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OPTIMAL MANAGEMENT OF PRODUCTION STOCKS WITH THE NEUTROSOPHIC FUZZY NUMBERS

Abstract. *The paper aims to shape the companies' production stocks using fuzzy neutrosophic numbers. The stock modeling using neutrosophic and triangular fuzzy numbers improves the company's financial performance, new indicators being proposed, such as: the optimal quantity of stocks, the cost minimization function, the cost of placing order, and the cost of storing a purchased unit of product. The significant advantage of shaping stocks management using fuzzy neutrosophic numbers is that it allows companies to determine the optimal quantities of stocks to be supplied using nonlinear mathematical programming algorithms. The innovative models for determining the optimal quantity of stocks are structured in two categories, namely: optimal stock supply models with fuzzy neutrosophic variables, with a single product, a constant demand, with out-of-stock, as well as models of optimal supply of stocks, with fuzzy neutrosophic variables, with several products, a constant demand and with out-of-stock. Both models use algorithms specific to nonlinear mathematical programming and provide a complete picture of the companies' stocks acquisition strategies needed for its operational activity. Finally, the results obtained from the modeling/simulation validate the operationalisation of the stock models presented and the modern foundation of companies' decisions on stock performance indicators.*

Keywords*: fuzzy neutrosophic numbers, production management, optimisation stock models, triangular fuzzy numbers, storage management*

JEL Classification: D24, E23, M11, O15

1. Introduction

The optimal management of companies' current production assets, especially those in the stock's category, is a company's strategic objective. For the operational activity, most of a company's expenditure is accounted for by stocks expenditure because it underpins the technological processes of companies to obtain finished products. Furthermore, the stock decisions directly impact the performance

DOI: 10.24818/18423264/57.2.23.02

of companies in terms of cash or benefits (Bogdan et al., 2017). Stocks expenses also account for a significant share of the total operating expenses of companies. Hence, the study of optimisation issues is of strategic importance for the foundation of any company's decisions. The high importance of the information provided by financial and managerial accounting in the stocks management decision-making process is shown by Scorțe and Farcaș (2013).

The stocks management decisions are of high importance when talking about factors that affect the performance of companies. The literature has developed stock models that allow the calculation of stock performance indicators. However, these performance indicators do not contain enough information on the probability of achievement or non-achievement performance indicators using neutrosophic fuzzy numbers, and the probability of their uncertainty, also determined using neutrosophic fuzzy numbers.

In the category of their stocks acquisition, the costs of placing an order (c_l), storing a unit of purchased products (c_s), and the penalty cost for the lack of a stock unit $\left(c_p\right)$. These cost categories determine the optimal quantities of stocks to be released by an order (q_c^*), the number of orders (k) placed over a period T. The optimal amount of stocks originally existing (q_i^*), cost minimisation functions $fC(q_c^*,\ q_i^*)$, allow the minimum cost value of a company to be determined so that the costs generated by stock formation, regardless of their nature, have a positive impact on the company's performance.

As a result, the literature has established and founded the stock models (Paterson et al., 2010) consisting of a single product or several products, with or without out-of-stock, modeled in the present paper using neutrosophic fuzzy numbers. Their general objective is to minimise the overall cost function $fC(q_c^*,\ q_i^*)$, conditioned on company-specific constraints. For example, the storage capacity depends on the storage area (c_d) for each product and the total storage capacity available to the company.

In this context, this research paper aims to improve the foundation of the decision to optimally train stocks by shaping them using fuzzy neutrosophic triangular numbers. Therefore, the performance parameters of the stocks are in the form $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle$ for the optimal quantity of stocks in order, or $\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle$ for the optimal quantity of stocks originally existing. Also, the cost minimisation function $fC\left(\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, u_{\tilde{q}_c^*} \rangle; \langle \tilde{q}_i^*; w_{\tilde{q}_i^*}, u_{\tilde{q}_i^*}, y_{\tilde{q}_i^*} \rangle\right)$ as a result, the stock modeling using neutrosophic fuzzy numbers must improve the company's financial performance. In addition, other specific components of the calculation such as the cost of placing order (c_l), the cost of storing a purchased unit of product (c_s), or penalty cost $\left(c_p\right)$ are modeled using fuzzy triangular numbers to improve the foundation of decisions in the process of forming current assets.

The performance indicators of the stocks are supplemented with additional information specific to the fuzzy neutrosophic numbers, respectively, with the probability of achievement$\left(w_{q^*}\right)$ their probability of uncertainty $\left(u_{q^*}\right)$ or non-achievement$\left(y_{q^*}\right)$. Thus, fuzzy triangular neutrosophic numbers solve the problem of completing stock performance indicators with information specific to neutrosophic fuzzy numbers on probabilities of achievement/non-achievement and uncertainty necessary for the decision-making process. In contrast, simple triangular fuzzy numbers solve the problem of belonging specific costs to the cost classes assessed by the linguistic terms.

The stock models presented in this article have an innovative approach because they use the fuzzy neutrosophic triangular numbers and nonlinear mathematical programming to model performance indicators specific to production stocks. Finally, the results obtained from the modeling/simulation validate the operationalisation of the stock models presented and the modern basis of decisions on stock performance indicators.

2. Literature review

The optimal stocks management is an important research topic because it directly affects the financial performance of companies in the medium and long term. Over the years, researchers worldwide have dedicated their attention to studying this topic for studying this critical field of operational activity of companies, with 825 ISI scientific papers already being recorded in the WoS database in April 2023, including the keywords in their subject: "*optimal management of production stocks*". The first article published on this topic of research appeared in 1992 and, as of 2007, their number increased significantly, as shown in Figure no.1.

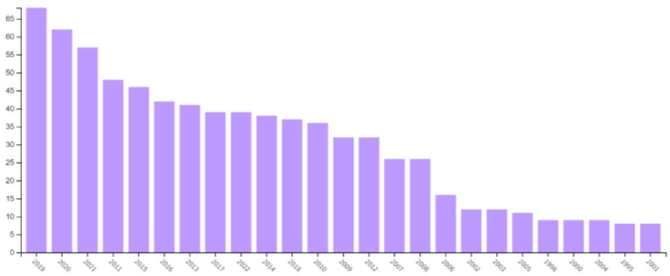

Figure 1. The evolution of studies published in ISI journals on the topic 'optimal management of production stocks' from 1991 to 2021

Most of the ISI articles shown in figure 1 fall into the following areas of research: operations research management science (30%), engineering manufacturing (15%), engineering industrial (12%), management (11%), economics (7%). About the country where the authors of the articles come from, the USA ranks first with 24% of all scientific studies published in WoS, followed by China (13%), Australia (9%), France (6%), Canada (5%), Spain, Germany, Italy, England (4%).

