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COST AND REVENUE EFFICIENCY MODELS FOR HETEROGENEOUS INPUTS AND OUTPUTS IN DATA ENVELOPMENT ANALYSIS

Abstract. *In the realm of data envelopment analysis (DEA), the crucial concern is to gauge the efficiency of firms that are differentiated by their inputs and outputs, and the prices associated with them. Traditional DEA operates under the assumption that all inputs and outputs are quantifiable as real numbers. However, this may not be a valid assumption in practical settings where integer variables and parameters are required. In this study, we explore three distinct models of cost and revenue efficiency that are suitable for situations where production possibilities are limited to integer values. We compare these models by means of introducing theorems. Further, we extract the target function values from these models using LINGO programming, with the help of a case study.*

Keywords: *Data Envelopment Analysis, Cost Efficiency, Revenue Efficiency, Integer Values, Directional Distance Function*

JEL Classification: C44, C67, C02, D24, D57, C60, H21, Y10

1. Introduction

Charnes et al. introduced Data Envelopment Analysis (DEA) in 1978, and it was later expanded upon by Banker et al. in 1984. DEA is a technique used in operations research to evaluate the efficiency and productivity of Decision Making Units (DMUs) that have multiple inputs and outputs. It is a nonparametric and linear programming approach that has become a popular method for benchmarking and measuring the relative efficiency of DMUs with multiple inputs and outputs. However, the traditional DEA models may produce

DOI: 10.24818/18423264/57.3.23.06

non-integer values for certain targets of inefficient DMUs. The purpose of DEA is to establish a reference technology that can be used to assess the efficiency of individual DMUs. The most widely used models are the output-oriented or input-oriented Farrell (1957) models, which are based on Farrell's radial measurement technique.

The Directional Distance Function (DDF), introduced by Chambers, Chung, and Färe (1957), has been demonstrated to be a crucial tool in production theory, yielding Shepard-type input and output distance functions as special cases. The DDF expands the output and contracts the input simultaneously, and its evaluation of any DMU depends on the directional vector and the production possibility set. Directional distance functions are widely used in modelling production technologies. Since DMUs usually have both input and output values, some of which are integers, such as the number of magazines, passengers, or universities, the DDF allows for the evaluation of these DMUs, regardless of the integer values. Moreover, it is used to generate a distance measure that aligns with the production system's objectives and facilitates the analysis of the efficiency of the DMUs in a unified framework. The DDF can also determine the optimal combination of inputs and outputs for a given production system, identify the best DMUs for a given production environment, and gauge a DMU's performance relative to its peers.

In 2006, Lozano and Villa were the first to explicitly tackle this problem in DEA. They suggested estimating the Production Possibility Set (PPS) by intersecting standard DEA technology with the non-negative integer set, and they presented a linear programming model (MILP) to restrict the calculated objectives to integers. Kuosmanen and Kazemi Matin (2009) disputed Lozano and Villa's work and introduced new axioms and a model to resolve the issues. Ylvinger (2000) employed DEA models to measure technical efficiency, while Farrell formulated the notion of cost efficiency, which requires precise input, output, and input price values. In multi-output environments, Cherchye et al. (2013,2014) proposed a new DEA methodology to evaluate cost efficiency. Färe et al. (1985) developed cost-efficiency and revenue-efficiency DEA models that demand not only input and output quantity information, but also their prices for each firm. Sahoo et al. (2013) broadened Tone's (2002) value-based models in a directional DEA framework to create novel measures of directional cost-efficiency and revenue-efficiency.

By considering the background information, a research gap has been identified where there is insufficient investigation on integer data models that measure cost efficiency and revenue efficiency in a particular direction under the constant returns-to-scale principle. Addressing this gap would allow researchers to create new models to evaluate the effectiveness and productivity of decision-making units with numerous inputs and outputs. Current research has found that the Directional Distance Function (DDF) is a crucial tool in production theory, as it provides more recognisable Shepard-type output and input distance functions as special cases. In addition, several techniques have been proposed by researchers to estimate value-based technical inefficiency, including the production possibility set and the directional distance function.

This paper explores three different models for cost and revenue analysis, all of which involve limiting the production possibility set to integer values. The first model uses conventional models with real values, while the second model rewrites the cost-efficiency and revenue models. The third model, which is the focus of the paper, also incorporates the Directional Distance Function (DDF) to move in a specific direction. This model seeks to achieve the minimum cost and maximum revenue by finding a new level of inputs and outputs based on defined axioms.

The paper is organised as follows: Section 2 outlines the main cost and revenue efficiency models with real values, while Section 3 presents the same models with integer values. In Section 4, cost and revenue efficiency models are described with integer data in a specific direction under the axiom of constant returns to scale. Finally, Section 5 provides a case study.

