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CROSS-EFFICIENCY EVALUATION UNDER NON-DECREASING RETURNS TO SCALE AND WEIGHT RESTRICTIONS

Abstract. *Data Envelopment Analysis (DEA) is a powerful technique for estimating the efficiency of decision-making units (DMUs) by assigning appropriate weights. To enhance the discriminative capabilities of DEA in evaluating DMUs, the cross-efficiency evaluation method has been introduced, based on peer evaluations. However, a significant challenge in cross-efficiency analysis lies in the non-uniqueness of optimal weights. In this paper, we propose a novel approach to cross-efficiency evaluation, emphasising the selection of weighted profiles within non-decreasing returns to scale (NDRS) production technologies. This approach introduces conditions to ensure non-zero weights and reduce weight disparities. Additionally, we present two optimisation models from benevolent and aggressive viewpoints as secondary goals. To illustrate the effectiveness of the new approach, we provide numerical examples and an empirical application involving the performance evaluation of 14 airline companies.*

Keywords*: Cross-efficiency; multiple optimal weights; secondary goals; aggressive and benevolent models*

JEL Classification: C61, C67

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1. Introduction

Data Envelopment Analysis (DEA), introduced by Charnes, Cooper, and Rhodes in 1978, stands as a powerful non-parametric tool for evaluating the efficiency of homogeneous production units. It leverages linear programming techniques to assess the efficiency of Decision-Making Units (DMUs) involved in multifaceted production processes that encompass various inputs and outputs. What sets DEA apart is its flexibility to accommodate diverse input-output combinations, along with its ability to account for varying levels of production efficiency – be it increasing or decreasing – relative to the scale and output levels.

DEA's versatility finds widespread application today, with organizations across diverse sectors, such as banking, postal services, healthcare facilities, educational centers, power generation plants, and refineries, harnessing its analytical prowess (Cook et al., 2013; Yang, 2014; Oukil and Amin, 2015).

Yet, within DEA's power lies a potential pitfall. DMUs are free to choose arbitrary weights for their inputs and outputs, enabling them to maximise their efficiency scores. Although this flexibility is advantageous, it also poses challenges in the realm of performance evaluations. To address these issues and ensure equitable assessments, weight restrictions have emerged as an integral component of DEA models. Podinovski and Bouzdine-Chameeve (2015) have played a pivotal role in pioneering and advancing this concept within the DEA framework.

Expanding upon the DEA paradigm, the cross-efficiency model, introduced by Sexton et al. (1986) and further refined by Doyle and Green in 1994, introduces a transformative shift from self-evaluation to peer-evaluation. This innovation has garnered considerable acclaim, offering a solution to one of DEA's inherent challenges: the presence of multiple efficient DMUs. In the cross-efficiency model, a second computational stage comes into play. Here, each unit shares its optimal weights, which are then used to assess the efficiency of other units. This approach levels the playing field, making it easier to rank all units within the system. For a more in-depth exploration of the practical applications of cross-efficiency, consider the valuable insights provided by Podinovski's work in 2016 and the recent contribution of Kiaei et al. (2023).

However, the use of DEA linear programs introduces a potential stumbling block – the inevitability of alternate optimal solutions. This can leave the cross-efficiency scores derived from the original DEA method somewhat ambiguous. Depending on the choice of these alternate solutions, a DMU's cross-efficiency rating may improve, albeit at the cost of potentially inflating the ratings of other DMUs, a concern voiced by Doyle and Green (1994) and Sexton et al. (1986).

To mitigate the issue of non-unique DEA solutions, Sexton et al. (1986) and Doyle and Green (1994) introduced the concept of secondary goals. Additional methods for addressing non-uniqueness in DEA optimal weights within the cross-efficiency context include the use of constant units (Cook & Zhu, 2014) and cross-efficiency evaluation based on Pareto's improvement. A host of researchers have further explored means to minimise the impact of non-uniqueness in DEA optimal weights in cross-efficiency assessments, exemplified in the works of Jahanshahloo et al., 2011; Dimitrov and Sutton, 2010; Guo et al.,2013; Wu et al., 2016; Liang et al., 2008; Rodder and Reucher,2011, 2012; Moeini et al., 2015, and others.