In this section, the most important articles related to optimal stock management are presented. Fahimnia et al. (2015) reviewed the existing quantitative models in the literature to manage stock supply chain risks. The authors conclude that supply chain risk is being thoroughly researched. Also, in his work on liquidity management, Michalski (2010) states that there must be a sufficient balance between cash and production assets in the form of receivables and stocks. If the level of production assets is not appropriate for its business, it increases its operational risk through loss of liquidity (Chen et al., 2011).

Although Su and Lin (2015) analysed the problems caused by order-based production by developing a fuzzy multi-objective linear programming model, the authors managed to solve the problems of procurement, production, planning decisions with fuzzy, multi-component, multi-supplier, multi-source, and multi-machine environments, within recoverable remanufacturing systems. Altendorfer (2018) developed an analytical model to optimise the planning parameters of the stochastic production batch with several items and limited capacity. The main findings are that the optimal size of the production batch and the reserve stock are directly correlated with the storage capacity and the rate order related to those items. A few years later, the authors Tavaghof-Gigloo and Minner (2020) conducted an experimental study by comparing the performance of the integrated model of solving the problem of stochastic sizing of the production batch with the sequential approach. The authors conclude that the performance of the integrated model is indirectly influenced by the target service level.

In their paper, Sodenkamp et al. (2016) integrate multi-criteria decision analysis and linear programming into stock management. The authors maximise the total value of purchases by optimising the allocation of orders to suppliers, applying the proposed model on selecting contractors, and allocating orders in an agricultural commodity trading company. Also, in their research paper, Woerner et al. (2018) develop an algorithm to find the optimal capacity allocation and the level of the reserve stock to minimise the costs of storing the products. Mula et al. (2014) proposed fuzzy mathematical programming to plan the supply chain's production in a multi-product, multi-plant environment with fuzzy capacity constraints. Innovative stocks management models have been developed by Bouchery et al. (2012), Mousavi et al. (2012), Keshavarz Ghorabaee et al. (2015), He et al. (2018, 2020), Pan and Yang (2008).

So, the models that allow calculations using mathematical programming, fuzzy logic, and neutrosophic numbers regarding the companies' performance indicators have aroused the researchers' interest and are described by the specialised literature. Thus, the evolution of using neutrosophic sets starts from the concept of a neutrosophic set that goes beyond the difficulties of fuzzy and fuzzy intuitive sets and becomes helpful in all fields of science and engineering.

There are several situations where the multi-criterial decision-making process works with uncertain, imprecise, and inconsistent information. However, Bausys et al. (2015) and Bausys and Zaadskas (2015) developed the COPRAS method to solve decision-making problems in the context of the neutrosophic set. Wang and Wang (2019) propose and validate in their research an extended VIKOR method for the interval of neutrosophic numbers. Aydin et al. (2018) presented the use of neutrosophic logic to evaluate the most profitable investment options for a company. In order to overcome the difficulties in defining the functions belonging to the investment parameters, they developed a simplified method of analysing the neutrosophic present value. In their paper, Khalifa and Kumar (2020) solved a neutrosophic linear programming problem with single value neutrosophic numbers. The score-based method applied to a portfolio of shares was proposed.

Neutrosophic numbers can also be applied in economics to decision-making when the ambiguity or complexity of attributes make it impossible to assess problems with real numbers (Khan et al. (2018); Stanujkic et al. (2019); Bolos et al. (2019).

3. Prerequisites

The indicators specific to production stocks are diverse such as the total quantity of products $\{\tilde{Q}\}$, the number of products in an order $\{\tilde{q}\}$, the number of specific orders (k), the period-specific to an order (t), the categories of costs specific to the supply with specific costs (c_i) as well as other categories of specific indicators that are used in the modeling process. Two fuzzy number categories will be used to model stocks management-specific indicators: neutrosophic fuzzy numbers and fuzzy triangular numbers.

3.1. Neutrosophic fuzzy numbers used in stocks production modeling

Neutrosophic fuzzy numbers, as opposed to classical fuzzy numbers, have specifics that they contain additional information about the probability of achieving a specific stocks management indicator denoted by $\left(w_{\tilde{I}_i}\right)$, the probability of uncertainty of a specific stocks management indicator denoted by $\left(u_{\tilde{I}_i}\right)$, as well as the probability of non-achievement of a specific stocks management indicator denoted by $\left(y_{\tilde{I}_i}\right)$. These neutrosophic fuzzy number probabilities are attached to the determined membership functions for stocks management-specific indicators and provide additional information for the optimal stock management decision-making process.

Definition 1: Either the universe of speech, the multitude of total quantities of products $\{\tilde{Q}\}$ and the quantities of products in an order $\{\tilde{q}\}$ and $\mathrm{F}[0,1]$ set of rules for all fuzzy neutrosophic triangular numbers. We say the fuzzy number $\langle\tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}}\rangle$ and respectively $\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle$ is called a neutrosophic triangular number of the total quantity of products/quantity of products in order of $\tilde{Q}/\tilde{q}_c = \{\langle\tilde{Q}/\tilde{q}_c, \mu_{\tilde{Q}/\tilde{q}_c}, \vartheta_{\tilde{Q}/\tilde{q}_c}, \lambda_{\tilde{Q}/\tilde{q}_c}\rangle\, Q/q_c \in \tilde{Q}/\tilde{q}_c\}$ where $\mu_{\tilde{Q}/\tilde{q}_c}: Q/q_c \to [0,1]$; $\vartheta_{\widetilde{Ra}}: Q/q_c \to [0,1]$ și $\lambda_{\tilde{Q}/\tilde{q}_c}: Q/q_c \to [0,1]$ for which membership functions are defined following the relationships below:

Membership function for the total quantities/quantities in a product order most likely to be needed (achievement) $\left(\mu\tilde{Q}/\tilde{q}_{(x)}\right)$ (truth membership).

$$\mu\tilde{Q}/\tilde{q}_{c_{(x)}} = \begin{cases} \frac{w_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_x} - \tilde{Q}/\tilde{q}_{c_{a1}})}{\tilde{Q}/\tilde{q}_{c_{b1}} - \tilde{Q}/\tilde{q}_{c_{a1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{a1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{b1}} \\ w_{\tilde{Q}/\tilde{q}_c} \; for \; \tilde{Q}/\tilde{q}_{c_x} = \tilde{Q}/\tilde{q}_{c_{b1}} \\ \frac{w_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_x})}{\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_{b1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{b1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{c1}} \\ 0 \; for \; any \; values \; outside \; the \; range \; [\tilde{Q}/\tilde{q}_{c_{c1}}; \tilde{Q}/\tilde{q}_{c_{a1}}] \end{cases} \quad (1)$$