2. Preliminaries: General cost, revenue efficiency in DEA

The non-parametric methodology of DEA is utilised for measuring and analysing a variety of efficiency models, including cost and revenue efficiency. The concept of efficiency evaluates the ability of a Decision Making Unit (DMU) to produce the current outputs at a minimal cost given the input prices of each DMU. Traditional cost-efficiency analysis methods require accurate values for inputs, outputs, and input prices. Let us consider n DMUs, denoted by $\{(x_j, y_j): j = 1, \dots, n\}$. We assume that each DMU produces the outputs by consuming the inputs. For each DMU_j where j=1,…,n , the non-negative input and output vectors are denoted by $x_j = (x_{1j}, \dots, x_{mj})$ and $y_j = (y_{1j}, \dots, y_{sj})$. $X_j = (X_1, \dots, X_n)^T$ and $Y_j = (Y_1, \dots, Y_n)^T$ are used to denote, respectively, the input matrix and the output matrix. The production technology is given by: $= \{(x, y): x\ can\ produce\ y\}$, which we assume satisfies standard regularity conditions, as well as the strong disposability of inputs and outputs in particular. Let $c = (c_1, \dots, c_M) \in \mathbb{R}_+^M$ be a row vector of input prices and $p = (p_1, \dots, p_S) \in \mathbb{R}_+^S$ be a row vector of output prices. Cost efficiency and its model will be discussed here. In this type of efficiency, outputs are in the best situation (highest price), and the model minimises the cost of inputs as shown below:

$$A1 = cost1 : Min \sum_{i=1}^{m} c_i \widehat{x_i}$$

$$s.t. \qquad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \widehat{x_i} \qquad i = 1, \dots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, \dots, s \tag{2.1}$$
$$\lambda_j, \widehat{x_i} \geq 0 \qquad j = 1, \dots, n$$

The unit price of the input i of Decision Making Unit o (DMU_o) is denoted by w_i, which produces output y at the lowest cost. The minimum cost is denoted by $c\widehat{x}^*$, while the observed cost of DMU_o is denoted by $c\widehat{x_o}$. Cost efficiency is then defined as the ratio of the minimum cost to the observed cost and is, therefore, a value between 0 and 1.

Revenue efficiency is a type of efficiency in which outputs are in their most optimal situation. The unit price of the output r of Decision Making Unit o (DMUo) is denoted as p_r. So, revenue efficiency is defined as the ratio of the observed revenue $p\widehat{y_o}$ to the maximum revenue $p\widehat{y}^*$ and is therefore bounded by 0 and 1. With the model aimed at maximising the revenue of outputs. This can be expressed as follows.

$$B1 = REVENUE1 : Max \sum_{r=1}^{s} p_r \widehat{y_r}$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \qquad i = 1, \dots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \widehat{y_r} \qquad r = 1, \dots, s \tag{2.2}$$
$$\lambda_j, \widehat{y_r} \geq 0 \qquad j = 1, \dots, n$$

3. Methodology: General cost, revenue efficiency with integer-valued data

In this part, cost- and revenue-efficiency models with a more limited set of production possibilities are examined. The previous model examined the production possibility set on real numbers, whereas this part limits the production possibility set to integer values. The following model seeks to minimise input cost, such that c_i is the price of the ith input unit. The model is expressed as follows:

$$A2 = cost2 : Min \sum_{i=1}^{m} c_i \widehat{x_i}$$
$$s.t. \qquad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \widehat{x_i} \qquad i = 1, \dots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, \dots, s \tag{3.1}$$
$$\hat{x_i} \leq x_{io} \qquad i = 1, \dots, m$$
$$\lambda_j, \hat{x_i} \geq 0 \qquad j = 1, \dots, n$$
$$\hat{x_i} \in \mathbb{Z} \qquad i \in I$$

In model A1, the minimum cost was discussed using real values; however, this model considers integer values. Model A2 takes into account the lowest price to produce y and belongs to the integer valued. As described in Section 2, the ratio of the minimum cost to observed cost is known as the cost efficiency. This value exists in the range of 0 and 1. The following model presents the revenue efficiency model in the presence of integer data. Here, p_r is the price of the rth output unit.

$$B2 = REVENUE2 : Max \sum_{r=1}^{s} p_r \hat{y_r}$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \qquad i = 1, \dots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \hat{y_r} \qquad r = 1, \dots, s \tag{3.2}$$
$$\hat{y_r} \geq y_{ro} \qquad r = 1, \dots., s$$
$$\lambda_j, \hat{y_r} \geq 0 \qquad j = 1, \dots, n$$
$$\hat{y_r} \in \mathbb{Z} \qquad r \epsilon I$$

In model B1, the revenue efficiency is given with the real data set. However, in model B2, the production possibility set is restricted to integer-valued data. In other words, takes into account the highest revenue at the same cost and belongs to the integer valued. As described in Section 2, the ratio of the observed revenue to maximum revenue is known as the revenue efficiency. This value exists within the range of 0 and 1. Models A1 and B1 considered the production possibility set on real value sets, whereas models A2 and B2 limited the production possibility set to the integer value set. As a result, the minimum cost in model A2 is expected to be greater than in model A1, and the maximum revenue value in model B2 is less than in model B1. However, these differences are due to the application of models with different data sets. In conclusion, note that, in most cases, the values of the models are equal and, in some cases, they are obtained with very little differences. The following two theorems support the preceding statements.

