in output and input- oriented, respectively.

$$
\begin{array}{lll}
max \ \varphi & & \\
s.t \quad \sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, & d = 1, \ldots, l & \\
\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq \varphi y_{ro}, & r = 1, \ldots, s & (6) \\
\lambda_j^2 \geq 0, & j = 1, \ldots, n &
\end{array}
$$

$$
\begin{array}{lll}
min \ \theta & & \\
s.t \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta x_{io}, & i = 1, \ldots, m & \\
\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, & d = 1, \ldots, l & (7) \\
\lambda_j^1 \geq 0 & j = 1, \ldots, n &
\end{array}
$$

Also, the changes in the estimation of the components of the vector z are restricted such that for every $d = 1,\dots,l$ we have $(0.9)z_{do} \leq z_{do} + \gamma_d \leq (1.1)z_{do}$. Note that the lower and upper bounds of 0.9 and 1.1 are respectively selected according to the opinion of the managers or decision makers who have a good knowledge of the system under their supervision. The values of 0.9 and 1.1 for the lower and upper bounds are determined by the manager or decision maker. The following models (8) and (9) are introduced by considering the leader-follower method.

$$
\begin{array}{llll}
& \min \sum_{d=1}^{l} \gamma_d & & \\
s.t & \sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq (z_{do} + \gamma_d), & d = 1,\dots,l & \\
& \sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq \varphi^* y_{ro}^N, & r = 1,\dots,s & (Leader) \\
& (0.9)z_{do} \leq z_{do} + \gamma_d \leq (1.1)z_{do}, & d = 1,\dots,l & \\
& \lambda_j^2 \geq 0, & j = 1,\dots n & (8) \\
& \gamma_d \; free \;\; in \;\; sign, & d = 1,\dots,l &
\end{array}
$$

Regarding the increase of the output applied by the decision maker, the intermediate products are estimated within the pre-introduced bound interval. As a result, the estimation of each component of the intermediate product vector is obtained as $(z_{do} + \gamma_d^*)$, $d = 1,\dots,l$.

As the intermediate products change from z to $z^N = z + \gamma^*$. Therefore, if the input-oriented technical efficiency of the first component of the network remains constant or improves, to estimate the independent inputs of the first stage of the network, model (13) is introduced as follows. Also, the changes in the estimation of vector components are limited so that for each $i = 1,\dots,m$ we have $(0.9)x_{io} \leq x_{io} + \beta_i \leq (1.1)x_{io}$. Note that the lower and upper bounds 0.9 and 1.1 are selected according to the opinion of the managers or decision makers who sufficiently know the system they lead.

$$
\begin{array}{llll}
& \min \;\; \sum_{j=1}^{m} \beta_i & & \\
s.t & \sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^*(x_{io} + \beta_i), & i = 1,\dots,m & \\
& \sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq (z_{do} + \gamma_d^*), & d = 1,\dots,l & \text{(Follower)} \\
& (0.9)x_{io} \leq x_{io} + \beta_i \leq (1.1)x_{io}, & i = 1,\dots,m & (9) \\
& \lambda_j^1 \geq 0, & j = 1,\dots,n & \\
& \beta_i \;\; free \;\; in \;\; sign, & i = 1,\dots,m &
\end{array}
$$

From the optimal solution obtained from model (9), the estimated input vector with components $(x_{io} + \beta_i^*)$, $i = 1,\dots,m$ is obtained. The leader-follower method has other modes in which the inverse model of data envelopment analysis in network

structure can be evaluated under the conditions mentioned above considering the general conditions of the system set by the manager.
In model (11) inputs are divided into controllable and non-controllable categories, $i_1 \in D, i_2 \in ND$.

$$\begin{aligned}
&\max\ \varphi \\
&s.t \quad \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le z_{do}, && d = 1, \dots, l \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge \varphi y_{ro}, && r \in R_1 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}, && r \in R_2 \\
&\quad \lambda_j^2 \ge 0, && j = 1, \dots, n \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\min \theta \\
&s.t \quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le \theta x_{io}, && i \in I_1 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le x_{io}, && i \in I_2 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge z_{do}, && d = 1, \dots, l \\
&\quad \lambda_j^1 \ge 0\ , && j = 1, \dots, n \qquad (11)
\end{aligned}$$

In this case, at the discretion of the manager, the output vector increases from y to yN. To improve the output-oriented technical efficiency, the following relation should be used. φ is the amount considered for efficiency improvement.