Weight constraints also emerge as vital components in cross-efficiency when considering the concept of peer evaluation within secondary goals. Orkcu et al. (2011) proposed a method aimed at exploring alternative optimal solutions to curb unusual input and output weights. Wang and Chin (2010a) introduced a neutral model to select optimal weights, while Jahanshahloo et al. (2011) built upon the concept of symmetric weights, first introduced by Dimitrov and Sutton (2010), as part of a secondary goal approach. Ruiz and Sirvent (2010, 2011) further enriched the field by suggesting various weight-restriction methods as secondary goals within the cross-efficiency framework. Guo et al. (2013) introduced a novel cumulative method for cross-efficiency, which incorporates the preferences of decision makers. Jahanshahloo et al. (2014) proposed a secondary goal model to determine appropriate weights for DMUs, and Wu et al. (2015) explored a max-min model within the DEA framework to select optimal weights based on specific criteria.

It should be noted that most DEA studies on cross-efficiency, including those mentioned above, predominantly adopt the assumption of Constant Returns to Scale (CRS) in production technology. This preference arises because Variable Returns to Scale (VRS) models can yield negative cross-efficiency scores when assessing unit performance. Nevertheless, Lim and Zhu (2014) have introduced a novel approach for cross-efficiency in VRS cases, grounded in a geometric interpretation of both CRS and VRS scores. These novel approaches, while promising, present challenges in their practical implementation, warranting further development for real-world use.

This paper revolves around two pivotal concepts in cross-efficiency evaluation. First, it delves into the imposition of weight constraints on profile weight selection to eliminate unusual weights and ensure the presence of non-zero, more sensible weight values. Second, it explores the adoption of a more realistic representation of technology in real-world applications, known as non-decreasing returns to scale (NDRS), in contrast to the commonly used Constant Returns to Scale (CRS) assumption. Furthermore, given the persistent challenge of multiple weight solutions, we delve into the application of secondary goal models within this context, introducing benevolent and aggressive models under the NDRS framework.

The structure of the remainder of this paper is as follows: Section 2 offers an in-depth exploration of preliminary concepts in cross-efficiency evaluation, particularly within the framework of NDRS. Section 3 is dedicated to the presentation of novel benevolent and aggressive models designed for cross-efficiency assessment. Section 4 provides a concrete example and discusses the empirical application of the newly proposed models within the context of the airline industry. Section 5 concludes the paper, summarising its key findings and contributions.

2. Preliminaries

This section provides a concise introduction to the Non-Decreasing Returns to Scale (NDRS) model in the context of cross-efficiency evaluation. Through a simple numerical example, we illustrate the presence of multiple optimal solutions in cross-efficiency assessment when operating under the NDRS assumption.

Consider a scenario where there exist n production units awaiting evaluation. Each DMU_j (where j ranges from 1 to n) generates s outputs (denoted as y_{rj}, where r spans from 1 to s) while utilising m inputs (represented as x_{ij}, with i ranging from 1 to m). We adopt the NDRS output-oriented DEA model in multiplier mode, which can be formulated as the following optimisation model:

$$\max E_{dd} = \sum_{r=1}^{s} u_{rd} y_{rj} + u_0$$

Subject to

$$\begin{aligned} &\sum_{i=1}^{m} v_{id} x_{id} = 1, \\ &\sum_{r=1}^{s} u_{rd} y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 \leq 0, \quad j = 1, \dots, n, \\ &\forall r, \forall i: u_{rd}, v_{id} \geq 0,\ u_0 \geq 0. \end{aligned} \tag{1}$$

In this model, the non-negative variable u_0 represents the NDRS technology, while v_i and u_r denote the weights assigned to the ith input and the rth output, respectively. To calculate the cross-efficiency scores for DMU_j, we consider the following ratio, in which DMU_j is evaluated using the optimal weights of DMU_d:

$$E_{dj} = \frac{\sum_{r=1}^{s} u^*{}_{rd} y_{rj} + u_0}{\sum_{i=1}^{m} v^*{}_{id} x_{ij}} \ . \ j = 1.2. \dots . n \tag{2}$$

where * denotes an optimal solution for Model (1). It is important to note that under the Non-Decreasing Returns to Scale (NDRS) assumption, E_{dj} values are non-negative. The cross-efficiency score for DMU_j is defined as the simple average of all E_{dj} values for $d = 1, 2, \dots, n$, as shown in Eq. (3).

$$\overline{E_j} = \frac{1}{n} \sum_{d=1}^{n} E_{dj} \tag{3}$$

The computation of cross-efficiency scores presents a formidable challenge due to the proliferation of multiple optimal weight solutions stemming from Model (1) when assessing DMU_d. This intricacy frequently engenders ambiguity in the resulting cross-efficiency values, leading to instances where their definition becomes elusive. Furthermore, the process of computing these optimal weights may sporadically yield values of zero or, more problematically, values deemed unacceptable. Such occurrences add an additional layer of complexity to the assessment procedure. Notably, the challenge is further compounded by the vast range of variation observed in optimal weight solutions. Navigating this extensive spectrum to identify a suitable value becomes a Herculean and intricate endeavour.