Theorem 3.1. $A2 \geq A1$.

Proof. See the appendix.

Theorem 3.2. $B2 \leq B1$.

Proof. The proof of this theorem is similar to the proof of theorem 3.1, so its presentation is omitted.

4. Directional Distance Function (DDF)

Consider a technology involving n observed decision-making units (DMUs), each of which uses m inputs to produce s outputs. The Directional Distance Function (DDF) seeks to simultaneously minimise inputs and maximise outputs of a given DMU by using a direction vector $g = (-g^{-}, g^{+})$. Consequently, the input-oriented DDF ($g = (-g^{-}, 0)$) and the output-oriented DDF ($g = (0, g^{+})$) were presented. The Directional Distance Function allows the analyst to select the direction in which the inefficient DMU is projected onto the frontier. Let the vector $(g^x, g^y) \in \mathbb{R}_{+}^{m+s}$, with $(g^x, g^y) \neq 0$, be any arbitrary direction in which the (x_o, y_o) is to be projected. Then the corresponding efficient projection is $(x_o - \beta g^x, y_o + \beta g^y)$, and the scalar is known as the Directional Distance Function. The Data Envelopment Analysis (DEA) problem for measuring the DDF is (see Färe and Grosskopf, 2000):

$$
\begin{aligned}
DDF : & Max \ \beta \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta g_x \qquad i = 1, \dots, m \qquad (4.1) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta g_y \qquad r = 1, \dots, s \\
& \lambda_j \geq 0 \qquad j = 1, \dots, n
\end{aligned}
$$

β^* can be interpreted as the directional value-based measure of technical inefficiency.

5. Main section

Now, we represent the cost and revenue efficiency in the presence of integer data with the Directional Distance Function. In this section, two models are introduced for cost efficiency and revenue efficiency toward a suitable vector g, and aim to obtain the $\hat{x}$ and $\hat{y}$ values that are integer-valued. The main aim of providing a new cost efficiency model is to achieve at least the same revenue as the previous one with the lowest cost, and the aim of providing a new revenue efficiency model is to determine the maximum revenue with the cost specified in the model. These models are proposed under constant returns to scale in the specified direction of g.

The lowest cost is generated at the output level when the model is considered with constant returns to scale. By multiplying the input and output levels by a factor of φ, the model can be evaluated at different levels of inputs and outputs. The aim of the model is to achieve the maximum revenue (py_o) while minimising the associated costs. Thus, we can reach the desired revenue at a different level of output and, at this level of output, reduce the inputs so that the overall cost is minimised. The axiom of constant returns to scale allows the model to be evaluated at a range of input and output levels. This axiom states that there is no difference between (x_o, y_o) and$(\varphi x_o, \varphi y_o)$, where φ is the coefficient that indicates the new input and output level at which the maximum revenue and minimum cost are achieved. In essence, the model can be resized, and we aim to find the minimum cost and maximum revenue at various levels of inputs and outputs.

The axiom of constant returns to scale determines the frontier of the model, but we must determine which part of the frontier is being utilised. The values of φ^* can be obtained by applying any optimisation software like Lingo. The model is allowed to search for the lowest cost at different levels of output. In other words, the cost determined at the level φy_o is less than the cost determined at levely_o. This model provides the manager with the ability to increase or decrease the level of production while maintaining minimum revenue and minimising the cost. Previous models did not offer this capability. Furthermore, this axiom allows the model to modify the input and output sizes simultaneously. Additionally, a directional distance function is utilised to find a particular direction at the lowest cost for production.

In the models outlined below, not all inputs are integers. The input set is divided into two sets I_1 and I_2, such that $I = I_1 \cup I_2$, where I_1 denotes the real-valued input set and I_2 is the integer-valued input set. Under constant returns to scale, towards the CCR frontier, the production possibility set is limited from the real-valued set to the integer-valued set, creating a more stringent set. The model then proceeds in a specific direction g, where the target function is realised. In other words, in the production possibility set towards the CCR, it moves in the direction that reaches the lowest cost with the integer value. Also, with this limitation, the costs do not increase from a constant value, such as cx_o. First, the proposed cost model is studied:

$$\begin{aligned}
A3 = cost3 : Min \quad & \sum_{i=1}^{m} c_i \hat{x}_i \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \hat{x}_i \qquad i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro} \qquad r = 1, \dots, s \\
& \hat{x}_i \leq \varphi x_{io} - \beta g_i^I \qquad i = 1, \dots, m \qquad (5.1)
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j\, py_{rj} \geq py_{ro} \qquad r = 1, \dots, s$$
$$\lambda_j \geq 0 \ \ , \beta \geq 0 \ , \widehat{x_i} \in \mathbb{Z} \ \ , i \in I_2$$
$$\widehat{x_i} \geq 0 \ , \qquad \varphi \geq 1$$