Membership function for total quantities/quantities in a product order with the uncertain probability of need (achievement) (indeterminacy membership)

$$\vartheta\tilde{Q}/\tilde{q}_{c_{(x)}} = \begin{cases} \frac{u_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_x} - \tilde{Q}/\tilde{q}_{c_{a1}})}{\tilde{Q}/\tilde{q}_{c_{b1}} - \tilde{Q}/\tilde{q}_{c_{a1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{a1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{b1}} \\ u_{\tilde{Q}/\tilde{q}_c} \; for \; \tilde{Q}/\tilde{q}_{c_x} = \tilde{Q}/\tilde{q}_{c_{b1}} \\ \frac{u_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_x})}{\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_{b1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{b1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{c1}} \\ 0 \; for \; any \; values \; outside \; the \; range \; [\tilde{Q}/\tilde{q}_{c_{c1}}; \tilde{Q}/\tilde{q}_{c_{a1}}] \end{cases} \quad (2)$$

Membership function for the total quantities/quantities in a product order with the lowest probability of need (achievement) $(\lambda\tilde{Q}/\tilde{q}_{(x)})$(falsity membership)

$$\lambda\tilde{Q}/\tilde{q}_{c_{(x)}} = \begin{cases} \frac{y_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_x} - \tilde{Q}/\tilde{q}_{c_{a1}})}{\tilde{Q}/\tilde{q}_{c_{b1}} - \tilde{Q}/\tilde{q}_{c_{a1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{a1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{b1}} \\ y_{\tilde{Q}/\tilde{q}_c} QC \; for \; \tilde{Q}/\tilde{q}_{c_x} = \tilde{Q}/\tilde{q}_{c_{b1}} \\ \frac{y_{\tilde{Q}/\tilde{q}_c}(\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_x})}{\tilde{Q}/\tilde{q}_{c_{c1}} - \tilde{Q}/\tilde{q}_{c_{b1}}} \; for \; \tilde{Q}/\tilde{q}_{c_{b1}} \leq \tilde{Q}/\tilde{q}_{c_x} \leq \tilde{Q}/\tilde{q}_{c_{c1}} \\ 0 \; for \; any \; values \; outside \; the \; range \; [\tilde{Q}/\tilde{q}_{c_{c1}}; \tilde{Q}/\tilde{q}_{c_{a1}}] \end{cases} \quad (3)$$

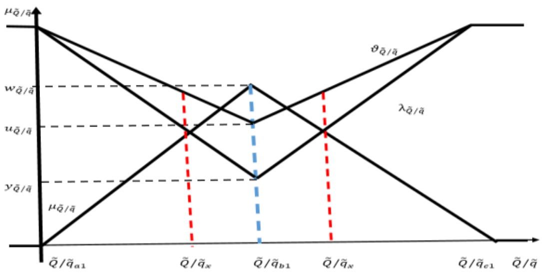

Figure 2. Fuzzy triangular neutrosophic number for shaping product quantities $\tilde{Q}/\tilde{q}$

Based on the membership functions, and the degrees of belonging of the two specific indicators are established, namely the total quantity of products, as well as the specific quantities in an order that will contain additional information regarding the probability $(u_{\tilde{Q}/\tilde{q}_c})(y_{\tilde{Q}/\tilde{q}_c})\left(\mu\tilde{Q}/\tilde{q}_{c_{(x)}}\right)\left(\vartheta\tilde{Q}/\tilde{q}_{c_{(x)}}\right)\left(\lambda\tilde{Q}/\tilde{q}_{c_{(x)}}\right)$ $\{\tilde{Q}\}\{\tilde{q}\}$ of realization, $(w_{\tilde{Q}/\tilde{q}_c})$ the probability of uncertainty $(u_{\tilde{Q}/\tilde{q}_c})$ as well as the probability of non-realisation $(y_{\tilde{Q}/\tilde{q}_c})$.

3.2. Triangular fuzzy numbers used in stocks production modeling

The use of fuzzy triangular numbers in modeling stock management has several advantages over the classical methods, among which we mention: allowing the transformation of stocks' indicators with different units of measurement into specific indicators comparable to the same unit of measurement; allowing the use of non-financial indicators in modeling the production stocks with the linguistic terms; allowing the clustering of indicators specific to production stocks by classes of importance/size.

Definition 2: Let the discourse universe be the set of specific costs for sourcing production stocks and F[0,1] set of rules for all $\{c_i\}$ fuzzy triangular numbers. We say that the triangular fuzzy number $[0,1]c_i\colon\{c_i, \mu_{c_i}\}$ with has the function of belonging defined by the form:$\mu_{c_i}\colon c_i \to [0,1]$

$$\mu c_{i_{(x)}} = \begin{cases} \frac{c_{i_x}-c_{i_{a1}}}{c_{i_{b1}}-c_{i_{a1}}}\ pentru\ c_{i_{a1}} \leq c_{i_x} \leq c_{i_{b1}} \\ c_{i_{b1}}\ pentru\ c_{i_x} = c_{i_{b1}} \\ \frac{c_{i_{c1}}-c_{i_x}}{c_{i_{c1}}-c_{i_{b1}}}\ pentru\ c_{i_{b1}} \leq c_{i_x} \leq c_{i_{c1}} \\ 0\ for\ any\ values\ outside\ the\ range\ \left[c_{i_{c1}}; c_{i_{a1}}\right] \end{cases} \tag{4}$$

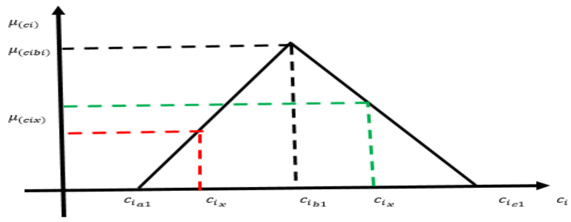

Figure 3. Triangular fuzzy number used in modeling optimal stock management

From Figure 3, it is noticed that for each fuzzy triangular number of the shape $c_i\colon\{\, c_i, \mu_{c_i}\}$ one can define its function of belonging, which has the advantage that for each value of the specific cost for the supply of stocks, one can determine the degree of belonging to a class of fuzzy numbers with the help of the function of belonging $\mu_{c_i}\colon c_i \to [0,1]\mu_{c_i}$.

4. Stocks optimisation model with neutrosophic fuzzy variables

4.1. A single product, and out of stock

The stocks optimisation model has a single product in its structure, the demand is constant, and the optimisation variables are neutrosophic fuzzy variables. The ultimate goal of the model is to determine the optimal order $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*}\rangle$ quantity, the initial quantity existing in stock $\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*}\rangle$ as well as the optimal cost function of the form $fC\big(\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*}\rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*}\rangle\big)$. The model with fuzzy neutrosophic variables, with out of stock, with a single product, considers the following hypotheses:

Hypothesis 1: The total quantity by which the supply of products is made over a period (T) is denoted with $\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle$, where demand for products on the market is considered constant. The quantity in order is unknown, but the same each time is marked with $\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle$ and takes place over a period (t), of (k) times. Periods (t) specific to order are part of period (T).