These inherent intricacies in the evaluation process underscore the compelling need for robust methodologies capable of effectively addressing these issues.

Example 1. To illustrate these complexities, consider a dataset comprising six DMUs, each characterised by two inputs and two outputs as detailed in Table 1. Notably, the table reveals the existence of two alternative sets of optimal weights in the last five columns, computed as part of the DMU evaluation process within Model (1). This example vividly demonstrates the presence of multiple solutions, emphasising the importance of a systematic approach to tackle the intricate landscape of optimal weights and their impact on cross-efficiency assessments.

Table 1. Data and Multiple Optimal Weights from Model (1)

	Inputs		Outputs		Alternative optimal weights				
DMU	x_1	x_2	y_1	y_2	u^*_0	v^*_1	v^*_2	u^*_1	u^*_2
A	1.5	0.2	1.4	0.35	0.433333 (1)	0.666667 (0.5333)	0 (1)	0.379006 (0)	0.103024 (0)
B	4	0.7	1.4	2.1	0 (0.175)	0.137615 (0.075)	0.642202 (1)	0.144168 (0)	0.380079 (0.39286)
C	3.2	1.2	4.2	1.05	0.0506522 (0.2031)	0.3125 (0.3125)	0 (0)	0.212739 (0.12612)	0.053185 (0.2545)
D	5.2	2	2.8	4.2	0.0350953 (0)	0.192308 (0.19231)	0 (0)	0.096875 (0)	0.165156 (0.23809)
E	3.5	1.2	1.9	2.5	0.2470046 (0)	0.285714 (0.2857)	0 (0)	0.067149 (0)	0.250165 (0.34637)
F	3.2	0.7	1.4	1.5	0.2483801 (0.2191)	0.194384 (0.0938)	0.539957 (1)	0 (0)	0.431965 (0.34729)

Table 2 presents the cross-efficiency scores, derived from the optimal weight solutions generated by Model (1), and these scores are presented in the bottom row of the table. The optimisation tasks associated with Model (1) were conducted using Lingo 18.0 software, leveraging the computational capabilities of a laptop equipped with an 11th Gen Intel® Core™ i5 processor operating at a clock speed of 2.40 GHz.

Table 2. Cross-Efficiency Scores with Non-Decreasing Returns to Scale

DMU	A	B	C	D	E	F
A	1 (1)	0.4426 (0.3529)	1 (0.3440)	0.5559 (0.2095)	0.6047 (0.3261)	0.5243 (0.4155)
B	1 (1)	1 (1)	0.8296 (0.4080)	1 (0.7636)	0.9775 (0.7912)	0.8675 (0.8131)
C	0.7832 (1)	0.3681 (0.7313)	1 (1)	0.5374 (1)	0.5374 (0.9864)	0.4283 (0.7614)
D	0.7922 (0.2889)	0.6728 (0.650)	1 (0.4063)	1 (1)	0.93904 (0.8844)	0.6800 (0.5804)
E	1 (0.2829)	0.7581 (0.6365)	1 (0.3978)	0.7581 (0.9792)	0.8659 (0.8659)	1 (0.5683)
F	1 (1)	1 (0.8822)	0.5527 (0.3892)	0.9866 (0.6744)	1 (0.7115)	0.8963 (0.8963)
CROSS EFFICIENCY	0.9292 (0.8648)	0.7069 (0.7319)	0.8970 (0.5313)	0.8063 (0.7746)	0.8208 (0.7833)	0.7327 (0.7080)

As demonstrated in Table 1, the presence of alternative weights leads to non-uniqueness in the computed scores, resulting in the emergence of zero values within the optimal weight solutions. Additionally, the computed weights exhibit substantial dissimilarity. To illustrate this point, consider the output weight values obtained for Unit C: ($u_1^* = 0.212739$ and $u_2^* = 0.053185$), contrasted with those for Unit D: ($u_1^* = 0.096875$ and $u_2^* = 0.165156$). These values indicate a marked disparity in preference between the first and second outputs. To establish a definitive ranking order of DMUs and effectively discriminate among efficient units, Doyle and Green (1994) and Sexton et al. (1986) introduced and refined the concept of "secondary goals" within cross-efficiency evaluation. The objective of these secondary goal models is to mitigate the proliferation of multiple optimal weights. In this context, they introduced aggressive and benevolent models, where secondary goals either minimise or maximise the cross-efficiency scores of the other peer DMUs. The formulation of their introduced models can be expressed as follows:

$$\min_{u,v} \max_{j \neq d} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \right\} \ or \ \max_{u,v} \min_{j \neq d} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \right\}$$

$$s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \qquad (4)$$

$$\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^*$$

$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 \leq 0 \qquad j = 1, \dots, n$

$\forall r, \forall i: u_r, v_i \geq 0, \qquad u_0 \geq 0.$

where θ_d^* represents the efficiency score of DMU_d under the Non-Decreasing Returns to Scale (NDRS) assumption obtained from Model (1).

