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AMERICAN-STYLE WARRANT PRICING — A MODEL BASED ON KoBoL PROCESS

Abstract. *Most of the research on the American-style warrant pricing is carried out under the framework of the B-S model, and the randomness of the financial market is limited. Under the KoBoL process, the American-style warrant pricing model is a partial differential equation with fractions, but there is a free boundary problem. The first-order fully implicit model is constructed using the coordinate transformation method and the penalty method, which converts the free boundary of the American warrant pricing model under the KoBoL process into a fixed boundary, and proves that the value of American-style warrant is not less than the exercise value. The theoretical analysis conclusion is verified by numerical simulation, and the influence of parameters on the contract value and the optimal exercise price is analysed.*
Keywords: *American-style warrant, pricing model, KoBoL process, penalty function.*

JEL Classification: G12, G11

Introduction

As a type of option, the American-style warrant has many similarities to the American option in terms of characteristics. Research on the pricing of American-style warrant is carried out under the framework of the B-S model (Rong, 2008; Du, 2009; Wang, 2016). However, the B-S model has very limited reflection on the real financial market, and the continuous-time *Brownian* motion cannot fully reflect the randomness of the financial market (Chen and Lin, 2018). In order to improve the pricing model, some scholars have proposed KoBoL model based on the general $Le\acute{v}y$ process as the basic model for the pricing of option risk assets.

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This model introduces the "*Damping*" effect in the density function of the characteristic equation based on the α steady-state process, which ensures that all conditional moments for realising risk assets are limited. Therefore, under the assumption that asset price follow the KoBoL process (Boyarchenko and Levendorski, 2012), the valuation of risky asset can be set as:

$$dx_t = (y - \pi)dt + dL_t^{KoBoL} \quad (1)$$

It can be solved as:

$$\begin{cases} Y_T = Y_t e^{(y-\pi)(T-t)} + \int_t^T dL_\vartheta^{KoBoL} \\ \pi = \frac{1}{2}\sigma^\alpha[p(\lambda-1)^\alpha + q(\lambda+1)^\alpha - \lambda^\alpha - \alpha\lambda^{\alpha-1}(q-p)] \end{cases} \quad (2)$$

Where, $\mathbf{x_t}$ is the logarithmic value of the risky asset $\mathbf{x_t = lnY_t}$, $\mathbf{y}$ is the risk-free interest rate, $\mathbf{T}$ is the expiry time of the risky asset, $\mathbf{L_t^{KoBoL}}$ is the increment of $Le\acute{v}y$ process under the equivalent martingale measure, and $\boldsymbol{\pi}$ is the convexity adjustment to make the expectation of $\mathbf{Y_T}$ become $\mathbf{E[Y_T] = e^{y(T-t)}Y_t}$, $\boldsymbol{\sigma}$ is the asset price volatility, $\boldsymbol{\alpha}$ is the parameter of the stable $Le\acute{v}y$ process, $\mathbf{p}$ is the probability of asset price rise, and $\boldsymbol{\lambda}$ is the asset risk price.

The KoBoL process can not only solve the problem of "asymmetric distribution" of the pricing of risky asset, but also retain the best features of the B-S model. Some scholars have researched the pricing of risky asset under this framework. For example, the research of Marom and Momoniat (2009) shows that the $Le\acute{v}y$ process can describe different types and stages of the market, and the KoBoL process shows good boundary convergence, which makes it perform better in practical applications. Based on the KoBoL process, Meng, et al. (2014) proposed a fast preprocessing iterative method for European option pricing using a band preprocessor. Cartea (2017) believes that the KoBoL process has achieved an appropriate balance between solving the characteristics of stock price evolution and the difficulty of mathematical processing, he used the KoBoL process to study the pricing of European option and barrier option, and a numerical analysis method of fractional differential equations is proposed. Zhang and Yin (2020) studied American option pricing based on the KoBoL process, and proposed a numerical discrete method for solving fractional partial differential equation, and theoretically analysed the sufficient conditions for the stability of the numerical format.

Under the KoBoL process, the American-style warrant pricing model has a free boundary problem, and the governing equation is a fractional partial differential equation. For the free boundary problem, academia has proposed some methods, among which the penalty method is very effective in solving the American option pricing problem and the infinite-dimensional nonlinear problem, and this method has been highly recognised by academia. After coordinate transformation American-style warrant can be regarded as an American call option under the KoBoL process. Therefore, under the KoBoL process, this paper uses coordinate transformation method and penalty method to improve the American-style warrant pricing model, and set up a first-order fully implicit model that suitable for American-style warrant pricing, and numerical simulation of the conclusion of the analysis is carried out, and the influence of parameters on the contract value of American-style warrant and the optimal exercise price is analysed.

2. Model

2.1 Governing equation

The payment function of the American-style warrant contract can be written as:

$$\Gamma(x_T, T) = \max(e^x - \psi e^{\gamma T}, 0) \tag{3}$$

Where, $\boldsymbol{\gamma}(\boldsymbol{\gamma} > \boldsymbol{y})$ is the cost of capital, and $\boldsymbol{\psi}$ is the contract price. According to the no-arbitrage pricing principle, the contract value $\boldsymbol{\Omega}(\mathbf{x}, \mathbf{t})$ at time **t** satisfies:

$$\Omega(x, t) = e^{-y(T-t)} E^G[\Gamma(x_T, T)] \tag{4}$$

Where, $\mathbf{E^G}$ is the conditional expectation operator under the measure $\mathbf{G}$. With the research conclusions of Zhang and Yin (2020), the contract value $\boldsymbol{\Omega}(\mathbf{x}, \mathbf{t})$ can be obtained to satisfy the formula (5).

$$\frac{\partial\Omega(x,t)}{\partial t} + [y - \pi - \lambda^{\alpha-1}(p-q)]\frac{\partial\Omega(x,t)}{\partial t} + \frac{1}{2}\sigma^\alpha[p e_x^{\lambda x} H_{xh}^\alpha e^{-\lambda x}\Omega(x,t) + q e_{-\infty}^{-\lambda x} H_x^\alpha e^{\lambda x}\Omega(x,t)] = \left(y + \frac{1}{2}\sigma^\alpha\lambda^\alpha\right)\Omega(x,t) \tag{5}$$