In order to evaluate the cost efficiency of the model, the ratio of the minimum cost to the observed cost is determined. Thus, the cost efficiency, which ranges from 0 to 1, is calculated. As demonstrated in Section 2, maximum revenue can be achieved with the observed cost (cx_o) mentioned in the constraints. Consequently, the inputs and outputs are proportionally increased and transmitted to point in the direction of g, which improves the values of x and y. Furthermore, the model maximises revenue by limiting the input cost (cx_o). Here, we present the proposed revenue model:

$$B3 = REVENUE3 : Max \ \sum_{r=1}^{s} p_r \widehat{y_r}$$

$$
\begin{aligned}
s.t. \qquad & \sum_{j=1}^{n} \lambda_j\, x_{ij} \leq \varphi x_{io} && i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j\, y_{rj} \geq \widehat{y_r} && r = 1, \dots, s \\
& \widehat{y_r} \geq \varphi y_{ro} + \beta g_r^O && r = 1, \dots., s \\
& \sum_{j=1}^{n} \lambda_j\, cx_{ij} \leq cx_{io} && i = 1, \dots, m \\
& \lambda_j \ \geq 0 \ \ , \beta \geq 0 \, , \ \widehat{y_r} \in \mathbb{Z} \ , r \in O_2 \\
& \widehat{y_r} \geq 0 \, , \ \varphi \geq 1
\end{aligned}
\tag{5.2}
$$

To assess the revenue efficiency of the model, the ratio of the observed revenue to the maximum revenue is computed. Thus, the revenue efficiency is calculated, which ranges from 0 to 1. As demonstrated in models A2 and B2, the production possibility set is limited to the integer value, and in the models A3 and B3, a more complex situation is considered. To represent this, the directional distance function is used in the new model. Consequently, it can be expected that the minimum cost in model A2 will be higher than that of model A3, and also that the maximum revenue value in model B2 will be lower than that of model B3. This is validated through the two theorems presented below.

Theorem 4.1. $A3 \leq A2$.

Proof. See the appendix.

Theorem 4.2. $B3 \geq B2$.

Proof. The proof of this theorem is similar to the proof of theorem 4.1, so its presentation is omitted.

6. Case study

Our proposed model was applied to the case of the university departments of IAUK. Characterised by three input values (the number of postgraduate students, bachelor's students, and master's students) and four outputs (the number of graduations, number of scholarships, number of research products, and level of managerial satisfaction), all of these variables are integer values. To demonstrate the practical applicability of our proposed model, we conducted an empirical analysis based on a real-life data set from the university departments of IAUK obtained from the study by Kousmanen and Kazemi Matin (2009). In this paper, three models for cost efficiency and three models for revenue efficiency were presented. We applied our proposed model to the data set and compared the results with those of the existing models. The results showed that our proposed model achieved higher efficiency when compared to the existing models. Our findings suggest that our proposed model can be applied to various situations in order to improve cost and revenue efficiency.

The models presented in Section 2 are equivalent to the conventional minimum-cost and maximum-revenue models. In Section 3, the production possibility set is limited to integer values, leading to cost efficiency and revenue efficiency models. In Section 4, in addition to limiting the set of production possibilities, inputs and outputs must be moved in a particular direction with the application of a directional distance function. In the following, six tables are provided, the first three containing the values obtained from the A1, A2, and A3 models, respectively, through a Lingo program.

The three subsequent tables contain the values obtained from the B1, B2, and B3 models, respectively, using a Lingo program. As noted in the theorems of the previous sections, the values in the tables are compared with each other.

Table 1. Optimal inputs, minimum cost & cost efficiency values in model (2.1)

DMU	x_1^*	x_2^*	x_3^*	**A1**	**efficiency**
1	44.42	176.74	5.25	**226.42**	**0.867**
2	0	213	0	**213**	**0.942**
3	0	326	0	**326**	**0.928**
4	0	159	0	**159**	**0.929**
5	0	23.71	37.03	**60.75**	**0.37**
6	0	1014	0	**1014**	**0.916**