$$0 < \frac{1}{\varphi^*} + \varphi \le 1 - \frac{1}{\varphi^*} \qquad (12)$$

Where $0 < \frac{1}{\varphi^*} \le 1$ and if $\varphi^* = 1$ then there is no chance to improve the efficiency and the result is $\varphi = 0$. If the manager aspires to improve the efficiency, then the improvement interval is $(0, 1 - \frac{1}{\varphi^*}]$, $\varphi \in (0, 1 - \frac{1}{\varphi^*}]$ and clearly $(\frac{1}{\varphi^*} + \varphi) \le 1$.

$$\begin{aligned}
&\min \quad \textstyle\sum_{d=1}^{l} \gamma_d \\
&s.t \quad \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le (z_{do} + \gamma_d), && d = 1, \dots, l \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge \varphi^* y_{ro}^N, && r = 1, \dots, s \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}^N, && r = 1, \dots, s \qquad (Leader) \\
&\quad (0.9) z_{do} \le z_{do} + \gamma_d \le (1.1) z_{do}, && d = 1, \dots, l \qquad (13) \\
&\quad \lambda_j^2 \ge 0, && j = 1, \dots n \\
&\quad \gamma_d\ free\ \ in\ \ sign, && d = 1, \dots, l
\end{aligned}$$

$$\begin{aligned}
&\min \textstyle\sum_{j=1}^{m} \beta_i \\
&s.t \quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le (\theta^* + \theta\ )(x_{io} + \beta_i), && i \in I_1 \\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le x_{io}, && i \in I_2
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq (z_{do} + \gamma_d^*), \qquad d = 1, \ldots, l \qquad (Follower)$$
$$(0.9)x_{io} \leq x_{io} + \beta_i \leq (1.1)x_{io}, \qquad i \in I_1 \qquad (14)$$
$$\lambda_j^1 \geq 0\,, \qquad j = 1, \ldots, n$$
$$\beta_i \ \ free \ \ in \ \ sign, \qquad i \in I_1$$

Considering the optimal solution obtained from models (13) and (14), the estimated value of the controllable components of the input vector is obtained as $x_{io} + \beta_i^*, i \in I_1$. The estimates of the components of the vector x are constrained so that for every $i \in I_1$we have $(0.9)x_{io} \leq x_{io} + \beta_i \leq (1.1)x_{io}$. Note that the lower and upper bounds, 0.9 and 1.1 are selected at the discretion of the managers or decision makers who have a thorough knowledge of the system under their control. The first and second bundles of constraints are considered for discretion and non-discretion inputs of stage 1. The third bundle of constraint is considered for the outputs of stage 1. The fourth bundles of constraint show the bound for input estimations.

3. Application

In this subsection, we evaluate the efficiency in the Iranian hospitals. Iran Social Security is a service-oriented organisation that enjoys the largest number of hospitals in Iran. In this sub-section, we use the presented models to evaluate the social security hospitals in Iran. Indexes that are used to evaluate the hospitals include financial performance evaluation indexes, obtained from the previous studies, or through interviews with the experts and information available from the Social Security Organisation. These indices are as follows: input, output, and intermediate: Input variable: Personnel costs (including salaries and wages of employees working in each hospital). Administrative costs (including the costs of purchasing medical equipment in the hospital to serve the patients). The cost of transfer payments (costs that the government allocates to the hospital for public health). Intermediate variable: The cost of doctors' benefits and case-based payments (including rewards and fee-for-service allocated to physicians to provide the best possible service). The cost of employees' motivational benefits and their case-based payments (including rewards and fee-for-service allocated to the employees to provide the best possible service). Output variable: Inpatients' medical bill total (the amount of money received to serve the hospitalised patients in each hospital). Outpatients' medical bill total (the amount of money received to serve the patients treated on an outpatient basis). Medical function total (income derived from the sale of medical devices to patients).

Here, the decision-making units responsible for efficiency evaluation are the hospitals of the Social Security Organisation, which include 70 hospitals. The information related to the input, intermediate, and the output indexes of the hospitals is for the year 2019. This information was extracted from the database of the Social Security Organisation. The descriptive statistics of these indexes are given in Table (1) below.