Where, $\mathbf{x} \in (-\infty, \mathbf{x_h}]$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, $\mathbf{1} < \boldsymbol{\alpha} < \mathbf{2}$,

$$\pi = \frac{1}{2}\sigma^\alpha[p(\lambda-1)^\alpha + q(\lambda+1)^\alpha - \lambda^\alpha - \alpha\lambda^{\alpha-1}(q-p)]$$

and

$$e_x^{\lambda x} H_{xh}^\alpha e^{-\lambda x}\Omega(x,t) = \frac{e^{\lambda x}}{\Pi(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{xh}\frac{e^{-\lambda\eta}\Omega(\eta,t)}{(\eta-x)^{\alpha+1-n}}d\eta$$
$$e_{-\infty}^{-\lambda x} H_x^\alpha e^{\lambda x}\Omega(x,t) = \frac{e^{-\lambda x}}{\Pi(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{x}\frac{e^{\lambda\eta}\Omega(\eta,t)}{(\eta-x)^{\alpha+1-n}}d\eta$$

2.2 Boundary conditions

According to the characteristics of the American-style warrant contract, the boundary conditions of the mathematical model can be obtained (Wang and Chen, 2020; Chen et al., 2015).

$$\begin{cases} \lim_{x\to-\infty}\Omega(x,t) = 0 \\ \Omega(x_h,t) = e^{xh} - \psi e^{\gamma t} \\ \frac{\partial\Omega(x_h,t)}{\partial x} = Y_h = e^h \\ \Omega(x,T) = \max(e^x - \psi e^{\gamma T}, 0) \end{cases} \quad (6)$$

To sum up, formulas (5) and (6) are the American-style warrant pricing model obtained under the KoBoL process. Since the contract can be executed in advance, the contract value must meet the following condition:

$$\Omega(x_t,t) \geq \max(e^{xt} - \psi e^{\gamma t}, 0) \quad (7)$$

2.3 Coordinate transformation

In order to eliminate the influence of the time variable $\mathbf{e_t}$ in the boundary conditions on the numerical results, coordinate transformation is carried out, let:

$$\theta = x - \gamma t,\ \theta_h = x_h - \gamma t,\ e^{-\gamma t}\Omega(x,t) = \Psi_\mu(\theta,t) \quad (8)$$

After variable transformation, the pricing models (5) and (6) can be transformed into:

$$\frac{\partial\Psi_\mu(\theta,t)}{\partial t} + \alpha\frac{\partial\Psi_\mu(\theta,t)}{\partial\theta} + \frac{1}{2}\sigma^\alpha\left[pe_\theta^{\lambda\theta}H_{\theta t}^{\alpha}e^{-\lambda\theta}\Psi_\mu(\theta,t)\right] + qe_{-\infty}^{-\lambda\theta}H_\theta^\alpha e^{\lambda\theta}\Psi_\mu(\theta,t) = b\Psi_\mu(\theta,t) \quad (9)$$

Where, $\boldsymbol{\theta \in (-\infty,\theta_h]}$, $\mathbf{t \in [0,T]}$, $\mathbf{1 < \alpha < 2}$, $\boldsymbol{\alpha = y - \pi - \lambda - \lambda^{\alpha-1}(q-p)}$, $\boldsymbol{b = y - \gamma + \sigma^\alpha\lambda^\alpha/2}$, $\boldsymbol{\lim_{\theta\to\infty}\Psi_\mu(\theta,t) = 0}$, $\boldsymbol{\Psi_\mu(\theta_h,t) = e^{\theta h} - \psi}$,$\boldsymbol{\frac{\partial\Psi_\mu(\theta_h,t)}{\partial\theta} = e^{\theta h}}$, $\boldsymbol{\Psi_\mu(\theta,T) = \max(e^\theta - \psi, 0)}$. Therefore, formula (7) is transformed as follows:

$$\Psi_\mu(\theta,t) \geq \max(e^\theta - \psi, 0) \quad (10)$$

We can see that $\boldsymbol{\Psi_\mu(\theta,t)}$ can be regarded as an American call option under the KoBoL process, where $\boldsymbol{e^\theta}$ is the price of the underlying asset, $\boldsymbol{\psi}$ is the exercise price, and $\boldsymbol{e^{\theta h}}$ is the optimal execution boundary. In addition, the right boundary $\boldsymbol{\theta_h}$ of the equation is unknown, which brings difficulties to the solution of the model. The penalty method can be used to transform the free boundary problem into a fixed boundary problem to facilitate the solution.

2.4 Penalty function

Adding a penalty function term to the governing equation (9) can solve the problem of free boundary. The penalty function term in this paper is $\frac{\boldsymbol{\mu R}}{\boldsymbol{\Psi_\mu(\theta,t)+\mu-q(\theta)}}$, where, $\mathbf{q(\theta)=e^{\theta}-L}$, $\mathbf{0<\mu\leq 1}$ is an infinitesimal constant, and $\mathbf{R}$ is a pending parameter. By adding the penalty function term to the fractional differential equation (9), the original equation is transformed into:

$$\frac{\partial\Psi_\mu(\theta,t)}{\partial t}+\alpha\frac{\partial\Psi_\mu(\theta,t)}{\partial\theta}+\frac{1}{2}\sigma^\alpha\left[pe_\theta^{\lambda\theta}H_{\theta\max}^{\alpha}e^{-\lambda\theta}\Psi_\mu(\theta,t)\right]+qe_{-\infty}^{-\lambda\theta}H_\theta^\alpha e^{\lambda\theta}\Psi_\mu(\theta,t)$$
$$+\frac{\mu R}{\Psi_\mu(\theta,t)+\mu-q(\theta)}=b\Psi_\mu(\theta,t) \qquad (11)$$

Where, $\boldsymbol{\theta\in(-\infty,\theta_{max}]}$, $\mathbf{t\in[0,T]}$, $\boldsymbol{1<\alpha<2}$, $\boldsymbol{\lim_{\theta\to\infty}\Psi_\mu(\theta,t)=0}$, $\boldsymbol{\Psi_\mu(\theta_{max},t)=e^{\theta max}-\psi}$, $\boldsymbol{\Psi_\mu(\theta,T)=\max(e^{\theta}-\psi,0)}$, $\boldsymbol{e^{\theta max}}$ is the maximum value of the underlying asset price. According to the conclusions of Yan and Qin (2018), the maximum stock price is generally 3 to 4 times the exercise price, so the numerical analysis below will also refer to this range.