Between the total quantity of products $\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle$ required over the period (T), and the number of products in an order $\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle$ the equational models of the shape are formed: $k = \frac{\langle T \rangle}{\langle t \rangle} = \frac{\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{\langle \tilde{q}; w_{\tilde{q}}, u_q, y_{\tilde{q}} \rangle}$ (5)

Hypothesis 2: The quantity of products in stock at the beginning of each supply period is the same and shall be noted $\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle$. The quantity of products in stock is quantified with a specific frequency after replenishing the stocks necessary for the company's current activity. The probability of achievement is $(w_{\tilde{q}_i})$, the probability of uncertainty is $(u_{\tilde{q}_i})$, and the probability of non-achievement $(y_{\tilde{q}_i})$.

Hypothesis 3: The cost of placing an order is noted with (c_l) and is a simple triangular fuzzy number of the shape $c_l\colon \{c_l, \mu_{c_l}\}$ with $\mu_{c_l}\colon c_l \rightarrow [0,1]$ with the membership function defined by the form:

$$\mu c_{l(x)} = \begin{cases} \frac{c_{l_x} - c_{l_{a1}}}{c_{l_{b1}} - c_{l_{a1}}} \; for \; c_{l_{a1}} \leq c_{l_x} \leq c_{l_{b1}} \\ c_{l_{b1}} \; for \; c_{l_x} = c_{l_{b1}} \\ \frac{c_{l_{c1}} - c_{l_x}}{c_{l_{c1}} - c_{l_{b1}}} \; for \; c_{l_{b1}} \leq c_{l_x} \leq c_{l_{c1}} \\ 0 \; for \; any \; values \; outside \; the \; range \; [c_{l_{c1}}; c_{l_{a1}}] \end{cases} \tag{6}$$

Hypothesis 4: The average cost of storing the number of products is determined by the average quantity of products existing on stock $\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle$, the unit cost of storage c_s which is a triangular fuzzy number, as well as the storage period (t_1) after a formula of form: $\bar{C}_s = \frac{1}{2} \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle \times c_s \times t_1$ (7)

The triangular fuzzy number specific to the cost of storage is in the form $c_s\colon \{c_s, \mu_{c_s}\}$ cu $\mu_{c_s}\colon c_s \rightarrow [0,1]$ with the form membership function:

$$\mu c_{s(x)} = \begin{cases} \frac{c_{s_x} - c_{s_{a1}}}{c_{s_{b1}} - c_{s_{a1}}} \; for \; c_{s_{a1}} \leq c_{s_x} \leq c_{s_{b1}} \\ c_{s_{b1}} \; for \; c_{s_x} = c_{s_{b1}} \\ \frac{c_{s_{c1}} - c_{s_x}}{c_{s_{c1}} - c_{s_{b1}}} \; for \; c_{s_{b1}} \leq c_{s_x} \leq c_{s_{c1}} \\ 0 \; for \; any \; values \; outside \; the \; range \; [c_{s_{c1}}; c_{s_{a1}}] \end{cases} \tag{8}$$

Hypothesis 5: The average cost due to the lack of stock is determined by the number of products missing from stock $\frac{1}{2}\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle - \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle$, the missing period in stock $(t - t_1)$ as well as the cost of the penalty per unit of stock (c_p). As with the cost of storage and the cost of placing an order, the penalty cost per unit of stock is a triangular fuzzy number of the shape $c_p\!:\left\{c_p, \mu_{c_p}\right\}$ cu $\mu_{c_p}\!:\ c_p \to [0,1]$ with the form membership function:

$$\mu c_{p_{(x)}} = \begin{cases} \frac{c_{s_p} - c_{p_{a1}}}{c_{p_{b1}} - c_{p_{a1}}}\ for\ c_{p_{a1}} \le c_{p_x} \le c_{p_{b1}} \\ c_{p_{b1}}\ for\ c_{p_x} = c_{p_{b1}} \\ \frac{c_{p_{c1}} - c_{p_x}}{c_{p_{c1}} - c_{p_{b1}}} for\ c_{p_{b1}} \le c_{p_x} \le c_{p_{c1}} \\ 0\ for\ any\ values\ outside\ the\ range\ \left[c_{p_{c1}}; c_{p_{a1}}\right] \end{cases} \tag{9}$$

The average cost of the penalty for the lack of stock will be determined with the calculation formula:

$$C_p = \frac{1}{2}\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle - \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right) \times (t - t_1) \times c_p \tag{10}$$

Total cost function for an order $fC\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle; \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right)$ is given by the relationship:

$$fC\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\widetilde{q_c}}\rangle; \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right) = \Big[c_l + \frac{1}{2}\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle \times c_s \times t_1 + \frac{1}{2}\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle - \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right) \times (t - t_1) \times c_p\Big] \times k \tag{11}$$

The figure 4 shows the main components of the stock management model without-a-stock consisting of a single product.

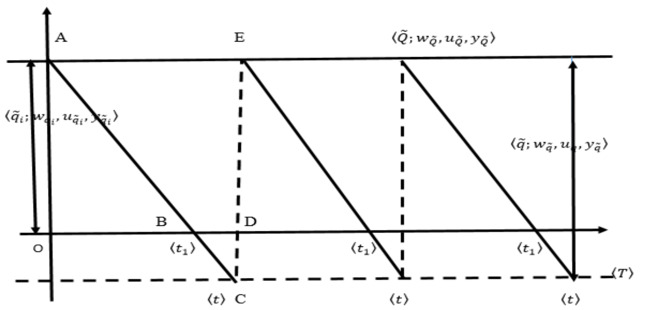

Figure 4. Stock model with a single product, with out-of-stock, and neutrosophic components

According to figure 5, from the likeness of the triangle $\Delta AOC \sim \Delta BCD$, both being rectangular triangles we will have:

$$\frac{t - t_1}{t} = \frac{\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle - \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle}{\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle} \tag{12}$$

From the above relationship it follows that we will have:

$$\frac{t_1}{t} = \frac{\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \tag{13}$$

In the expression of the total cost function we replace the values for $(k), (t_1)\ and\ (t - t_1)$ depending on the values of $\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle$ și $\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle$ so that we will obtain:

$$fC\big(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\big) = \Big[c_l + \frac{1}{2}\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle \times \frac{\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times t \times c_s + \frac{1}{2}\big(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle - \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\big) \times \frac{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle - \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times t \times c_p\Big] \times k \tag{14}$$

Performing the calculations as well as from the replacement of t with the expression $t = \frac{T}{k}$, and the replacement of k with the expression $k = \frac{\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle}$ will result in:

$$fC\big(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\big) = c_l \times \frac{\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} + \frac{\left[\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right]^2}{2\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times T \times c_s + \frac{\left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle - \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right]^2}{2\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times T \times c_p \tag{15}$$