Table 1. Descriptive statistics of Input, intermediate and output indicator

Inputs and outputs	SD	Min	max	Median	Mean	Index
Personnel Cost	217900066432.00	100308729308.00	1010636663692.00	357353755868.00	409331610535.00	I_1
Administrative Cost	129826599850.00	47666689822.00	675940502602.00	148456431631.00	182022508509.00	I_2
Transfer Payments Cos	19478592316.00	304460000.00	114833025662.00	13416920061.00	18155001554.00	I_3
Cost of doctors' benefits and case-based payments	57482124764.00	21568066941.00	275430849907.00	97254081332.00	111803444016.00	z_1
Cost of employees' motivational benefits and case-based payments	43337442003.00	11157885649.00	201450841612.00	69297624895.00	74927109139.00	z_2
Inpatients' medical bill total	256713963000.00	23174020848.00	1258965108626.00	293494582437.00	333811465916.00	o_1
Outpatients' medical bill total	136411237002.00	35756944246.00	1055619199512.00	140554743156.00	167767197235.00	o_2
Medical function total	350280036308.00	106842397844.00	1620573606349.00	449377178901.00	523403639849.00	o_3

After the necessary data are collected, we will analyse the data and will implement the research model. To this end, we use a process similar to the numerical example given in the previous subsection and analyse the obtained results accordingly.

Consider the output data as listed in Table (2). According to the opinion of the decision-makers in the social security organisation, outputs are increased. The index is changed into y^N. Consider the Descriptive statistics data of increased outputs shown in Table (2).

Table 2. Descriptive statistics data

Index	Mean	Median	Max	Min	SD
o_1	333811465916	293494582437	1258965108626	23174020848	256713963000
o_2	167767197235	140554743156	1055619199512	35756944246	136411237002
o_3	523403639849	449377178901	1620573606349	106842397844	350280036308
O_1^N	358280464370	320644623401	1133068597763	25491422933	269068854414
O_2^N	180877277545	154069232518	950057279561	39332638671	132183948547
O_3^N	559975907039	494000803792	1525616461669	117526637628	359237926342

We first calculate the initial efficiency of the hospitals. Considering the efficiency score constant, and the increased outputs, we wish to obtain the amount of change in the inputs. All Descriptive statistics of the obtained results are given in Table (3) below.

Table 3. Descriptive statistics Optimal Results

Index	Mean	Median	Max	Min	SD
Efficiency	0.88	0.89	1.00	0.57	0.092
o_2	405224037162.39	350142584736.00	1010636663692.00	100308729308.00	216766026222.038
I_1^*	180907291743.68	147498958588.00	675940502602.00	47666689822.00	130430675143.401
I_2^*	18165424296.14	13398029921.00	114833025662.00	304460000.00	19619039998.846
I_3^*	0.88	0.89	1.00	0.57	0.092

In the second column of Table (3), the efficiency of each hospital has been calculated. In the third to fifth columns, the initial outputs, and in the sixth to eighth columns, the modified outputs have been given. The inputs corresponding to this output change are then calculated by maintaining the efficiency constant, the results of which are given in Table (3).

In Table (2) only the changes of input indexes are calculated according to the changes of output indices, and the intermediate indexes are considered to be constant.

Table 4. The Descriptive statistics values of data

Index	SD	Min	Max	Median	Mean
I_1^*	182361067244.49	60801000000.00	961290000000.00	201870000000.00	249425942028.99
I_2^*	87773557051.03	28893000000.00	477270000000.00	95929000000.00	119009028985.51
I_3^*	4106860402.37	701358000.00	33242000000.00	2397180000.00	3398642057.97
z_1^*	58872723495.82	23935000000.00	283910000000.00	75135000000.00	88437507246.38
z_2^*	20167386844.56	2550550000.00	136070000000.00	15454000000.00	21655812753.62

The values of the new inputs are given, and the new intermediate indexes are also presented. It should be noted that the efficiency scores are assumed constant.

Using model (8) and restricting the intermediate indices by adding constraint$(0.9)z_{do} \leq z_{do} + \gamma_d \leq (1.1)z_{do}$, the amount of the change of the intermediate indices is obtained. Also, using model (9) and restricting the input indices and by adding constraint$(0.9)z_{do} \leq z_{do} + \gamma_d \leq (1.1)z_{do}$, the amount of changes of input indexes is obtained. The results of the model implementation are given in Table (5). In Table (5), "Output Effi" and "Input Effi" show the efficiency scores respectively in output and input orientations.