3. Theoretical Derivation

In this part, we will show the derivation process of the first-order fully implicit model. Assuming that the positive integers $\mathbf{m}$ and $\mathbf{n}$ make the space step $\boldsymbol{\Delta\theta}$ satisfy $\boldsymbol{m\Delta\theta=\theta_{max}}$ and the time step $\boldsymbol{\Delta t}$ satisfy $\mathbf{n\Delta t=T}$ respectively, then $\boldsymbol{\theta_j=(j-1)\Delta\theta}$, $\mathbf{t_i=(i-1)\,\Delta t}$, Where, $\mathbf{j=\ldots,-2,-1,0,1,2,\ldots m+1}$, $\mathbf{i=1,2,\ldots,n+1}$. The first-order space and time derivatives use forward and backward difference methods respectively. Using the $Grünwal-Letnikov$ formula to approach the left and right fractional derivatives:

$$e_\theta^{\lambda\theta}H_{\theta\max}^{\alpha}e^{-\lambda\theta}\Psi_\mu(\theta_j,t_i)=\frac{1}{(\Delta\theta)^\alpha}\sum_{\rho=0}^{M-j+2}G_{\rho,\lambda}^{\alpha}\Psi_\mu(\theta_{j+\rho-1},t_i)+o(\Delta\theta^2) \qquad (12)$$

$$e_{-\infty}^{-\lambda\theta}H_\theta^\alpha e^{\lambda\theta}\Psi_\mu(\theta_j,t_i)=\frac{1}{(\Delta\theta)^\alpha}\sum_{\rho=0}^{\infty}G_{\rho,\lambda}^{\alpha}\Psi_\mu(\theta_{j-\rho+1},t_i)+o(\Delta\theta^2) \qquad (13)$$

Where, $\boldsymbol{G_{\rho,\lambda}^{\alpha}}$ is the coefficient of the fractional derivative and satisfies the following conditions:

$$G_{0,\lambda}^{\alpha}=\gamma_1\varphi_0e^{\Delta\theta\lambda},\ G_{1,\lambda}^{\alpha}=\gamma_1\varphi_1+\gamma_2\varphi_0$$
$$G_{\rho,\lambda}^{\alpha}=(\gamma_1\varphi_\rho+\gamma_2\varphi_{\rho-1}+\gamma_3\varphi_{\rho-2})e^{-(\rho-1)\Delta\theta\lambda}\ (\rho\geq 2)$$

The parameter $\boldsymbol{\varphi_1}$ of $\mathbf{G^{\alpha}_{\rho,\lambda}}$ satisfies the following iterative relationship, that is $\boldsymbol{\varphi_0 = 1}$ and $\boldsymbol{\varphi_\rho} = (\mathbf{1} - \frac{1+\alpha}{\rho})\boldsymbol{\varphi_{\rho-1}}$ $(\boldsymbol{\rho \geq 2})$, and the parameter $\boldsymbol{\gamma_i}$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3})$ of $\mathbf{G^{\alpha}_{\rho,\lambda}}$ satisfies $\boldsymbol{\gamma_1 + \gamma_2 + \gamma_3 = 1}$ and $\boldsymbol{\gamma_1 - \gamma_3 = \alpha/2}$.

In summary, the fully implicit difference form of equation (11) can be written as:

$$\frac{\Psi_j^{i+1} - \Psi_j^i}{\Delta t} + \alpha \frac{\Psi_j^i - \Psi_{j-1}^i}{\Delta\theta} + \frac{1}{2}\sigma^\alpha\left[\frac{p}{(\Delta\theta)^\alpha}\sum_{\rho=0}^{m-j+2} G^{\alpha}_{\rho,\lambda}\Psi^i_{j+\rho-1} + \frac{q}{(\Delta\theta)^\alpha}\sum_{\rho=0}^{\infty} G^{\alpha}_{\rho,\lambda}\Psi^i_{j-\rho+1}\right] + \frac{\mu R}{\Psi_j^i + \mu - q(\theta_j)} = b\Psi_j^i \tag{14}$$

The corresponding boundary conditions are approximated as $\lim\limits_{\mathbf{j}\to\infty} \boldsymbol{\Psi_j^i = 0}$, $\boldsymbol{\Psi^i_{m+1}} = \mathbf{e^{\theta max}} - \boldsymbol{\psi}$ and $\boldsymbol{\Psi_j^{n+1}} = \mathbf{max}(\mathbf{e^{\theta j}} - \boldsymbol{\psi}, \mathbf{0})$, Where, $\boldsymbol{\Psi_j^i} = \boldsymbol{\Psi_\mu}(\boldsymbol{\theta_i}, \mathbf{t_i})$.

The penalty function term is used to approximate the governing equation to obtain the fully implicit form of the fractional partial differential equation. Based on this, $\boldsymbol{\Psi_j^i}$ still satisfies the inequality (10), which verifies the effectiveness of the difference scheme. Before the proof, combined with the research of Xi and Cao (2014), Ma, et al. (2019) and other scholars, *Lemma 1* is proposed.

Lemma 1

If $\max\left\{\frac{(2-\alpha)(\alpha^2+\alpha-8)}{2(\alpha^2+3\alpha+2)}, \frac{(1-\alpha)(\alpha^2+2\alpha)}{2(\alpha^2+3\alpha+4)}\right\} \leq \gamma_3 \leq \frac{(2-\alpha)(\alpha^2+\alpha-3)}{2(\alpha^2+3\alpha+2)}$, for $\mathbf{1} < \boldsymbol{\alpha} < \mathbf{2}$, $\boldsymbol{\lambda} > 0$, then $\mathbf{G^{\alpha}_{\rho,\lambda}}$ satisfies $\varphi_0 = 1$, $\varphi_1 = -\alpha$, $0 \leq \ldots \leq \varphi_3 \leq \varphi_2 \leq \varphi_1 \leq 1$, $\sum_{\rho=0}^{\infty}\varphi_\rho = 0$, $G^{\alpha}_{1,\lambda} \leq 0$, $G^{\alpha}_{0,\lambda} + G^{\alpha}_{2,\lambda} \geq 0$, $G^{\alpha}_{\rho,\lambda} \geq 0$ $(\rho \geq 2)$.