3. Non-Decreasing Return to Scale Cross-Efficiency with Weight Restrictions and New Secondary Goals

In this section, we delve into the imposition of weight constraints when selecting weight profiles. Building upon the groundwork laid by Ruiz and Sirvent (2010) in the realm of weight restrictions, we introduce a novel approach to cross-efficiency evaluation. To achieve this, we outline a two-step procedure, wherein a fresh secondary goal is proposed for Non-Decreasing Returns to Scale (NDRS) cross-efficiency assessment, integrated with weight restrictions aimed at mitigating weight disparities.

Step 1: In the initial phase, we employ the Weight-Restricted Model (5). This model is designed to circumvent the occurrence of zero weights and excessive variations among the multiple optimal weight solutions, thus promoting more stable weight profiles.

$max \quad \varphi_d$

$s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \qquad (5)$

$\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta^*$

$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 \leq 0$

$z_i \leq v_i^d \leq h_i \qquad i = 1, \dots, m$

$z_o \leq u_r^d \leq h_o \qquad r = 1, \dots, s$

$\frac{z_i}{h_i} \geq \varphi_d \,, \frac{z_o}{h_o} \geq \varphi_d,$

$\forall i: z_i, h_i \geq 0, z_o, h_o \geq 0$

In this optimisation model, θ_d^* represents the NDRS efficiency score of DMU_d, as obtained from Model (1). The weight constraints implemented in Model (5) are introduced to restrict both the input and output multipliers, ensuring that they fall within the prescribed bounds of z_i, h_i, z_o, and h_o, respectively. It's worth noting that $\varphi_d \leq Min\left\{\frac{z_i}{h_i}, \frac{z_o}{h_o}\right\} \leq 1$, and any value exceeding φ signifies that the ratios $\frac{z_i}{h_i}$ and $\frac{z_o}{h_o}$ are approaching unity. Essentially, this configuration ensures that the computed lower and upper bounds are as close as possible, effectively mitigating disparities in optimal weights.

Additionally, φ_d serves as an indicator of the lack of correlation between the coefficients of the optimal solutions for DMU_d's input and output weights. For instance, a φ_d value of 0.4 implies that, in the performance assessment of DMU_d, the lowest value for any input (or output) weight cannot be less than 40% of its highest value.

Step 2: Subsequently, as part of the second step, we explore new benevolent and aggressive perspectives to formulate a bi-objective optimisation programming model. These models are tailored for the NDRS scenario and are designed to circumvent the emergence of zero weights and substantial weight disparities.

$$\max_{u,v} \min_{j\neq d} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj} + u0}{\sum_{i=1}^{m} v_i x_{ij}} \right\} \ or \min_{u,v} \max_{j\neq d} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj} + u0}{\sum_{i=1}^{m} v_i x_{ij}} \right\}$$

$$s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \tag{6}$$

$$\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^*$$

$$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 \le 0 \quad ,j = 1, \ldots, n$$

$$z_i^* \le v_i^d \le h_i^* \qquad i = 1, \ldots, m$$

$$z_o^* \le u_r^d \le h_o^* \qquad r = 1, \ldots, s$$

$$\forall i{:}\ z_i \ge 0,\ z_o, u_0 \ge 0.$$

It's essential to note that z_i^* , h_i^*, z_o^*, and h_o^* represent the computed optimal solutions from the first stage, while θ_d^* signifies the efficiency score obtained through Model (1).

A crucial observation is that Model (6) remains both feasible and bounded under all circumstances. It's worth emphasising that, in light of the acquired optimal solution, any DMU_j with a cross-efficiency score of $\frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}}$ less than 1 is deemed inefficient, whereas those with a score of $\frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}}$ equal to 1 are classified as efficient.