DMU	x_1^*	x_2^*	x_3^*	**A1**	**efficiency**
7	0	0	61	**61**	**1**
8	0	66.72	6.66	**73.39**	**0.358**
9	0	675	0	**675**	**0.928**
10	0	697	0	**697**	**0.901**
11	0	0	50	**50**	**0.757**
12	36.84	95.15	0	**132**	**0.243**
13	193.92	505.57	154.4	**853.91**	**0.864**
14	0	0	34	**34**	**1**
15	27.07	573.92	0	**601**	**0.755**
16	104.92	313.18	237.29	**655.4**	**0.975**
17	0	166	0	**166**	**1**
18	0	761	0	**761**	**1**
19	68.59	224.4	0	**293**	**0.924**
20	5.64	355.35	0	**361**	**0.745**
21	126.91	276.71	41.68	**445.31**	**0.861**
22	147.03	320.56	33.49	**501.09**	**0.858**
23	157.53	407.46	0	**565**	**0.828**
24	75.39	347.6	0	**423**	**0.748**
25	132.46	288.81	16.08	**437.36**	**0.725**
26	0	332	0	**332**	**0.89**
27	96.05	209.42	30.9	**336.38**	**0.969**
28	0	0	70	**70**	**1**
29	26.82	134.12	12.41	**173.36**	**0.528**
30	0	122.05	1	**123.05**	**0.46**
31	0	219	0	**219**	**0.835**
32	0	794	0	**794**	**0.776**
33	0	1111	0	**1111**	**0.816**
34	42.38	140.18	76.08	**258.65**	**0.92**
35	172	375	0	**547**	**1**
36	66.09	202.5	159.76	**428.36**	**0.931**
37	0	332	140	**472**	**0.622**
38	0	1158	0	**1158**	**0.919**
39	0	394	0	**394**	**0.362**

DMU	x_1^*	x_2^*	x_3^*	**A1**	**efficiency**
40	0	0	67	**67**	**0.971**
41	51.06	152.93	0	**204**	**0.649**
42	44.69	181.3	0	**226**	**0.609**

In the first three tables, the cost efficiency values are given. In Table 1, the values obtained from the primary minimum cost model (A1) are presented.

The first column in Table 1 displays the number of units, the second column shows the optimal values of the first input (x_1^*), the third column holds the optimal values of the second input (x_2^*), and the fourth column presents the optimal value of the third input (x_3^*). The next column displays the values of the objective function for the optimal inputs and their costs in model A1, calculated by the Lingo software, and the last column presents the cost efficiency. The objective function calculates the minimum cost in DMUs; however, the efficiency of each unit is equal to the value obtained from the objective function divided by the cost value of each unit. The values obtained from model A1 are provided in Table 1. As can be seen in the table, there is no integer-valued condition for the optimal inputs; thus, the real values have been obtained.

Table 2. Optimal inputs, minimum cost, cost efficiency values in model (3.1)

DMU	x_1^*	x_2^*	x_3^*	**A2**	**efficiency**
1	0	230	0	**230**	**0.869**
2	0	170	46	**216**	**0.942**
3	0	281	48	**329**	**0.928**
4	0	138	23	**161**	**0.929**
5	164	0	0	**164**	**0.371**
6	291	723	0	**1014**	**0.916**
7	0	0	61	**61**	**1**
8	0	83	0	**83**	**0.355**
9	0	675	0	**675**	**0.928**
10	0	697	0	**697**	**0.901**
11	0	0	50	**50**	**0.819**
12	36	96	0	**132**	**0.243**
13	0	872	0	**872**	**0.864**
14	0	0	34	**34**	**1**
15	0	603	0	**603**	**0.755**
16	0	672	0	**672**	**0.976**

DMU	x_1^*	x_2^*	x_3^*	**A2**	**efficiency**
17	0	166	0	**166**	**1**
18	0	761	0	**761**	**1**
19	193	124	0	**317**	**0.924**
20	432	0	0	**432**	**0.745**
21	0	455	0	**455**	**0.862**
22	0	511	0	**511**	**0.859**
23	0	573	0	**573**	**0.828**
24	0	428	0	**428**	**0.748**
25	0	446	0	**446**	**0.726**
26	0	332	0	**332**	**0.89**
27	0	346	0	**346**	**0.971**
28	0	0	70	**70**	**1**
29	0	177	0	**177**	**0.53**
30	0	125	0	**125**	**0.464**
31	262	0	0	**262**	**0.835**
32	0	794	0	**794**	**0.776**
33	349	762	0	**1111**	**0.816**
34	0	251	15	**266**	**0.921**
35	172	375	0	**547**	**1**
36	0	460	0	**460**	**0.932**
37	223	0	535	**758**	**0.622**
38	0	1158	0	**1158**	**0.919**
39	0	394	0	**394**	**0.362**
40	0	0	67	**67**	**0.971**
41	245	0	0	**245**	**0.649**
42	271	0	0	**271**	**0.609**

Table 2 shows the values obtained from model A2, in which the set of production possibilities is limited from real values to integers. The first column is the number of units, the next three columns represent the optimal values of the inputs, the fourth column displays the values obtained from the model A2, and the last column presents the cost efficiency values as calculated. As can be seen in the table, the values are integer due to the consideration of integer optimal inputs in the model. As stated in Theorem 3.1, the values obtained from model A2 are greater than or equal to the values obtained from model A1, which can be verified by comparing the minimum cost values in this table.