Lemma 2

When $\mathbf{1} < \boldsymbol{\alpha} < \mathbf{2}$, $\boldsymbol{\lambda} > 0$, then $\mathbf{G^{\alpha}_{\rho,\lambda}}$ satisfies the equation:

$$\sum_{\rho=0}^{\infty} G^{\alpha}_{\rho,\lambda} = \left(\gamma_1 e^{\Delta\theta\lambda} + \gamma_2 + \gamma_3 e^{-\Delta\theta\lambda}\right)(1 - e^{-\Delta\theta\lambda})^\alpha$$

Refer to the conclusions of Hao, et al. (2015), if the constant $\mathbf{z} \in (-\mathbf{1}, \mathbf{1}]$, then $(\mathbf{1} - \mathbf{z})^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\rho}=\mathbf{0}}^{\infty} \boldsymbol{\varphi_\rho} \mathbf{z}^{\boldsymbol{\rho}}$. On the basis of *Lemma 1*, we can get:

$$\begin{aligned}\sum_{\rho=0}^{\infty} G^{\alpha}_{\rho,\lambda} &= \sum_{\rho=2}^{\infty}\left(\gamma_1\varphi_\rho + \gamma_2\varphi_{\rho-1} + \gamma_3\varphi_{\rho-2}\right)e^{-(\rho-1)\Delta\theta\lambda} + G^{\alpha}_{0,\lambda} + G^{\alpha}_{1,\lambda} \\ &= \gamma_1 e^{\Delta\theta\lambda}\sum_{\rho=2}^{\infty}\varphi_\rho e^{-(\rho-1)\Delta\theta\lambda} + \gamma_2\sum_{\rho=2}^{\infty}\varphi_{\rho-1}e^{-(\rho-1)\Delta\theta\lambda} + \\ &\quad \gamma_3 e^{-\Delta\theta\lambda}\sum_{\rho=2}^{\infty}\varphi_{\rho-2}e^{-(\rho-2)\Delta\theta\lambda} + G^{\alpha}_{0,\lambda} + G^{\alpha}_{1,\lambda} \\ &= \gamma_1 e^{\Delta\theta\lambda}\left[\left(1 - e^{-\Delta\theta\lambda}\right)^\alpha + \alpha e^{-\Delta\theta\lambda} - 1\right] + \gamma_2\left[\left(1 - e^{-\Delta\theta\lambda}\right)^\alpha - 1\right] \\ &\quad + \gamma_3 e^{-\Delta\theta\lambda}\left(1 - e^{-\Delta\theta\lambda}\right)^\alpha + \gamma_1 e^{\Delta\theta\lambda} - \alpha\gamma_1 + \gamma_2 \\ &= \left(\gamma_1 e^{\Delta\theta\lambda} + \gamma_2 + \gamma_3 e^{-\Delta\theta\lambda}\right)\left(1 - e^{-\Delta\theta\lambda}\right)^\alpha\end{aligned}$$

Theorem 1

If $\mathbf{\Delta t} \leq \frac{\mathbf{1}}{\boldsymbol{\gamma}-\mathbf{y}}$, $\boldsymbol{\gamma_3} \geq \frac{\frac{1}{2}(\mathbf{e}^{\boldsymbol{\theta}\mathbf{max}}-\mathbf{1})-\mathbf{1}}{\mathbf{e}^{\boldsymbol{\theta}\mathbf{max}}+\mathbf{e}^{-\boldsymbol{\theta}\mathbf{max}}-\mathbf{2}}$, and $\mathbf{R}$ satisfies the following inequality:

$$R \geq |a| \frac{e^{\theta\max}(e^{\theta\max}-1)}{e^{\theta\max}} + \left(\frac{1}{2}\sigma^{\alpha}\lambda^{\alpha} + |b|\right)(e^{\theta\max} + \psi)$$

Then $\boldsymbol{\Psi_j^i}$ obtained by formula (14) satisfies the inequality $\boldsymbol{\Psi_j^i} \geq \mathbf{max}(\mathbf{e}^{\boldsymbol{\theta}\mathbf{j}} - \boldsymbol{\psi}, \mathbf{0})$.

For the proof of *Theorem 1*, first prove that $\boldsymbol{\Psi_j^i} \geq \mathbf{e}^{\boldsymbol{\theta}\mathbf{j}} - \boldsymbol{\psi}$, and then prove that $\boldsymbol{\Psi_j^i} \geq \mathbf{0}$ exists for any $\mathbf{i}$ and $\mathbf{j}$.Let $\mathbf{q_j} = \mathbf{e}^{\boldsymbol{\theta}\mathbf{j}} - \boldsymbol{\psi}$, $\boldsymbol{\vartheta_j^i} = \boldsymbol{\Psi_j^i} - \mathbf{q_j}$, we can get:

$$\vartheta_j^{i+1} - \frac{a\Delta t}{\Delta\theta}\vartheta_{j-1}^i + \frac{1}{2}\sigma^{\alpha}\Delta t\left[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha}\,\vartheta_{j+\rho-1}^i + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\,\vartheta_{j-\rho+1}^i\right] + \frac{\mu R\Delta t}{\vartheta_j^i+\mu} + \Delta tF = \left(1 - \frac{a\Delta t}{\Delta\theta} + b\Delta t\right)\vartheta_j^i$$

Where, $F = \frac{a}{\Delta\theta}(q_j - q_{j-1}) - bq_j + \frac{1}{2}\sigma^{\alpha}[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha}\, q_{j+\rho-1} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\, q_{j-\rho+1}]$.

As p+q=q, so $\left|\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\right| \leq \frac{\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}}{(\Delta\theta)^{\alpha}}$.

Based on the *Lemma 2*, when $\boldsymbol{\gamma_3} \geq \frac{\frac{\alpha}{2}(\mathbf{e}^{\boldsymbol{\theta}\mathbf{max}}-\mathbf{1})-\mathbf{1}}{\mathbf{e}^{\boldsymbol{\theta}\mathbf{max}}+\mathbf{e}^{-\boldsymbol{\theta}\mathbf{max}}-\mathbf{2}}$, we can get $\left|\frac{\sum_{\rho=0}^{\infty} \mathbf{G}_{\rho,\lambda}^{\alpha}}{(\boldsymbol{\Delta\theta})^{\alpha}}\right| \leq \boldsymbol{\lambda}^{\boldsymbol{\alpha}}$, then:

$$\frac{1}{2}\sigma^{\alpha}\left|\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha}\psi + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\psi\right| \leq \frac{1}{2}\sigma^{\alpha}\lambda^{\alpha}\psi \tag{15}$$

$$\frac{1}{2}\sigma^{\alpha}\left|\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha}e^{\theta_{j+\rho-1}} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}e^{\theta_{j+\rho-1}}\right| \leq \frac{1}{2}\sigma^{\alpha}\lambda^{\alpha}e^{\theta\max} \tag{16}$$