Theorem 1: Either the total cost function for the single-product stock model with variables $\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle$ and $\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle$ and $\tilde{q}_i/\tilde{q}_c = \{\langle \tilde{q}_i/\tilde{q}_c, \mu_{\tilde{q}_i/\tilde{q}_c}, \vartheta_{\tilde{q}_i/\tilde{q}_c}, \lambda_{\tilde{q}_i/\tilde{q}_c} \rangle\ q_i/q_c \in \tilde{q}_i/\tilde{q}_c\}$ where $\mu_{\tilde{q}_i/\tilde{q}_c}: q_i/q_c \to [0,1]$; $\vartheta_{\tilde{q}_i/\tilde{q}_c}: q_i/q_c \to [0,1]$ și $\lambda_{\tilde{q}_i/\tilde{q}_c}: q_i/q_c \to [0,1]$ consisting of a single product with fuzzy neutrosofic variable in the form:

$$fC\big(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\big) = c_l \times \frac{\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} + \frac{\left[\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right]^2}{2\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times T \times c_s + \frac{\left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle - \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right]^2}{2\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} \times T \times c_p \tag{16}$$

Total Cost Function $fC\big(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\big)$ admits **extreme points** $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle$ and $\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle$ known as optimal points for the single-product stock model, with fuzzy variables, determined with the calculation relationships:

$$\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle = \sqrt{\frac{2c_l\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{T \times c_s \times \frac{c_p}{c_s + c_p}}} \tag{17}$$

and respectively:

$$\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle = \sqrt{\frac{2c_l\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle \times \frac{c_p}{c_s + c_p}}{T \times c_s}} \tag{18}$$

Minimum point value $C\left(\langle\tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*}\rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*}\rangle\right)$ is given by the relation that represents the minimum total cost function in neutrosophic fuzzy variables:

$$fC\left(\langle\tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*}\rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*}\rangle\right) = \sqrt{2\langle\tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}}\rangle \times T \times c_l \times c_s \times \frac{c_p}{c_s+c_p}} \quad (19)$$

Demonstration:

In order to establish extreme points we will identify stationary points using partial derivatives of order 1 of the function: $fC\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle; \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right)$ so we'll have:

$$\begin{cases} \dfrac{\partial fC\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle; \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right)}{\partial\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle} = 0 \\ \dfrac{\partial fC\left(\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle; \langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right)}{\partial\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle} = 0 \end{cases} \quad (20)$$

From the first equation from (20), by performing the above calculations and rearranging the above terms we'll obtain that:

$$-\frac{2c_l\langle\tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}}\rangle + \left[\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right]^2 (T\times c_p + T\times c_s) - \left[\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle\right]^2 T\times c_p}{2\left[\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle\right]^2} = 0 \quad (21)$$

From the first equation from (20), we will get, after performing the calculations:

$$\frac{\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle (T\times c_s + T\times c_p) - \langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle \times T\times c_p}{\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle} = 0 \quad (22)$$

This forms a system of two equations with two unknowns, i.e. the quantity of products in an order $\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle$ and the original stock quantity $\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle$ by the form:

$$\begin{cases} -\dfrac{2c_l\langle\tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}}\rangle + \left[\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle\right]^2 (T\times c_p + T\times c_s) - \left[\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle\right]^2 T\times c_p}{2\left[\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle\right]^2} = 0 \\ \dfrac{\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle (T\times c_s + T\times c_p) - \langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle \times T\times c_p}{\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle} = 0 \end{cases} \quad (23)$$

From the second equation of the above system, we get:

$$\langle\tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i}\rangle = \langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle \frac{c_p}{c_s+c_p} \quad (24)$$

Replacing the above expression in the equation system and performing the operations in the above formula we will get:

$$\left[\langle\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\rangle\right]^2 \times T \times c_s \times \frac{c_p}{c_s+c_p} = 2c_l\langle\tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}}\rangle \quad (25)$$

In conclusion, **the optimal quantity of products** in an order $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle$ will be given by the expression:

$$\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle = \sqrt{\frac{2c_l\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle}{T \times c_s \times \frac{c_p}{c_s + c_p}}} \quad (26)$$

To get **the optimal number of products** $\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle$ for the initial stock we will use the relationship:

$$\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle = \langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle \frac{c_p}{c_s + c_p} \quad (27)$$

For the final form **of the optimal quantity of products in an order** $\langle q_i^*; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle$ we will use the calculation formula:

$$\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle = \sqrt{\frac{2c_l\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle \times \frac{c_p}{c_s + c_p}}{T \times c_s}} \quad (28)$$

To verify that the point $(\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle)$ is extreme point, determine the second-order derivatives as:

$$\begin{cases} \dfrac{\partial^2 fC\left(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right)}{\partial^2 \langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle} < 0 \\ \dfrac{\partial^2 fC\left(\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle; \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right)}{\partial \langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle} > 0 \end{cases} \quad (29)$$

From the first relationship above, we will obtain:

$$\frac{-\left[4\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle T \times c_p\right]\left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle\right]^2}{4\left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle\right]^4} +$$

$$-\frac{\left(2c_l\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle + \left[\langle \tilde{q}_i; w_{\tilde{q}_i}, u_{\tilde{q}_i}, y_{\tilde{q}_i} \rangle\right]^2 (T \times c_p + T \times c_s) - \left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle\right]^2 T \times c_p\right) 4\langle \tilde{q}_c \rangle}{4\left[\langle \tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c} \rangle\right]^4} < 0 \quad (30)$$

Also, from the second relationship, we will obtain:

$$\frac{\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}(T \times c_s + T \times c_p)}{\left[\tilde{q}_c; w_{\tilde{q}_c}, u_{\tilde{q}_c}, y_{\tilde{q}_c}\right]^2} > 0 \quad (31)$$

After the calculations have been made, it follows that the point $C\left(\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle\right)$ admits minimum, and **the value of the minimum point** is:

$$fC\left(\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle; \langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle\right) =$$

$$\sqrt{2\langle \tilde{Q}; w_{\tilde{Q}}, u_{\tilde{Q}}, y_{\tilde{Q}} \rangle \times T \times c_l \times c_s \times \frac{c_p}{c_s + c_p}} \quad (32)$$

4.2. Multiple products, and out of stock

The stock model with neutrosophic fuzzy variables, with several products necessary for the production activity, has as its ultimate objective to determine the order quantity required to be supplied $\langle\tilde{q}_{cj}^*; w_{\tilde{q}_{cj}^*}, u_{\tilde{q}_{cj}^*}, y_{\tilde{q}_{cj}^*}\rangle$, the quantity initially in stock $\langle\tilde{q}_{ij}^*; w_{\tilde{q}_{ij}^*}, u_{\tilde{q}_{ij}^*}, y_{\tilde{q}_{ij}^*}\rangle$, as well as the total cost function $fC\left(\langle\tilde{q}_{cj}^*; w_{\tilde{q}_{cj}^*}, u_{\tilde{q}_{cj}^*}, y_{\tilde{q}_{cj}^*}\rangle; \langle\tilde{q}_{ij}^*; w_{\tilde{q}_{ij}^*}, u_{\tilde{q}_{ij}^*}, y_{\tilde{q}_{ij}^*}\rangle\right)$ with the meaning already known for the specific probability of neutrosophic fuzzy numbers.