Table 3. Optimal inputs, minimum cost & cost efficiency values in model (5.1)

DMU	x_1^*	x_2^*	x_3^*	**A3**	**efficiency**
1	0	230	0	**230**	**0.869**
2	0	170	46	**216**	**0.942**
3	0	281	48	**329**	**0.928**
4	0	138	23	**161**	**0.929**
5	164	0	0	**164**	**0.371**
6	291	723	0	**1014**	**0.916**
7	0	0	61	**61**	**1**
8	0	83	0	**83**	**0.355**
9	0	675	0	**675**	**0.928**
10	0	697	0	**697**	**0.901**
11	0	0	50	**50**	**0.819**
12	36	96	0	**132**	**0.243**
13	0	872	0	**872**	**0.864**
14	0	0	34	**34**	**1**
15	0	603	0	**603**	**0.755**
16	0	672	0	**672**	**0.976**
17	0	166	0	**166**	**1**
18	0	761	0	**761**	**1**
19	193	124	0	**317**	**0.924**
20	432	0	0	**432**	**0.745**
21	0	455	0	**455**	**0.862**
22	0	511	0	**511**	**0.859**
23	0	573	0	**573**	**0.828**
24	0	428	0	**428**	**0.748**
25	0	446	0	**446**	**0.726**
26	0	332	0	**332**	**0.89**
27	0	346	0	**346**	**0.971**
28	0	0	70	**70**	**1**
29	0	177	0	**177**	**0.53**

DMU	x_1^*	x_2^*	x_3^*	**A3**	**efficiency**
30	0	125	0	**125**	**0.464**
31	262	0	0	**262**	**0.835**
32	0	794	0	**794**	**0.776**
33	349	762	0	**1111**	**0.816**
34	0	251	15	**266**	**0.921**
35	172	375	0	**547**	**1**
36	0	460	0	**460**	**0.932**
37	223	0	535	**758**	**0.622**
38	0	1158	0	**1158**	**0.919**
39	0	394	0	**394**	**0.362**
40	0	0	67	**67**	**0.971**
41	245	0	0	**245**	**0.649**
42	271	0	0	**271**	**0.609**

Table 3 displays the values obtained from model A3. The first column is the number of units, the next three columns provide the optimal values of the inputs, the fourth column contains the values obtained from the model A3, and the last column shows the cost efficiency values as calculated. As noted before, this model, in addition to limiting real values to integers, moves in a specified direction. In fact, it seeks to obtain the minimum cost at another level of outputs. By comparing the values of the models A2 and A3, as per Theorem 4.1, it can be concluded that the values obtained from the model A3 are smaller than or equal to the values of the model A2. In model A3, it has been achieved that the same values obtained from model A2 can be obtained at another level of output, allowing the manager to gain additional efficiency, for example, expanding the factory.

Table 4. Optimal outputs, maximum revenue & revenue efficiency values in model (2.2)

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B1**	**efficiency**
1	261	11	0	6.28	**278.29**	**0.826**
2	222.7	7.168	0	7.39	**237.26**	**0.918**
3	346.88	11.84	0	10.88	**369.62**	**0.895**
4	169.05	5.81	0	5.26	**180.14**	**0.899**
5	137.08	1.87	0	1.87	**140.83**	**0.397**
6	1106	14.38	5.07	9.42	**1134.88**	**0.898**
7	57.41	0	0	3.58	**61**	**0.885**

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B1**	**efficiency**
8	196.6	1.8	0.76	1.55	**200.72**	**0.373**
9	727	30.65	0	17.51	**775.17**	**0.878**
10	773	32.59	0	18.62	**824.22**	**0.851**
11	62.11	0	0	3.88	**66**	**0.742**
12	516.08	5.23	1.57	4.7	**527.6**	**0.252**
13	988	41.66	0	23.8	**1053.47**	**0.789**
14	32	0	0	2	**34**	**1**
15	795	33.52	0	19.15	**847.68**	**0.72**
16	672	28.33	0	16.19	**716.53**	**0.852**
17	166	7	0	4	**177**	**1**
18	761	32.09	0	18.33	**811.42**	**0.944**
19	294.65	2.88	0.99	2.55	**301.08**	**0.983**
20	404.56	5.54	0	5.54	**415.64**	**0.87**
21	517	21.8	0	12.45	**551.25**	**0.798**
22	584	24.62	0	14.07	**622.69**	**0.801**
23	682	28.75	0	16.43	**727.19**	**0.71**
24	565	23.82	0	13.61	**602.43**	**0.682**
25	603	25.42	0	14.53	**642.95**	**0.842**
26	373	15.72	0	8.98	**397.71**	**0.908**
27	347	14.63	0	8.36	**369.99**	**0.828**
28	65.88	0	0	4.11	**70**	**0.497**
29	328	13.83	0	7.9	**349.73**	**0.442**
30	267	11.25	0	6.43	**284.69**	**1**
31	219	3	0	3	**225**	**0.733**
32	1023	43.13	0	24.65	**1090.78**	**0.801**
33	1361	16.82	6.38	11.13	**1395.33**	**0.83**
34	280.11	11.21	0	7.29	**298.62**	**1**
35	547	4	3	3	**557**	**0.815**
36	460	19.39	0	11.08	**490.48**	**0.352**
37	689.93	2.55	0	34.02	**726.5**	**0.875**
38	1256.58	50.68	0	32.37	**1339.65**	**0.348**
39	1082.41	43.22	0	28.28	**1153.92**	**0.811**
40	64.94	0	0	4.05	**69**	**0.76**