Substituting formula (15) and formula (16) into equation $\mathbf{F}$, we can get:

$$|F| \leq |a|\frac{e^{\theta\max}(e^{\theta\max}-1)}{e^{\theta\max}} + (\frac{1}{2}\sigma^{\alpha}\lambda^{\alpha} + |b|)(e^{\theta\max} + \psi) \tag{17}$$

Let $\boldsymbol{\vartheta_j^i} = \mathbf{min_j}\boldsymbol{\vartheta_j^i}$ and $\boldsymbol{\vartheta_L^{i+1}} = \mathbf{min_j}\boldsymbol{\vartheta_j^{i+1}}$, then:

$$\left\{1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\Delta t[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}]\right\}\vartheta_j^i - \frac{\mu R\Delta t}{\vartheta_j^i+\mu} - \Delta tF \geq \vartheta_L^{i+1} \tag{18}$$

Define a new function:

$$\Phi(x) = \left\{1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\Delta t[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}]\right\} x - \frac{\mu R\Delta t}{x+\mu} - \Delta tF$$
(19)

According to the conclusion of *Lemma 2*, when $\boldsymbol{\gamma_3 \geq \frac{\frac{\alpha}{2}(e^{\theta max}-1)-1}{e^{\theta max}+e^{-\theta max}-2}}$ and $\boldsymbol{\Delta t \leq \frac{1}{\gamma-y}}$, we can get:

$$1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\Delta t\left[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\right] \geq 1 + b\Delta t \quad (20)$$

$$-\frac{1}{2}\sigma^{\alpha}\Delta t\frac{(\gamma_1 e^{\Delta\theta\lambda}+\gamma_2+\gamma_3 e^{-\Delta\theta\lambda})(1-e^{-\Delta\theta\lambda})^{\alpha}}{(\Delta\theta)^{\alpha}} \geq 1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\lambda^{\alpha}\Delta t \quad (21)$$

Determine the monotonicity of function $\boldsymbol{\Phi(x)}$ on the basis of $\boldsymbol{1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\lambda^{\alpha}\Delta t = 1 + (y-\gamma)\Delta t \geq 0}$, for the function $\boldsymbol{\Phi(x)}$, according to $\boldsymbol{\Phi'(\vartheta_j^i) = 1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\Delta t\left[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}\right] + \frac{\mu R\Delta t}{(\vartheta_j^i+\mu)^2} \geq 2}$ for conversion, it can be concluded that $\boldsymbol{\Phi(x)}$ is monotonically increasing.

Assuming $\boldsymbol{\vartheta_L^{i+1} \geq 0}$, then $\boldsymbol{\Phi'(\vartheta_j^i) \geq 0}$. According to the value range of **R** in *Theorem 1*, we know that **R+F≥0**, then $\boldsymbol{\Phi(0) = -\Delta t(R+F) \leq 0}$. As the monotonicity of $\boldsymbol{\Phi(x)}$, it can be judged that $\boldsymbol{\vartheta_j^i \geq 0}$, then $\boldsymbol{\vartheta_j^i \geq 0}$. Based on the boundary condition of $\boldsymbol{\vartheta_j^{n+1} \geq 0}$, using the mathematical induction, we can get $\forall_{\mathbf{i}}$, $\mathbf{j}$, $\boldsymbol{\vartheta_j^i \geq 0}$, and the first stage of the proof is completed. Then enter the second stage of the proof process to prove $\boldsymbol{\vartheta_j^i \geq 0}$. According to the previous method, redefine $\boldsymbol{\Psi_J^i = \min_j \Psi_j^i}$, $\boldsymbol{\Psi_L^{i+1} = \min_j \Psi_j^{i+1}}$.

Since it was proved in the first stage that $\forall_{\mathbf{i}}$, $\boldsymbol{j, \Psi_j^i \geq q_j}$, we can get $\boldsymbol{\frac{\mu R\Delta t}{\vartheta_j^i+\mu} \geq 0}$, then:

$$\left\{1 + b\Delta t - \frac{1}{2}\sigma^{\alpha}\Delta t[\frac{p}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{m-j+2} G_{\rho,\lambda}^{\alpha} + \frac{q}{(\Delta\theta)^{\alpha}}\sum_{\rho=0}^{\infty} G_{\rho,\lambda}^{\alpha}]\right\} \Psi_J^i \geq \Psi_L^{i+1} \quad (22)$$

In summary, we can get $\boldsymbol{\Psi_J^i \geq \Psi_L^{i+1}}$. Assuming $\boldsymbol{\Psi_L^{i+1} \geq 0}$, then $\boldsymbol{\Psi_J^i \geq 0}$. Based on the boundary condition $\boldsymbol{\Psi_j^{n+1} \geq 0}$, it can be concluded that $\forall_{\mathbf{i}}$, $\mathbf{j}$, $\boldsymbol{\vartheta_j^i \geq 0}$.

In the actual financial market, the risk-free interest rate $\mathbf{y}$ and the capital cost $\boldsymbol{\gamma}$ usually satisfy $\boldsymbol{0 < y < \gamma < 1}$ and $\boldsymbol{1/(\lambda - y) > 1}$. Therefore, the time step length that limit the previous proof process is in line with the actual situation, and it can also be satisfied in the following numerical simulation.

4. Numerical simulation

We use a numerical simulation example to test the correctness of the conclusions of previous theoretical analysis in this part. After verifying the previous conclusions through numerical calculations, we performed a sensitivity analysis of the parameters to compare whether there are significant differences under different parameter values.

4.1 Numerical calculation

In order to facilitate the simulation, the original semi-infinite area $(-\infty, \boldsymbol{\theta_{max}}] \times [\mathbf{0}, \mathbf{T}]$ is divided into a finite area $(\boldsymbol{\theta}, \mathbf{t}) \in (\boldsymbol{\theta_{min}}, \boldsymbol{\theta_{max}}] \times [\mathbf{0}, \mathbf{T}]$, Where $\boldsymbol{\theta_{min}} = \mathbf{ln}(\mathbf{0.01})$, the left boundary condition in the original model is changed to $\boldsymbol{\Psi_\mu}(\boldsymbol{\theta_{min}}, \mathbf{t}) = \mathbf{0}$. For the convenience of description, the subscripts $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ of $\mathbf{G}^{\boldsymbol{\alpha}}_{\boldsymbol{\rho},\boldsymbol{\lambda}}$ are omitted in the rest of this paper. Redefine the spatial step length as $\Delta\boldsymbol{\theta} = (\boldsymbol{\theta_{max}} - \boldsymbol{\theta_{min}})/\mathbf{m}$, then $\boldsymbol{\theta_j} = (\mathbf{j} - \mathbf{1})\Delta\boldsymbol{\theta} + \boldsymbol{\theta_{min}}$, **j=1,2,...,m**, then the matrix form of equation (14) is:

$$[\beta I + \eta N + \varpi_1(pM^T + qM)]{\Psi_\mu}^i - f({\Psi_\mu}^i) = {\Psi_\mu}^{i+1} + \varpi_2 E^i \qquad (23)$$