The stock model with fuzzy neutrosophic variables, with several products, presents a series of particularities, out of which we mention:

1. **Total quantity** of each product necessary for a company's production activity is $\langle\tilde{Q}_i; w_{\tilde{Q}_i}, u_{\tilde{Q}_i}, y_{\tilde{Q}_i}\rangle$ over the period (T), while the quantity in order for each product (i) required in the period (t) will be in the form $\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle$. Modeling the total quantity of each product $\langle\tilde{Q}_i; w_{\tilde{Q}_i}, u_{\tilde{Q}_i}, y_{\tilde{Q}_i}\rangle$, and the number of products in each order $\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle$, is made with the help of fuzzy triangular neutrosophic numbers.

2. **Cost of placing an order** (C_l) is dependent on the number of products required for an order and can be written in the form of:

$$C_l = \sum_{i=1}^{n} \frac{\langle\tilde{Q}_i; w_{\tilde{Q}_i}, u_{\tilde{Q}_i}, y_{\tilde{Q}_i}\rangle \times c_{li}}{\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle} \tag{33}$$

3. **Average cost of storing products** (i) will be marked with (C_s) and will be determined with the calculation formula:

$$C_s = \sum_{i=1}^{n} \frac{1}{2}\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle \times T \times c_{si} \tag{34}$$

4. **Purchase value** (V_a) products necessary for the operational activity of companies will be determined according to the quantity of products $\langle\tilde{Q}_i; w_{\tilde{Q}_i}, u_{\tilde{Q}_i}, y_{\tilde{Q}_i}\rangle$ necessary over the period (T) as well as the purchase price of the products (p_a) and is in the form of:

$$V_a = \sum_{i=1}^{n}\langle\tilde{Q}_i; w_{\tilde{Q}_i}, u_{\tilde{Q}_i}, y_{\tilde{Q}_i}\rangle \times p_{ai} \tag{35}$$

5. **Total storage capacity** of products (i) supply is dependent on the volume required for storage for each supplied product (c_{di}) as well as the number of products supplied for each order $\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle$ after a formal relationship:

$$C_d = \sum_{i=1}^{n}\langle\tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}}\rangle \times c_{di} \tag{36}$$

For the study of the stock model with fuzzy neutrosophic variables, with several products, with out-of-stock, it is based on the above-mentioned assumptions as well as from the form of the total cost function of the stock model presented in equation 16. We also know that each of the products purchased $\langle\tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle$ from a quantity of order takes up a volume of storage $\left(c_{dj}\right)$, and the company's total storage capacity is limited (C_d) so that the inequality to be achieved is in the form of:

$$\sum_{j=1}^{n}\langle\tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} \le C_d \tag{37}$$

Theorem 2: Either the total cost function for the multi-product stock model, with fuzzy neutrosophic variables $\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle$ and $\langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{ij}, y_{\tilde{q}_{ij}} \rangle$ are $\tilde{q}_{ij}/\tilde{q}_{cj} = \left\{ \langle \tilde{q}_{ij}/\tilde{q}_{cj}, \mu_{\tilde{q}_{ij}/\tilde{q}_{cj}}, \vartheta_{\tilde{q}_{ij}/\tilde{q}_{cj}}, \lambda_{\tilde{q}_{ij}/\tilde{q}_{cj}} \rangle \; q_i/q_c \in \tilde{q}_{ij}/\tilde{q}_{cj} \right\}$ where $\mu_{\tilde{q}_{ij}/\tilde{q}_{cj}}: q_{ij}/q_{cj} \to [0,1]$; $\vartheta_{\tilde{q}_{ij}/\tilde{q}_{cj}}: q_{ij}/q_{cj} \to [0,1]$ and $\lambda_{\tilde{q}_{ij}/\tilde{q}_{cj}}: q_{ij}/q_{cj} \to [0,1]$ consisting of several products with fuzzy neutrosophic variables of the form:

$$fC\left(\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle ; \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}} \rangle\right) = \sum_{j=1}^{n} c_{lj} \times \frac{\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle}{\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle} + \sum_{j=1}^{n} \frac{\left[\langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}} \rangle\right]^2}{2\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle} \times T \times c_{sj} + \sum_{j=1}^{n} \frac{\left[\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle - \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}} \rangle\right]^2}{2\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle} \times T \times c_{pj} \tag{38}$$

Total Cost Function $fC\left(\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle ; \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}} \rangle\right)$ admits extreme points $\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{c}j}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle$ și $\langle q^*_{ij}; w_{q^*_{ij}}, u_{q^*_{ij}}, y_{q^*_{ij}} \rangle$ known as optimal points for the multi-product stock model with neutrosophic fuzzy variables, determined with the calculation relationships specific to the minimisation problem:

<u>Case I:</u> If $C_d^* \le C_d$, then the optimal solution for the optimal quantity for an order $\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle$ and the optimal quantity of products in stock $\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle$ will be determined with the calculation formulas:

$$\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}}}} \tag{39}$$

and respectively:

$$\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}}}} \times \frac{c_{pj}}{c_{sj}+c_{pj}} \tag{40}$$

<u>Case II:</u> If $C_d^* > C_d$, then the optimal solution for the optimal quantity for an order $\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle$ and the optimal quantity of products in stock $\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle$ will be determined with the calculation formulas:

$$\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}} + \sum_{j=1}^{n} 2\lambda \times c_{dj}}} \tag{41}$$

and respectively:

$$\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}} + \sum_{j=1}^{n} 2\lambda \times c_{dj}}} \times \frac{c_{pj}}{c_{sj}+c_{pj}} \tag{42}$$

Demonstration:

To minimise the total cost function $fC\left(\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle ; \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle\right)$ to the stock model, with fuzzy variables, consisting of several products, with out of stock, the Lagrangian of the problem becomes:

$$L = \sum_{j=1}^{n} c_{lj} \times \frac{\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j}\rangle}{\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle} + \sum_{j=1}^{n} \frac{\left[\langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle\right]^2}{2\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle} \times T \times c_{sj} +$$
$$\sum_{j=1}^{n} \frac{\left[\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle - \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle\right]^2}{2\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle} \times T \times c_{pj} -$$
$$\lambda\left(\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d\right) \tag{43}$$