Table 5. The Descriptive statistics Input and intermediate values obtained from models (8) and (9) with YN

Index	SD	Min	Max	Median	Mean
Output Effi	0.43	1.00	3.38	1.54	1.59
z_1^*	61686015394.49	23725000000.00	284620000000.00	105540000000.00	121416710144.93
z_2^*	45193662658.33	5460780000.00	196990000000.00	70797000000.00	77832662753.62
Input Effi	0.10	0.62	1.10	0.96	0.96
I_1^*	217149959148.52	100310000000.00	1010600000000.00	348630000000.00	394455362318.84
I_2^*	116739890958.51	47667000000.00	608350000000.00	138480000000.00	176451159420.29
I_3^*	17578396010.96	334906000.00	103350000000.00	12208000000.00	16762236173.91

The values of change in the intermediate indices using model (8) are given in Table (4). To compare the efficiencies before and after the change in the indices, the output-oriented efficiencies are also given. As it is assumed, the efficiency remains constant. In addition, the amount of change in the input indices after the implementation of model (9) is given in Table (4). Also, the values of the input-oriented efficiency before and after this change are given, respectively.

Finally, the second and the third inputs are considered to be non-controllable. Using models (12) and (13), the changes in the values of the intermediate and the first input indices are calculated, respectively. The results of this calculation can be seen in Table (6) below. In Table (6), Descriptive statistics of "Output Effi" and "Input Effi" show the efficiency scores, respectively, in output and input orientations.

Table 6. The Descriptive statistics input and the intermediate values obtained from models (12) and (13) with Y^N

Index	SD	Min	Max	Median	Mean
Output Effi	1.66	1.00	9.63	1.91	2.46
z_1^*	62571703182.71	23724870000.00	302974100000.00	105535600000.00	121357952608.70
z_2^*	46546095071.77	5460783000.00	196994200000.00	70796880000.00	78093813378.26
Input Effi	0.10	0.59	1.10	0.89	0.90
I_1^*	216766042978.80	100308700000.00	1010637000000.00	350142600000.00	405224040579.71

As in Tables (4), (5) and (6), the values of the change of the intermediate indices are given and the values of the output-oriented efficiency before and after this change are also presented, respectively. As can be seen, the values of the efficiency in these two columns are equal indicating that efficiency does not change after the changes. Since the second and the third inputs are non-controllable, here we have to modify only the first input, as given in the Table (6). In addition, the input-oriented efficiencies before and after this change are given in the sixth and eighth columns, respectively, which are equal, indicating that the efficiency remains the same after the change.

4. Conclusions

The diversity of modelling in data envelopment analysis in general and network data envelopment analysis in particular is due to the real-world problem and the need for evaluation systems. In practical examples of network data envelopment analysis models, there exist non-controllable factors. Therefore, consideration of these factors in inverse data envelopment analysis models with network structure can also be of utmost importance. Because no case has been made for the inverse data envelopment analysis models with network structure in the presence of non-controllable in the previous studies, it is important to introduce this issue and consider these factors simultaneously so that a feasible and practical model can be achieved in terms of mathematical modelling in data envelopment analysis models.

In this paper, we presented models for inverse data envelopment analysis with a network structure with non-controllable inputs. Considering the changes in the outputs or inputs made by the manager, assuming a constant efficiency value, the estimates of optimal values of inputs, outputs, or intermediate products in a network was obtained. Due to the existence of non-controllable inputs, such indexless were considered in network models in inverse data envelopment analysis. Finally, the proposed models were implemented in an application of the proposed molds in the efficiency evaluation of hospitals was presented and the results were analysed. For limitations of the presented method, it can be mentioned that the presented model is in output orientation. For dealing with inputs and output estimations, an input orientation model should be considered. Also, it is important to conduct future research to estimate the inputs, outputs, or intermediates by considering the profitability being constant. It can also be interesting to consider stochastic data in the proposed models according to its variety of applications, as performed presence of fuzzy and stochastic data.