Where, $\boldsymbol{\beta} = \mathbf{1} + \mathbf{b}\boldsymbol{\Delta}\mathbf{t} - \frac{\mathbf{a}\Delta\mathbf{t}}{\Delta\boldsymbol{\theta}}$, $\boldsymbol{\xi} = \frac{\mathbf{a}\Delta\mathbf{t}}{\Delta\boldsymbol{\theta}}$, $\boldsymbol{\varpi_1} = \frac{-\frac{1}{2}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}\Delta\mathbf{t}}{(\Delta\boldsymbol{\theta})^{\boldsymbol{\alpha}}}$, $\boldsymbol{\varpi_2} = -\frac{1}{2}\boldsymbol{\sigma}^{\boldsymbol{\alpha}}\boldsymbol{\Delta}\mathbf{t}$, $\mathbf{I}$ is an **m-1** order identity matrix, ${\boldsymbol{\Psi_\mu}}^{\mathbf{i}} = (\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{2}}, \boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{3}}, \dots, \boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{m-1}}, \boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{m}})$, $\mathbf{f}({\boldsymbol{\Psi_\mu}}^{\mathbf{i}}) = [(\mathbf{f}(\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{2}}), \mathbf{f}(\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{3}}), \dots, \mathbf{f}(\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{m-1}}), \mathbf{f}(\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{m}})]$. At the same time, $\mathbf{f}(\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{j}}) = \frac{\boldsymbol{\mu}\mathbf{R}\Delta\mathbf{t}}{\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{j}} + \boldsymbol{\mu} - \mathbf{q}^{\mathbf{i}}_{\mathbf{j}}}$, $\mathbf{E}^{\mathbf{i}} = (\frac{\mathbf{p}}{(\Delta\boldsymbol{\theta})^{\boldsymbol{\alpha}}}\mathbf{G_m}, \frac{\mathbf{p}}{(\Delta\boldsymbol{\theta})^{\boldsymbol{\alpha}}}\mathbf{G_{m-1}}, \dots, \frac{\mathbf{p}}{(\Delta\boldsymbol{\theta})^{\boldsymbol{\alpha}}}\mathbf{G_2}, \frac{\mathbf{p}}{(\Delta\boldsymbol{\theta})^{\boldsymbol{\alpha}}}\mathbf{G_1})\boldsymbol{\Psi}^{\mathbf{i}}_{\mathbf{m+1}}$, both $\mathbf{M}$ and $\mathbf{N}$ are square matrices of order **m-1**.

$$M = \begin{bmatrix} G_1 & G_2 & 0 & \cdots & 0 & 0 \\ G_2 & G_1 & G_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ G_{m-2} & G_{m-3} & G_{m-4} & \cdots & G_1 & G_1 \\ G_{m-1} & G_{m-2} & G_{m-3} & \cdots & G_1 & G_1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Equation (23) is solved by *Newton* iteration as:

$$\begin{aligned} &[\beta I + \eta N + \varpi_1(pM^T + qM) - J_f(\varphi^{l-1})]\Delta\varphi^l \\ &= {\Psi_\mu}^{i+1} + \varpi_2 E^{i+1} - [\beta I + \eta N + \varpi_1(pM^T + qM)]\varphi^{l-1} + f(\varphi^{l-1}) \\ &\qquad \varphi^l = \varphi^{l-1} + \rho\Delta\varphi^l \end{aligned}$$

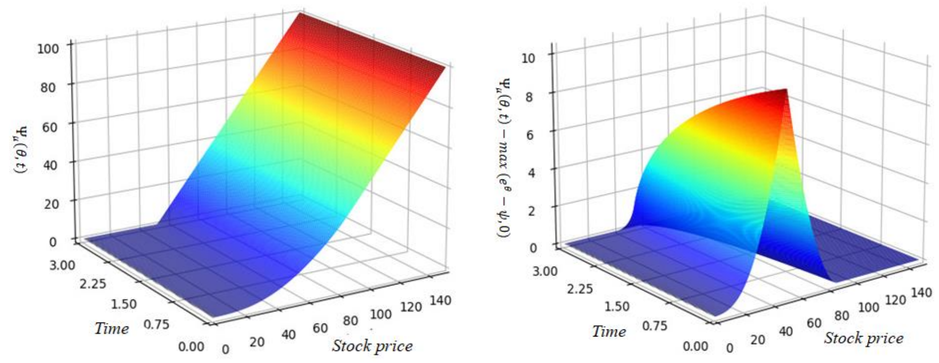
Where, **l=1,2.3…,** $\mathbf{J_f}(\boldsymbol{\varphi}^{\mathbf{l-1}})$ is the *Jacobian* matrix of the vector $\mathbf{f}(\boldsymbol{\varphi}^{\mathbf{l-1}})$, and $\boldsymbol{\rho} \in (\mathbf{0}, \mathbf{1})$ is the adjustment factor. In the numerical iteration process, it is assumed that the information of the current time $\mathbf{t_i}$ and the previous time $\mathbf{t_{i+1}}$ is known. Therefore, $\boldsymbol{\Psi_\mu}^{\mathbf{i+1}}$ can be used as the initial value of the iterative sequence $\boldsymbol{\varphi}^{\mathbf{l}}$, that is $\boldsymbol{\varphi}^{\mathbf{0}} = \boldsymbol{\Psi_\mu}^{\mathbf{i+1}}$. Set the amount of error $\boldsymbol{\mu}$, when $\left|\boldsymbol{\varphi}^{\boldsymbol{l}} - \boldsymbol{\varphi}^{\boldsymbol{l-1}}\right| \leq \boldsymbol{\mu}$, take $\boldsymbol{\Psi_\mu}^{\mathbf{i}} = \boldsymbol{\varphi}^{\mathbf{l}}$.

The following is a numerical simulation of the previous conclusions. The setting of each parameter value refers to existing researches and the actual situation of the financial market (Table 1). All simulation processes are implemented by MATLAB. It should be pointed out that this part aims to prove the general law of American-style warrant pricing through numerical simulation, even though the setting values of parameters may be different from the actual situation, it does not affect the basic conclusions of this paper.