In order to solve the optimisation problem, the conditions of the Khun-Tucken theorem specific to the square programming problems apply, according to which we will have:

$$\begin{cases} \dfrac{\partial L}{\partial \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle} - \lambda\left(\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d\right) = 0 \\ \dfrac{\partial L}{\partial \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle} - \lambda\left(\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d\right) = 0 \\ \lambda\left(\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d\right) = 0 \\ \sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d \leq 0 \\ \lambda \geq 0 \end{cases} \tag{44}$$

After the calculations are performed, the conditions for the Khun-Tucken theorem for the optimisation problem will be:

$$-\sum_{j=1}^{n} 2\,\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j}\rangle \times c_{lj} - \sum_{j=1}^{n} \left[\langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle\right]^2 \times T\left(c_{sj} + c_{pj}\right) +$$
$$\sum_{j=1}^{n} \left[\tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\right]^2 \times \left(T \times c_{pj} + 2\lambda \times c_{dj}\right) = 0 \tag{45.1}$$

$$\sum_{j=1}^{n} \langle \tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle \times T \times \left(c_{sj} + c_{pj}\right) - \sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times T \times c_{pj} = 0 \tag{45.2}$$

$$\lambda\left(\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d\right) = 0 \tag{45.3}$$

$\lambda \geq 0$

$$\sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times c_{dj} - C_d \leq 0 \tag{45.4}$$

It follows that we will have two situations:

Case I: If $\lambda = 0$

From the relationship (45.2) of Khun Tucken's conditions for the optimisation issue it follows that:

$$\langle \tilde{q}_{ij}^*; w_{\tilde{q}_{ij}^*}, u_{\tilde{q}_{ij}^*}, y_{\tilde{q}_{ij}^*}\rangle = \sum_{j=1}^{n} \langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle \times \frac{c_{pj}}{c_{sj}+c_{pj}} \tag{46}$$

By replacement in the relationship (45.1), we will get:

$$\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}}}} \tag{47}$$

The optimal quantity of products that exist on stock will be determined with the calculation formula:

$$\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}}}} \times \frac{c_{pj}}{c_{sj}+c_{pj}} \tag{48}$$

We check to see if the condition is met:

$$\langle \tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}} \rangle \times c_{dj} = C^*_d \leq C_d \tag{49}$$

From the above inequality check if $C^*_d \leq C_d$ then the optimal solution for the optimal quantity in an order $\langle \tilde{q}^*_{cj}; w_{\tilde{q}^*_{cj}}, u_{\tilde{q}^*_{cj}}, y_{\tilde{q}^*_{cj}} \rangle$ and the optimal quantity of products in stock $\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle$ will be determined with the calculation formulas previously established;

<u>Case II:</u> If the parameter $\lambda > 0$ and if $C^*_d > C_d$ then from the relationship (45.2 and 45.1) respectively from the conditions Khun Tucken, we will obtain:

$$\langle \tilde{q}^*_{ij}; w_{\tilde{q}^*_{ij}}, u_{\tilde{q}^*_{ij}}, y_{\tilde{q}^*_{ij}} \rangle = \sqrt{\frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj}}{\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}} + \sum_{j=1}^{n} 2\lambda \times c_{dj}}} \times \frac{c_{pj}}{c_{sj}+c_{pj}} \tag{50}$$

The value of the parameter (λ) will be achieved by determining the optimal storage capacity of the company in question from the condition of the Khun Tucken system:

$$\sum_{j=1}^{n} 2\,\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj} \times c_d^2 = C_d^2 \left(\sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}} + \sum_{j=1}^{n} 2\lambda \times c_{dj} \right) \tag{51}$$

It follows that the value of the parameter λ will be in the form:

$$\lambda = \frac{\sum_{j=1}^{n} 2\langle \tilde{Q}_j; w_{\tilde{Q}_j}, u_{\tilde{Q}_j}, y_{\tilde{Q}_j} \rangle \times c_{lj} \times c_{dj}^2 - C_d^2 \sum_{j=1}^{n} T \times c_{sj} \times \frac{c_{pj}}{c_{sj}+c_{pj}}}{2C_d^2 \sum_{j=1}^{n} c_{dj}} \tag{52}$$

5. Case Study with two products

A company plans to purchase two products (p_1) and respectively (p_2), and the cost of placing an order is known (c_l), as well as the cost of storing a product unit (c_s), as well as the storage volume of a product unit (c_d) presented in the form of intervals using fuzzy triangular numbers in Table 1:

Table 1. Production indicators

Product	Placement cost [lei/order]	Cost of storage [lei/order]	Storage volume sqm/product
p_1	$(1000, 2000, 3000)$	$(500, 750, 900)$	10
p_2	$(1500, 2500, 4000)$	$(700, 850, 1000)$	15

A production hall of 20,000 sqm represents the storage area available to the company. The penalty cost for lack of product (p_1) and the product respectively (p_2) from stock is in the form: $c_{p1} = (10\%, 20\%, 30\%)$ for values of $c_{p1} \in [10\%; 30\%]$

$c_{p2} = (5\%, 10\%, 15\%)$ for values of $c_{p2} \in [10\%; 30\%]$

It is required to determine with the help of neutrosophic fuzzy numbers the following indicators specific to stocks:

a) Quantity in an order $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle$ and number of orders (k) which will be achieved over the period (T);

b) Optimal Stock Quantity $\langle q_i^*; w_{q_i^*}, u_{q_i^*}, y_{q_i^*} \rangle$ over the period (T);

c) The amount of the minimum cost the company can get from placing an order.

Solution:

Determine the **costs of placing an order** $E_f(c_{li})$ and the cost of storing a product unit $E_f(c_{si})$ using the calculation formula: $E_f(c_i) = \frac{1}{6}(ci_{a1} + ci_{c1}) + \frac{2}{3} ci_{b1}$

Thus, we will have: $E_f(c_{l1}) = 1998\, u.m.$; $E_f(c_{l2}) = 2245\, u.m.$; $E_f(c_{s1}) = 731{,}9\, u.m.$; $E_f(c_{s2}) = 848{,}3\, u.m.$

We also determine **the value of the average cost by penalizing** for the two products concerned, $E_f(c_{p1})$ și $E_f(c_{p2})$, for which we will have:

$E_f(c_{p1}) = 19.980\, u.m.$; $E_f(c_{p2}) = 24.950\, u.m.$

a) Determination **of the order quantity** $\langle \tilde{\boldsymbol{q}}_{ci}^*; \boldsymbol{w}_{\tilde{q}_c^* i}, \boldsymbol{u}_{\tilde{q}_{ci}^*}, \boldsymbol{y}_{\tilde{q}_{ci}^*} \rangle$ and the number of orders (k) over the period (T) using the neutrosophic fuzzy numbers.