From a computational perspective, we can introduce non-negative slack variables s_j, transforming the inequality constraints in Model (6) into equality constraints:

$$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 + s_j = 0 \ \ , \ \ s_j \ge 0,\ j = 1, \ldots, n \tag{7}$$

Alternatively, we can express this as:

$$-\left(\sum_{r=1}^{s} u_{rd}\, y_{rj} + u_0\right) + \sum_{i=1}^{m} v_{id} x_{ij} = s_j \quad ,j = 1, \ldots, n.$$

Notably, for efficient DMU_j, an optimal weight profile exists in which $\sum_{r=1}^{s} u_{rd}^{*}\, y_{rj} - \sum_{i=1}^{m} v_{id}^{*} x_{ij} + u_0^{*} = 0$, or equivalently, $s_j^{*} = 0$. Conversely, for inefficient DMU_j not situated on the efficient frontier, we have $\sum_{r=1}^{s} u_{rd}^{*}\, y_{rj} - \sum_{i=1}^{m} v_{id}^{*}\, x_{ij} + u_0^{*} < 0$ in any optimal solution, corresponding to $s_j^{*} > 0$. Therefore, the efficiency score obtained relates inversely to the distance of its coordinates from the efficient frontier. Consequently, we can express this relationship as:

$$\min_{j\neq d} \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \equiv \max_{j\neq d} s_j \qquad (8)$$

This equivalence implies that the minimisation of the efficiency ratio $\frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}}$ as the objective function is synonymous with the maximisation of s_j for each evaluated DMU_j. Similarly, in a comparable manner:

$$\max_{j\neq d} \frac{\sum_{r=1}^{s} u_r y_{rj} + u_0}{\sum_{i=1}^{m} v_i x_{ij}} \equiv \min_{j\neq d} s_j \qquad (9)$$

Now, when considering the benevolent perspective, the corresponding model can be presented as follows:

$$\begin{aligned}
&\min_{u,v} \max_{j\neq d} s_j \\
&s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \qquad (10)\\
&\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^{*} \\
&\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 + s_j = 0 \qquad , j = 1, \dots, n \\
&z_i^{*} \leq v_i^{d} \leq h_i^{*} \qquad i = 1, \dots, m \\
&z_o^{*} \leq u_r^{d} \leq h_o^{*} \qquad r = 1, \dots, s \\
&\forall i : z_i \geq 0,\ z_o, u_0 \geq 0.
\end{aligned}$$

Let $\max_{j\neq d}\{s_j\} = \alpha$. In this case, the previously described minimax model can be equivalently reformulated as the following linear programming problem, which is considerably more straightforward to solve:

$$\begin{aligned}
&\min_{u,v,s} \alpha \\
&s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \qquad (11)\\
&\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^{*} \\
&\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 + s_j = 0 \qquad , j = 1, \dots, n \\
&\alpha \geq s_j \qquad j = 1, \dots, n \\
&z_i^{*} \leq v_i^{d} \leq h_i^{*} \qquad i = 1, \dots, m \\
&z_o^{*} \leq u_r^{d} \leq h_o^{*} \qquad r = 1, \dots, s \\
&z_o, z_i, u_0 \geq 0.
\end{aligned}$$

The benevolent perspective is integrated into Model (12).

$$\max_{u,v} \min_{j \neq d} s_j$$

$$s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \tag{12}$$

$$\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^*$$

$$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 + s_j = 0 \qquad ,j = 1, \dots, n$$

$$z_i^* \le v_i^d \le h_i^* \qquad i = 1, \dots, m$$

$$z_o^* \le u_r^d \le h_o^* \qquad r = 1, \dots, s$$

$$z_o, z_i, u_0 \ge 0$$

When we substitute $\min_{j \neq d}\{s_j\} = \beta$, we can proceed to solve the following linear model:

$$\max_{u,v} \beta$$

$$s.t \quad \sum_{i=1}^{m} v_{id} x_{id} = 1 \tag{13}$$

$$\sum_{r=1}^{s} u_{rd} y_{rj} + u_0 = \theta_d^*$$

$$\sum_{r=1}^{s} u_{rd}\, y_{rj} - \sum_{i=1}^{m} v_{id} x_{ij} + u_0 + s_j = 0 \qquad ,j = 1, \dots, n$$

$$\beta \le s_j \qquad j = 1, \dots, n$$

$$z_i^* \le v_i^d \le h_i^* \qquad i = 1, \dots, m$$

$$z_o^* \le u_r^d \le h_o^* \qquad r = 1, \dots, s$$

$$z_o, z_i, u_0 \ge 0.$$

To illustrate these concepts, we apply both benevolent and aggressive viewpoints to the data presented in Table 1. In the initial step, using Model (5), we establish the lower and upper bounds for both inputs and outputs. The results of this step are summarised in Table 3. The primary objective here is to mitigate excessive weight dispersion. The lower values for φ signify a broader weight range, while higher values indicate a more constrained weight range.