Table 1. Parameter value setting and setting basis

Parameter	Value	Setting basis
y	5%	National debt yield is generally regarded as the reference standard for risk-free interest rate in the capital market. According to the difference of the deadline, the interest rates of national debts in various countries are generally between 3.5% and 6.0%, the setting value in this paper is 5%.
γ	20%	The capital cost can be regarded as the average credit interest rate. According to the researches such as Schmitz (2005), Zhang and Yin (2020), the current annualised interest rate of commercial bank loans in various countries are generally between 16%-24%, this paper chooses 20%.
α	1.54	$\alpha \in (1,2)$, according to the research of Zhang and Yin (2020), Wang and Chen (2020) and other scholars, the maximum value is set to 1.54.
λ	1	$\lambda > 0$, according to the research of Chen and Lin (2018), Ma, et al. (2019), the maximum value is set to 1.
σ	0.8	According to the research of Chen et al. (2015), Xi and Cao (2014) and other scholars, the maximum value is set to 0.8.
T	2	The term of American-style warrant can be as long as 10 years, however, it is generally 1-3 years in actual economic activities, this paper takes 2 years.
ψ	50	For ease of analysis, set this value to 50 units.

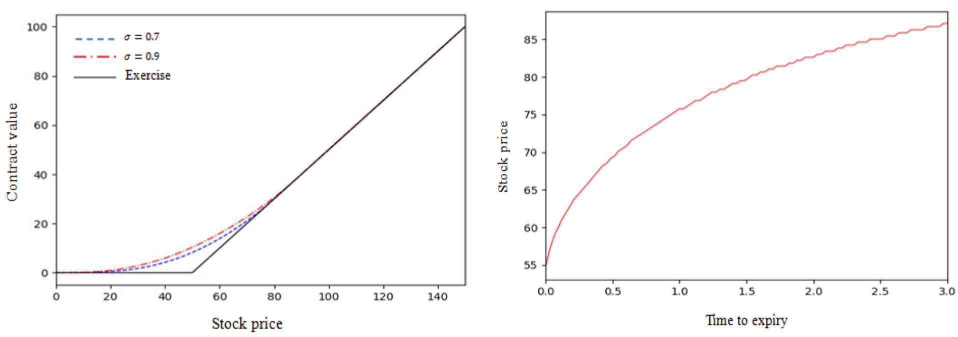
The entire time axis and the space axis in Figure 1 show the surface of $\boldsymbol{\Psi_\mu}(\boldsymbol{\theta},\mathbf{t})$ and $\boldsymbol{\Psi_\mu}(\boldsymbol{\theta},\mathbf{t}) - \mathbf{max}(\mathbf{e}^{\boldsymbol{\theta}} - \boldsymbol{\psi}, \mathbf{0})$. First, use numerical methods to generate a smooth and stable approximate solution (Figure 1(a)). The curved surface in Figure 1(b) shows that for all $\mathbf{i}$ and $\mathbf{j}$, the American-style warrant contract value $\boldsymbol{\Psi_j^i \geq \max(e^{\theta j} - \psi, 0)}$ is established, which is consistent with the conclusion of *Theorem 1*.



（a） （b）

Figure 1. Contract transaction characteristics after coordinate transformation

It can be seen in Figure 2(a) that the value of $\boldsymbol{\Psi_\mu(\theta, 0)}$ increases with the increase of stock price, and is always greater than the value payment function. It can be found from Figure 2(b) that the optimal execution boundary increases as the expiration time increases.



（a） （b）

Figure 2. Contract value and the optimal execution boundary after coordinate transformation

4.2 Sensitivity analysis

Use the inverse transformation of formula (8) to obtain the value of the American-style warrant and the optimal exercise price. As shown in Figure 3, the optimal exercise price will first increase and then decrease with the time to expiry. One of the reasons for this phenomenon is that the contract can be executed in advance, since the contract is still far from the expiration time, the time value of American-style warrant will increase, and make its exercise price gradually rise, such as the first half of the curve. Another important reason is that the final price in the pricing model will change over time. As the contract gets farther and farther from the expiry time, the contract holder needs to pay more capital cost, this leads to a decrease in the optimal exercise price.

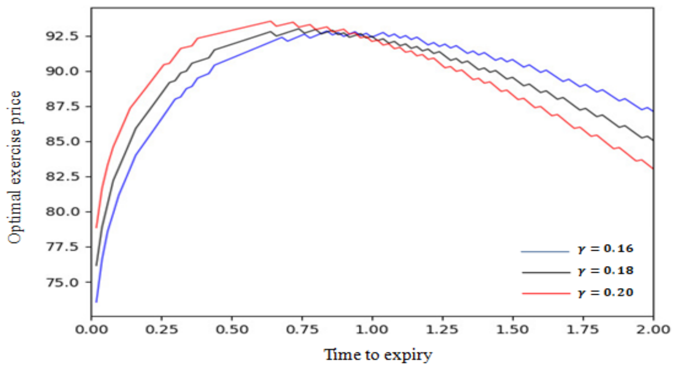
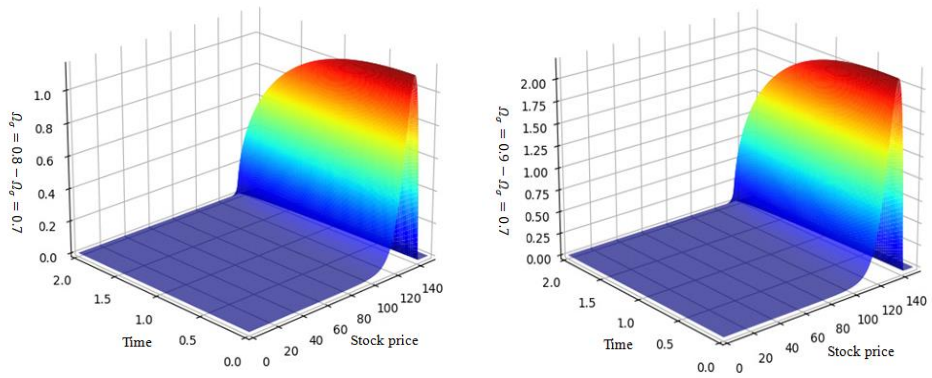


Figure 3. The optimal exercise price under different γ



(a) (b)

Figure 4. Contract value under different σ

Figures (a) and (b) in Figure 4 show the impact of different $\boldsymbol{\sigma}$ on the contract value of American-style warrant. It can be seen that the value of the American-style warrant monotonically increases relative to $\boldsymbol{\sigma}$. It can be seen from the stock price driving equation that when other parameters remain unchanged, $\boldsymbol{\pi(\sigma)dt}$ increases correspondingly with $\boldsymbol{\sigma}$ increases, which cause the fluctuation of the random variable **x** to increase. Moreover, since the statistical deviation of the payment function $\mathbf{max(e^{x} - \psi e^{\gamma t})}$ increases with the change of **x**, the contract holder has opportunities to obtain greater return. Therefore, the contract value of the American-style warrant increases monotonously with respect to $\boldsymbol{\sigma}$.