By replacing in the calculation formula no. 40 from the Theorem no. 2, we will have: $\langle \tilde{q}_{ci}^*; w_{\tilde{q}_c^* i}, u_{\tilde{q}_{ci}^*}, y_{\tilde{q}_{ci}^*} \rangle = \langle 3.167{,}6; 60{,}20{,}20 \rangle$

From the above inequality it is apparent that $C_d^* > C_d$ $(79.192{,}3 \geq 20.000)$, case II, theorem 2, the optimal solution of the problem will be given by the relationship no. 45, being $k = 4.402{,}51$ orders/year.

We check the condition: $\sum_{i=1}^{n} \langle \tilde{q}_{ci}; w_{\tilde{q}_{ci}}, u_{\tilde{q}_{ci}}, y_{\tilde{q}_{ci}} \rangle \times c_{di} = C_d^* \leq C_d$

For which we will get: $795 \times 10 + 795 \times 25 \leq 20.000$

It is apparent that inequality is verified, and therefore the optimal solution to the optimisation problem is: $\langle \tilde{q}_{ci}^*; w_{\tilde{q}_{ci}^*}, u_{\tilde{q}_{ci}^*}, y_{\tilde{q}_{ci}^*} \rangle = \langle 795{,}00\, u.m.; 60{,}20{,}20 \rangle$.

So, the optimal quantity for a product order, determined using neutrosophic fuzzy numbers, is $\langle \tilde{q}_c^*; w_{\tilde{q}_c^*}, u_{\tilde{q}_c^*}, y_{\tilde{q}_c^*} \rangle = \langle 795.00\, u.m.; 60{,}20{,}20 \rangle$/order with a probability of achievement of 60%, probability of uncertainty of 20%, probability of non-achievement of 20%. The probabilities specific to neutrosophic fuzzy numbers mean the achievement of the optimal order quantity in the percentage of 60%, the probability of uncertainty in the percentage of 20% as well as the probability of non-realisation of 20%.

b) Determining the optimal amount of stock $\langle\tilde{q}_{ij}^{*}; w_{\tilde{q}_{ij}^{*}}, u_{\tilde{q}_{ij}^{*}}, y_{\tilde{q}_{ij}^{*}}\rangle$ is also done using theorem No.2 using the calculation relationship no. 40, we will have:
$\langle\tilde{q}_{ij}^{*}; w_{\tilde{q}_{ij}^{*}}, u_{\tilde{q}_{ij}^{*}}, y_{\tilde{q}_{ij}^{*}}\rangle = \langle 1526{,}4\, u.m.; 60{,}20{,}20\rangle$

Therefore**,** the optimal quantity of products on stock will be $\langle\tilde{q}_{ij}^{*}; w_{\tilde{q}_{ij}^{*}}, u_{\tilde{q}_{ij}^{*}}, y_{\tilde{q}_{ij}^{*}}\rangle = \langle 1526{,}4\, u.m.; 60{,}20{,}20\rangle$ with a probability of achievement of 60%, and uncertain probability of 20%, a probability of non-achievement of 20%.

c) Determination of the minimum value of the total cost function for the total quantity of products required during a year will be done with the calculation formula no. 50 and we will get: $\langle 2.444.275{,}58\, u.m.; 60, 20, 20\rangle$ u.m./order.

So, we can state that during the period of time (T) total cost function

$$fC\left(\langle\tilde{q}_{cj}; w_{\tilde{q}_{cj}}, u_{\tilde{q}_{cj}}, y_{\tilde{q}_{cj}}\rangle; \langle\tilde{q}_{ij}; w_{\tilde{q}_{ij}}, u_{\tilde{q}_{ij}}, y_{\tilde{q}_{ij}}\rangle\right) = \langle 2.444.275{,}58; 60, 20, 20\rangle$$

u.m./order will achieve the minimum of $\langle 2.444.275{,}58\, u.m.; 60, 20, 20\rangle$ with a 60% probability of achievement, a 20% probability of uncertainty and a 20% probability of non-achievement.

6. Conclusions

The stock models presented are modeled both by nonlinear mathematical programming and fuzzy triangular neutrosophic numbers. Non-linear mathematical programming has the advantage of optimisation issues considering the constraints specific to the company's activities. In turn, fuzzy triangular neutrosophic numbers allow modeling so that stock performance indicators contain information on their probability of the probability specific to these categories of fuzzy numbers, i.e., probability achievement/non-achievement and uncertainty and their membership in cost classes specific to stock performance indicators.

Both categories of stock models presented in this article allow to determine the optimal quantity of products in an order $\langle\tilde{q}_{ci}^{*}; w_{\tilde{q}_{ci}^{*}}, u_{\tilde{q}_{ci}^{*}}, y_{\tilde{q}_{ci}^{*}}\rangle$, the optimal amount of existing stocks $\langle\tilde{q}_{ij}^{*}; w_{\tilde{q}_{ij}^{*}}, u_{\tilde{q}_{ij}^{*}}, y_{\tilde{q}_{ij}^{*}}\rangle$ as well as minimising the overall cost function of the stock formation process $fC\left(\langle\tilde{q}_{c}^{*}; w_{\tilde{q}_{c}^{*}}, u_{\tilde{q}_{c}^{*}}, y_{\tilde{q}_{c}^{*}}\rangle; \langle q_{i}^{*}; w_{q_{i}^{*}}, u_{q_{i}^{*}}, y_{q_{i}^{*}}\rangle\right)$. With the help of neutrosophic fuzzy numbers were modeled for the optimal quantity in an order $\langle\tilde{q}_{ci}^{*}; w_{\tilde{q}_{ci}^{*}}, u_{\tilde{q}_{ci}^{*}}, y_{\tilde{q}_{ci}^{*}}\rangle$, the optimal quantity of existing stocks $\langle\tilde{q}_{ij}^{*}; w_{\tilde{q}_{ij}^{*}}, u_{\tilde{q}_{ij}^{*}}, y_{\tilde{q}_{ij}^{*}}\rangle$ but also to minimise the overall cost function $fC\left(\langle\tilde{q}_{c}^{*}; w_{\tilde{q}_{c}^{*}}, u_{\tilde{q}_{c}^{*}}, y_{\tilde{q}_{c}^{*}}\rangle; \langle q_{i}^{*}; w_{q_{i}^{*}}, u_{q_{i}^{*}}, y_{q_{i}^{*}}\rangle\right)$. Thus, additional information is obtained to substantiate the specific probability of achievement/failure and uncertainty for stock performance indicators.

The constraints specific to the company's operational activity have been introduced to model stocks using nonlinear mathematical programming, constraints on the company's storage area. Since the company's overall cost function includes in addition to the specific costs of stock formation, we propose the modeling by fuzzy numbers of the process of purchasing stocks as a future direction of research, following the formation of the critical component of costs.

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