Table 3. Lower and Upper Bounds for Input and Output Weights

DMU	φ	u_0^*	z_i	h_i	z_o	h_o
A	1	0.2058824	0.4	0.625	0.4	0.5
B	0.257775	0.07034215	0.148908	0.577668	0.097381	0.377774
C	1	0	0.227273	0.227274	0.190476	0.190477
D	0.61256	0	0.095338	0.155639	0.103558	0.169056
E	0.242561	0.2288655	0.063982	0.263778	0.063982	0.260442
F	0.114602	0.1754169	0.0887	0.774397	0.0494	0.430899

In the second step, we utilise the weight ranges outlined in Table 3 as a basis to compute the optimal weights and the value of α within the aggressive viewpoint. This computation is accomplished by solving the linear optimisation Model (11). The outcomes of this computation are presented in Table 4.

Table 4. Computed Weights in the Aggressive Case

DMU	α	u_0^*	v_1^*	v_2^*	u_1^*	u_2^*
A	0.8642241	0.265000	0.608621	0.435345	0.400000	0.500000
B	0.2937	0.070000	0.148908	0.577667	0.097380	0.377774
C	0.401	0.000000	0.227273	0.227273	0.190476	0.190476
D	0.189	0.000000	0.155639	0.095338	0.103558	0.169057
E	0.2333196	0.155188	0.195276	0.263778	0.075511	0.224537
F	0.4555225	0.174517	0.185674	0.579775	0.049382	0.430899

As a result, unique cross-efficiency scores can be derived from the computed weights. It's important to emphasise that the diagonal elements in the cross-efficiency matrix represent the NDRS efficiency scores of the units obtained from Model (1).

Table 5. Cross-Efficiency Scores in the Aggressive Viewpoint

DMU	A	B	C	D	E	F
A	1.0000	0.6489	1.0000	0.7986	0.7967	0.6634
B	1.0000	1.0000	0.7489	1.0000	0.9880	0.8779
C	0.8627	0.6241	1.0000	0.8148	0.7846	0.6232
D	0.8084	0.7254	1.0000	1.0000	0.9397	0.7057
E	0.6578	0.6582	0.6841	0.9068	0.8600	0.6389
F	1.0000	1.0000	0.6469	0.9988	1.0000	0.8900
Cross Efficiency	**0.8882**	**0.7761**	**0.8466**	**0.9198**	**0.8948**	**0.7332**

The results of the benevolent case, computed using Model (12), are presented in Table 6. In this model, β serves a similar role as φ in Model (5). For instance, DMU_C and DMU_F exhibit β values of 0.623 and 0.649, respectively, indicating a greater dispersion of weights compared to DMU_A, which has a β value of 0.865.

Table 6. Computed Weights in the Benevolent Case

DMU	$\boldsymbol{\beta}$	$\boldsymbol{u_0^*}$	$\boldsymbol{v_1^*}$	$\boldsymbol{v_2^*}$	$\boldsymbol{u_1^*}$	$\boldsymbol{u_2^*}$
A	0.865	0.205882	0.588237	0.588230	0.453782	0.453782
B	0.749	0.070342	0.148908	0.577668	0.097381	0.377774
C	0.623	0.000000	0.227273	0.227273	0.190476	0.190476
D	0.705	0.000000	0.155639	0.095338	0.103556	0.169056
E	0.743	0.087328	0.195276	0.263778	0.0755111	0.260144
F	0.649	0.174517	0.185674	0.579775	0.049382	0.430899

Indeed, it is important to highlight that in both scenarios, both aggressive and benevolent, the new approach successfully avoids the selection of zero weights for both inputs and outputs. The computed cross-efficiency matrix and cross-efficiency scores in the benevolent case are presented in Table 7.

Table 7. Cross-Efficiency Scores in the Benevolent Viewpoint

DMU	A	B	C	D	E	F
A	1.0000	0.6489	1.0000	0.7986	0.7967	0.6634
B	1.0000	1.0000	0.7489	1.0000	0.9880	0.8779
C	0.8627	0.6241	1.0000	0.8148	0.7846	0.6232
D	0.8084	0.7254	1.0000	1.0000	0.9397	0.7057
E	0.9821	0.7584	0.7522	0.8488	0.8600	0.7383
F	1.0000	1.0000	0.6469	0.9988	1.0000	0.8900
Cross Efficiency	**0.9422**	**0.7928**	**0.8580**	**0.9102**	**0.8948**	**0.7498**

Additionally, it is worth observing that the cross-efficiency value for any unit in the benevolent case is always greater than or equal to the corresponding value in the aggressive case. Naturally, each case yields its own unique ranking order for the units. For instance, in the aggressive case, DMU_D attains the highest rank, whereas in the benevolent scenario, DMU_A secures the top position in the ranking order.