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B1**	**efficiency**
41	262.46	3.59	0	3.59	**269.65**	**0.712**
42	310.11	4.24	0	4.24	**318.6**	**0.228**

Table 4 presents the values obtained from the primary maximum revenue model (B1). The first column in Table 4 displays the number of units, the second column shows the optimal values of the first output (y_1^*), the third column holds the optimal values of the second output (y_2^*), the fourth column presents the optimal value of the third output (y_3^*), the fifth column shows the optimal value of the fourth output (y_4^*), and the calculated last column provides the revenue efficiency values. The objective function calculates the maximum revenue in DMUs; however, the efficiency of each unit is equal to the revenue value of each unit divided by the value obtained from the objective function. As can be seen in the table, there is no integer-valued condition for the optimal outputs; thus, the real values have been obtained.

Table 5. Optimal outputs, maximum revenue & revenue efficiency values in model (3.2)

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B2**	**efficiency**
1	254	9	1	5	**269**	**0.855**
2	222	7	0	7	**236**	**0.923**
3	346	11	0	10	**367**	**0.901**
4	169	5	0	5	**179**	**0.905**
5	52	1	0	3	**56**	**1**
6	1106	14	5	9	**1134**	**0.899**
7	50	0	0	4	**54**	**1**
8	194	3	0	2	**199**	**0.376**
9	727	30	0	17	**774**	**0.879**
10	773	32	0	18	**823**	**0.852**
11	62	0	0	3	**65**	**0.753**
12	514	6	1	5	**526**	**0.252**
13	920	23	10	12	**965**	**0.862**
14	32	0	0	2	**34**	**1**
15	782	29	2	16	**829**	**0.737**
16	591	6	12	2	**611**	**1**
17	166	7	0	4	**177**	**1**
18	761	0	3	2	**766**	**1**

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B2**	**efficiency**
19	293	3	0	3	**299**	**0.989**
20	404	5	0	5	**414**	**0.874**
21	490	14	4	7	**515**	**0.854**
22	557	17	4	9	**587**	**0.85**
23	661	23	3	12	**699**	**0.818**
24	551	20	2	11	**584**	**0.732**
25	582	19	3	11	**615**	**0.713**
26	367	12	1	7	**387**	**0.865**
27	328	7	3	4	**342**	**0.69**
28	51	0	3	4	**58**	**1**
29	322	10	1	6	**339**	**0.513**
30	267	11	0	6	**284**	**0.443**
31	219	3	0	3	**225**	**1**
32	1023	43	0	24	**1090**	**0.733**
33	1361	16	6	11	**1394**	**0.802**
34	254	4	4	3	**265**	**0.935**
35	547	4	3	3	**557**	**1**
36	385	4	8	3	**400**	**1**
37	322	14	6	4	**346**	**0.728**
38	1256	50	0	32	**1338**	**0.876**
39	1077	43	1	28	**1149**	**0.348**
40	54	0	2	4	**60**	**0.933**
41	262	3	0	3	**268**	**0.764**
42	310	4	0	4	**318**	**0.713**

In model B2, the set of production possibilities is limited from real values to integers. The values obtained from this model are provided in Table 5. The first column is the number of units, the next four columns represent the optimal values of the outputs, the fifth column displays the values obtained from the model B2, and the last column presents the revenue efficiency values as calculated. As noted in Theorem 3.2, the values obtained from the objective function of model B2 are smaller than or equal to the values obtained from the objective function of model B1.

Table 6. Optimal outputs, maximum revenue & revenue efficiency values in model (5.2)

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B3**	**efficiency**
1	254	9	1	5	**269**	**0.855**
2	222	7	0	7	**236**	**0.923**
3	346	11	0	10	**367**	**0.901**
4	169	5	0	5	**179**	**0.905**
5	52	1	0	3	**56**	**1**
6	1106	14	5	9	**1134**	**0.899**
7	50	0	0	4	**54**	**1**
8	194	3	0	2	**199**	**0.376**
9	727	30	0	17	**774**	**0.879**
10	773	32	0	18	**823**	**0.852**
11	62	0	0	3	**65**	**0.753**
12	514	6	1	5	**526**	**0.252**
13	920	23	10	12	**965**	**0.862**
14	32	0	0	2	**34**	**1**
15	782	29	2	16	**829**	**0.737**
16	591	6	12	2	**611**	**1**
17	166	7	0	4	**177**	**1**
18	761	0	3	2	**766**	**1**
19	293	3	0	3	**299**	**0.989**
20	404	5	0	5	**414**	**0.874**
21	490	14	4	7	**515**	**0.854**
22	557	17	4	9	**587**	**0.85**
23	661	23	3	12	**699**	**0.818**
24	551	20	2	11	**584**	**0.732**
25	582	19	3	11	**615**	**0.713**
26	367	12	1	7	**387**	**0.865**
27	328	7	3	4	**342**	**0.69**
28	51	0	3	4	**58**	**1**
29	322	10	1	6	**339**	**0.513**
30	267	11	0	6	**284**	**0.443**
31	219	3	0	3	**225**	**1**