REFERENCES

[1] **Charnes, A., Cooper. W.W., Rhodes. E, (1987),** ***Measuring the Efficiency of Decision Making Units***. *European Journal of Operational Research*, 2, 429–444;

[2] **Chen, Y, Cook, W.D., Chiang, K., Zhu, J. (2013),** ***Network DEA pitfalls: Divisional Efficiency and Frontier Projection Under General Network Structures***. *European Journal of Operational Research*, 226, 507–515;

[3] **Farell, M.J. (1957),** ***The Measurement of Productive Efficiency***. *Journal of Royal Statistical Society*, 120, 253–281;

[4] **Fare, R, Grosskopf, S. (2000),** ***Network DEA***. *Socio-Economic Planning Sciences*, 34, 35–49;

[5] **Ghiyasi, M. (2017),** ***Inverse DEA based on cost and revenue efficiency.*** *Computers & Industrial Engineering*, 114, 258–263;

[6] **Ghobadi, S. (2018),** ***Inverse DEA Using Enhanced Russell Measure in the Presence of Fuzzy Data***. *International journal of industrial mathematics*, 10(2);

[7] **Ghiyasi, M., Khoshfetrat, S. (2019),** ***Preserve the relative efficiency values: an inverse data envelopment analysis with imprecise data***. *International Journal of Procurement Management*, 12(3), 243–257;

[8] **Hadi-Vencheh, A., Foroughi. A.A., Soleimani-Damaneh, M. (2008),** ***A DEA model for resource allocation***. *Economic Modelling*, 25(5), 983–993;

[9] **Jahanshahloo G.R., Hosseinzadeh Lotfi F, Shoja. N, Tohidi G., Razavyan, S. (2004),** ***Input estimation and identification of extra in inverse DEA models.*** *Applied Mathematics and Computation*, 156, 427–437;

[10] **Jahanshahloo G.R., Hosseinzadeh Lotfi F., Rostamy-Malkhalifeh M., Ghobadi, S. (2014),** ***Using Enhanced Russell Model to Solve Inverse Data Envelopment Analysis Problems***. *The Scientific World Journal*, 2014(1);

[11] **Kao, C. (2017),** ***Network Data Envelopment Analysis: a review.*** *European Journal of Operation Research*, 239, 1–16;

[12] **Kao, C., Hwang, S.N. (2008),** ***Efficiency Decomposition In Two-Stage Data Envelopment Analysis. An Application to Non-Life Insurance Companies In Taiwan***. *European Journal of Operational Research*, 185, 418–429;

[13] **Kao, C. (2009),** ***Efficiency Decomposition in Network Data Envelopment Analysis: A Relational Model***. *European Journal of Operational Research*, 192, 949–962;

[14] **Lertworasirikul, S., Charnsethikul, P., Fang, S. (2011),** ***Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale***. *Computers & Industrial Engineering*, 61, 1017–1023;

[15] **Lozano, S., Gutiérrez, E., Moreno, P., (2013),** ***Network DEA Approach to Airports Performance Assessment Considering Undesirable Outputs***. *Applied Mathematical Modeling*, 37(4), 1665–1676;

[16] **Lozano, S. (2011),** ***Scale and Cost Efficiency Analysis of Networks of Processes.*** *Expert System with Applications*, 38, 6612–6617;

[17] **Liu, W., Zhongbao, Z., Chaoqun, M., Debin L., Wanfang S. (2015),** ***Two-stage DEA Models with Undesirable Input-Intermediate-Outputs***. *Omega*, 56, 74–87;

[18] **Mirsalehy, A., Abu Bakar Mohd R., Lai, S.L., bin Jaafar, A., Heydar, M. (2014),** ***Directional Slack-Based Measure for the Inverse Data Envelopment Analysis***. *The Scientific World Journal*, 2014 (6), Article ID 138923;

[19] **Soltanifar, M., Ghiyasi, M., Sharafi, M. (2022),** ***Inverse DEA-R models for merger analysis with negative data***. *IMA Journal of Management Mathematics*, dpac001, https://doi.org/10.1093/imaman/dpac001;

[20] **Wegener, M., Amin R. (2019),** ***Minimizing greenhouse gas emissions using inverse DEA with an application in oil and gas***. *Expert Systems with Applications*, 122, 369–375;

[21] **Wei, Q.L., Zhang, J., Zhang, X. (2000),** ***An inverse DEA model for input/output estimate.*** *European Journal of Operational Research*, 121(1), 151–163;

[22] **Yan. H, Wei. Q, Hao, G. (2002),** ***DEA models for resource reallocation and production input/output estimation.*** *European Journal of Operational Research*, 136, 19–31;

[23] **Zhu, J. (2003),** ***Quantitative Models for Performance Evaluation and Benchmarking: Data Envelopment Analysis with Spreadsheets***. *Kluwer Academic Publishers*, Boston.