4. Illustrative application

In today's competitive market landscape, the survival and success of businesses hinge on their ability to ensure customer satisfaction. This imperative becomes even more pronounced as competitive pressures continue to intensify. Consider the airline industry, where effective operations are essential for success. In this context, the management of customer needs and preferences takes on paramount significance. Prioritising service quality and aligning it with customer expectations, as assessed through their satisfaction with the services provided, can play a pivotal role in the resilience and success of passenger airlines operating in a highly dynamic and turbulent market (Tofallis, 1997).

In this section, we employ the new proposed approach to illustrate its effectiveness by considering a dataset comprising 14 major international passenger airlines from the year 1990. This dataset encompasses three inputs and two outputs, and the reference data have been sourced from company annual reports, as published in IATA (Tofallis, 1997).

- Inputs:

x_1: Aircraft capacity in ton-kilometers

x_2: Operating cost

x_3: Non-flight assets, include reservation systems, facilities, and current assets.

- Outputs:

y_1: Passenger-kilometers

y_2: Non-passenger revenue

Table 8 provides the input and output data, along with the efficiency scores computed under the NDRS assumption. In this evaluation, units 7, 10, 11, 12, and 13 are identified as efficient.

Table 8. Input/Output Data for 14 Passenger Airlines

DMU	x_1	x_2	x_3	y_1	y_2	efficiency
1	5723	3239	2003	26677	697	0.8932
2	5895	4225	4557	3081	539	0.778
3	24099	9560	6267	124055	1266	0.9475184
4	13565	7499	3213	64734	1563	0.959
5	5183	1880	783	23604	513	1

DMU	x_1	x_2	x_3	y_1	y_2	efficiency
6	19080	8032	3272	95011	572	0.97
7	4603	3457	2360	22112	969	1
8	12097	6779	6474	52363	2001	0.858
9	6587	3341	3581	26504	1297	0.97
10	5654	1878	1916	19277	972	1
11	12559	8098	3310	41925	3398	1
12	5728	2481	2254	27754	982	1
13	4715	1792	2485	31332	543	1
14	22793	9874	4145	122528	1404	1

Table 9 lists the values of φ, α, and β, along with the benevolent and aggressive cross-efficiency scores for the 14 passenger airlines under the NDRS assumption. These scores are presented in two different scenarios: with and without weight control.

Table 9. Benevolent and Aggressive Scores and Rankings for 14 Passenger Airlines

Aggressive and Benevolent score without weight control			Aggressive and Benevolent score with weight control				
DMU	Aggressive score	Benevolent score	φ	α	Aggressive score	β	Benevolent score
1	0.6943	0.7906	0.000285	1.000000	0.755	0.303060	0.7575
2	0.2371	0.2413	0.023983	1.000000	0.2367	0.224215	0.2379
3	0.482	0.7533	0.038609	0.475203	0.7296	0.047184	0.7304
4	0.572	0.8088	0.006265	0.530454	0.75	0.000433	0.7513
5	0.9453	0.954	1	1.000000	0.9244	0.328284	0.9274
6	0.5291	0.7474	0.046091	1.000000	0.7298	0.033606	0.7307
7	0.7491	0.8571	0.048076	1.000000	0.7941	0.259322	0.7969
8	0.4964	0.7097	0.008932	0.615680	0.6545	0.274520	0.6556
9	0.6392	0.7689	0.014908	1.000000	0.7168	0.401931	0.7201
10	0.7718	0.8075	0.040625	1.000000	0.775	0.173061	0.7770
11	0.6231	0.8811	0.080436	0.558041	0.7596	0.172363	0.7609
12	0.776	0.9079	0.039736	1.000000	0.8663	0.202492	0.8684
13	0.854	0.9663	1	1.000000	0.9629	0.069579	0.9652
14	0.5625	0.8446	0.754313	0.395271	0.8087	0.067428	0.8094

The comparison between the cross-efficiency values in both aggressive and benevolent modes, without weight control, reveals relatively small differences. However, some notable distinctions are evident, as exemplified by DMU_3.

Conversely, with the imposition of weight control, the computed results demonstrate that cross-efficiency scores for both the aggressive and benevolent modes become significantly closer for each unit. For instance, consider DMU11; in the aggressive (benevolent) models without weight control data (Table 9), the scores are 0.6231 (0.8811), respectively. In contrast, the aggressive (benevolent) models with weight control data (Table 9) yield scores of 0.7596 (0.7609) for the same DMU, respectively. This indicates that, with weight control data, the potential cross-efficiency scores for this DMU fall within the range of [0.7596, 0.7609], whereas without weight control data, they span a wider interval of [0.6231, 0.8811].