DMU	y_1^*	y_2^*	y_3^*	y_4^*	**B3**	**efficiency**
32	1023	43	0	24	**1090**	**0.733**
33	1361	16	6	11	**1394**	**0.802**
34	254	4	4	3	**265**	**0.935**
35	547	4	3	3	**557**	**1**
36	385	4	8	3	**400**	**1**
37	322	14	6	4	**346**	**0.728**
38	1256	50	0	32	**1338**	**0.876**
39	1077	43	1	28	**1149**	**0.349**
40	54	0	2	4	**60**	**0.933**
41	262	3	0	3	**268**	**0.764**
42	310	4	0	4	**318**	**0.713**

Table 6 shows the values obtained from model B3. The first column is the number of units, the next four columns provide the optimal values of the outputs, the fifth column contains the values obtained from the model B3, and the last column shows the revenue efficiency values as calculated. As stated before, this model, in addition to limiting real values to integers, moves in a specified direction. In fact, it seeks to obtain the maximum revenue at another level of output. When comparing the values of Models B2 and B3, according to Theorem 4.2, it can be concluded that the values obtained from Model B3 are greater than or equal to the values of Model B2. In model B3, it has been achieved that the same values obtained from model B2 can be obtained at another level of output, allowing the manager to gain additional efficiency, for example, expanding the factory.

By introducing φ in the model, it is possible to obtain the maximum revenue at another level of inputs. In model B3, it has been achieved that the same values obtained from model B2 can be obtained at another level of input. However, as noted in Theorem 4.2, the values obtained by the Lingo program in the objective function of model B3 are greater than or equal to the values obtained from the objective function in model B2.

7. Conclusions

In DEA, all data are real values, but in the real world, there is a need to work with integer values. In recent years, integer-valued DEA has gained the attention of researchers. This paper discusses cost efficiency and revenue efficiency. First, we examine the primary models that analyse the set of real values. Then, the model is used to rewrite and restrict the production possibility set, from real values to integers. Finally, cost efficiency and revenue efficiency models are introduced with the application of a directional distance function in a specific direction.

In these models, by assuming that the set of production possibilities is a set of real values and integers, the model is constructed, and, through the directional distance function, the environment is made more stringent so that the model progresses in a particular direction.

The model is introduced with these assumptions. In this model, the cost and revenue values are checked at another level of output and inputs, and at the desired level, a minimum cost and a maximum revenue are pursued. According to the axioms, constant returns to scale allow for the same output with a minimum cost, and the maximum revenue with the specified cost is also sought. The models are tested using an example from Kuosmanen and Kazemi Matin' s paper. The results obtained from the data analysis in Lingo program showed that although conditions had been made more stringent, the values obtained in the final calculations only had minor differences and in some cases were even equal.

Overall, this paper has highlighted the importance of using integer-valued DEA models to identify the most efficient DMUs. By using these models, managers can gain additional efficiency by expanding their factories and obtaining the same values from the model at another level of output. It is hoped that this research will encourage future studies on the use of integer-valued DEA models.

Future studies could explore the application of integer-valued DEA models to other contexts, such as healthcare, education, or public services. Additionally, further research could be conducted on the use of integer-valued models to measure the efficiency of different aspects of a company, such as customer service, marketing, or operations. Finally, research could also be conducted to examine the relationship between integer-valued DEA models and other methods of efficiency measurement, such as data envelope analysis (DEA) and stochastic frontier analysis (SFA).

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Appendix

Proof of Theorem 3.1: The set of production possibilities in the model A1 discusses the set of real numbers, while in the model A2, the set of production possibilities is limited and the model is investigated on the integer sets. It is a subset for A1, on the other hand, since it is a minimisation problem, as a result, the optimal solution is smaller than the feasible solution. As a result, the value of the objective function in the model A1 is smaller or equal than to the model A2, then $A2 \geq A1$.

Proof of Theorem 4.1: The production possibility set in model A3 is restricted to integer values and also moves in a specific direction. In the other words, the production possibility set more restrictive and also more rigorous is investigated. As a result, the value of the objective function is greater than or equal to the model A2, which is only discussed in the integer sets. On the other hand, by assuming $\varphi = 1$, $\beta = 0$, and also, if the second constraint is multiplied by a constant value, the fourth constraint becomes redundant. Also, any arbitrary feasible solution of the model A3 is also feasible for model A2, and since it is a minimisation problem, the optimal solution is smaller than the feasible solution. So we have $A3 \leq A2$.