The results also reveal distinct rank orders for the units when weight control is imposed. The ranking outcomes in the aggressive cross-efficiency method for this dataset are as follows.

<u>Without weight control:</u>

$$DMU_{13} \succ DMU_5 \succ DMU_{12} \succ DMU_{11} \succ DMU_7 \succ DMU_{14} \succ DMU_4 \succ DMU_{10} \succ DMU_1 \succ DMU_9 \succ DMU_3 \succ DMU_6 \succ DMU_8 \succ DMU_2$$

<u>With weight control:</u>

$$DMU_{13} \succ DMU_5 \succ DMU_{12} \succ DMU_{14} \succ DMU_7 \succ DMU_{10} \succ DMU_{11} \succ DMU_1 \succ DMU_4 \succ DMU_6 \succ DMU_3 \succ DMU_9 \succ DMU_8 \succ DMU_2$$

It is important to note that approximately 72% of the units experience a change in their ranking when weight control is introduced in the aggressive mode. For instance, DMU_5 achieves the second rank without weight control, while it occupies the first position in terms of performance with weight control in the cross-efficiency evaluation. Only for DMU_{12}, DMU_7, DMU_{11}, and DMU_2, the ranking remains consistent between the aggressive mode with and without weight control.

A similar comparison was conducted in the benevolent mode, both with and without weight control. Based on the results presented in Table 9:

<u>Without weight control:</u>

$$DMU_5 \succ DMU_{13} \succ DMU_{12} \succ DMU_{10} \succ DMU_7 \succ DMU_1 \succ DMU_9 \succ DMU_{11} \succ DMU_4 \succ DMU_{14} \succ DMU_6 \succ DMU_8 \succ DMU_3 \succ DMU_2$$

<u>With weight control:</u>

$$DMU_{13} \succ DMU_5 \succ DMU_{12} \succ DMU_{14} \succ DMU_7 \succ DMU_{10} \succ DMU_{11} \succ DMU_1 \succ DMU_4 \succ DMU_6 \succ DMU_3 \succ DMU_9 \succ DMU_8 \succ DMU_2$$

It is evident that the ranking undergoes changes when weight control is applied in the benevolent mode. For instance, DMU_{10}, which ranks fourth without weight control, drops to the sixth position in performance when weight control is integrated into the cross-efficiency evaluation. Notably, only for DMU_{12}, DMU_7, DMU_4, and DMU_2, the ranking remains consistent between the benevolent mode with and without weight control.

In conclusion, it appears that the cross-efficiency values derived from the aggressive and benevolent modes with weight control offer more accurate and reliable predictions of the efficiency status of production systems compared to the assessments conducted without weight control.

5. Conclusions

Cross-efficiency analysis serves as a powerful tool for evaluating unit performance, offering valuable insights into its effectiveness. The incorporation of weight control within the cross-efficiency framework addresses significant issues related to zero or unrealistic weight assignments. Our proposed approach places a primary emphasis on the selection of weight profiles, thus mitigating the potential for DMUs to employ impractical weight schemes during cross-efficiency assessments. The notion of unrestrained weights for all DMUs lacks logical coherence, making the adoption of weight profiles a sound strategy for curbing excessive variations in weight allocations within the cross-efficiency evaluations. Through the establishment of upper and lower bounds for input and output weights, as prescribed by our methodology, we effectively restrain weight dispersion.

Our approach strategically addresses the challenges posed by unrealistic or zero-weighting profiles, particularly in the context of efficient DMUs. Although we have presented our method using small-scale examples in this study, its potential to yield more comprehensive cross-efficiency scores, while avoiding implausible weight assignments, has been demonstrated.

While cross-efficiency analysis has traditionally been associated with constant returns to scale efficiency (CRS), the significance of exploring variable returns to scale (VRS) efficiency should not be understated. Many real-world systems require VRS considerations rather than CRS, which emphasises the relevance of studying VRS within the cross-efficiency framework. In this paper, we have introduced a cross-efficiency model that selects weight profiles while accommodating non-decreasing returns to scale (NDRS) principles, effectively circumventing the challenges of negative, zero, and unrealistic weight assignments. Our numerical and empirical examples underscore the practicality and superiority of the proposed model.

In the realm of cross-efficiency evaluation, various secondary goals have been introduced to address inherent limitations. The issue of weight multiplicity remains a prominent concern, prompting the development of pessimistic and optimistic models as extensions of our proposed approach. Through empirical applications, we have demonstrated the efficacy of both aggressive and benevolent strategies, reinforcing the superiority of our model in enhancing the accuracy and reliability of cross-efficiency assessments.

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