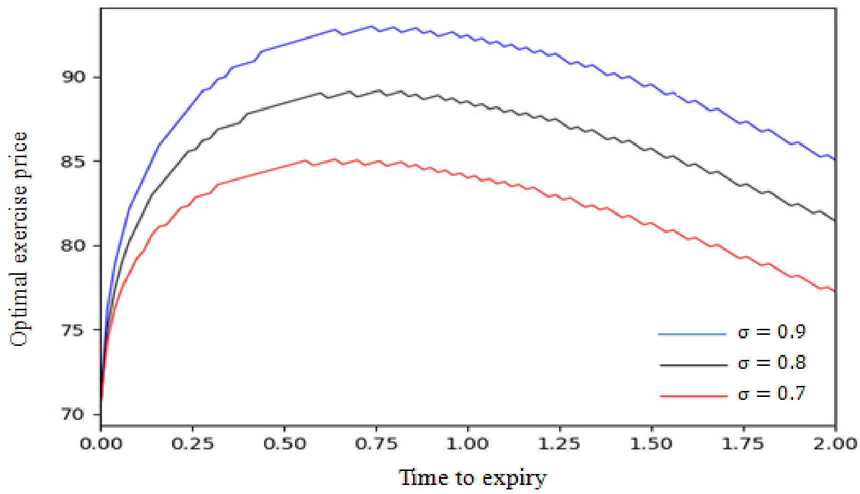
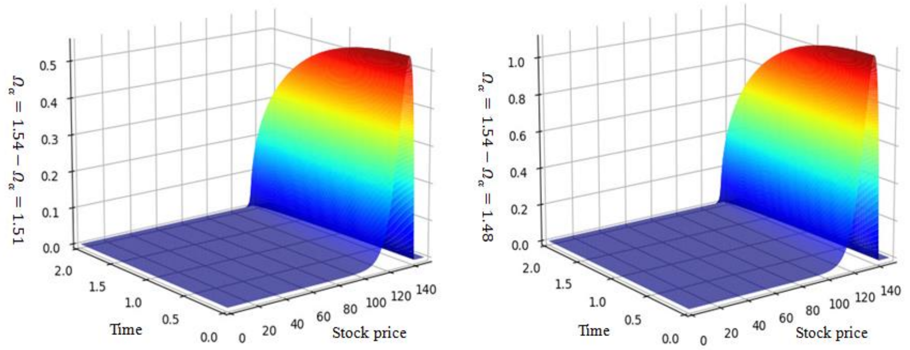


Figure 5. The optimal exercise price under different $\boldsymbol{\sigma}$

Figure 5 shows the effect of different $\boldsymbol{\sigma}$ on the optimal exercise price. It can be analysed from two perspectives. First, at the optimal exercise boundary, the contract value is equal to the exercise price. Therefore, the contract value will increase with the increase of $\boldsymbol{\sigma}$, which increases the optimal exercise price. Second, as the parameter $\boldsymbol{\sigma}$ becomes larger, the stock may obtain a higher price, which makes investors believe that the contract may have a higher value and will not choose to terminate the contract, which result in a higher exercise price. In summary, the optimal exercise price increases monotonically with respect to the parameter $\boldsymbol{\sigma}$.



（a） (b)

Figure 6. Contract value under different α

Figure 6 shows the effect of different α on the value of American-style warrant contract. It can be seen from Figures 6(a) and 6(b) that the contract value increases monotonously with respect to α. According to the non-standard steady-state random variable α, it can be known that when there is no abnormal deviation in the stock price, it is possible to obtain greater stock value with α increases. Since the contract value changes in the same direction as the stock price, the contract value increases monotonically with respect to the parameter α.

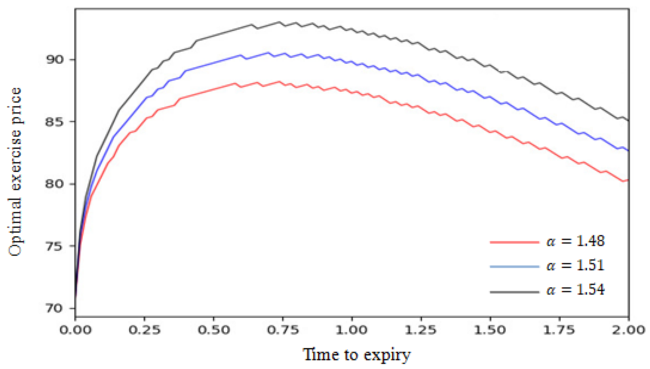


Figure 7. The optimal exercise price under different α

Figure 7 shows the effect of different α on the optimal exercise price. In the execution domain, the value of American-style warrant contract is equal to the exercise price. It can be seen from the foregoing that the optimal exercise price increases monotonously with respect to α. From the financial perspective, it can be considered that an increase in α makes the stock price more likely to rise, which makes investors believe that the contract may have a higher value, and leading to a higher exercise price.

5. Conclusions

Reasonable pricing of the American-style warrant has obvious significance for financial market risk control. This paper studies the pricing model of American-style warrant under the KoBoL process. Under the KoBoL process, the free boundary problem is transformed into a fixed boundary problem using coordinate transformation method and penalty method, the first-order fully implicit form of the American-style warrant pricing model is constructed, and it is proved that the penalty function difference method is useful for solving the governing equation is still valid, and finally a numerical simulation is carried out. The study found that the value of American-style warrant contract is not less than the exercise value, and the optimal exercise price decreases as $\boldsymbol{\gamma}$ increases. At the same time, due to the existence of $\boldsymbol{\gamma}$, the optimal execution boundary is no longer a monotonic function on the time to expiry. For the parameters $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$, with the value of the contract increases, investors' expectation of greater value for the stock price increases, which leads to an increase in the contract value and the optimal exercise price. The american-style warrant is an important derivative financial instrument in the financial market, its price not only involves the interests of investors, but also has an important impact on the stock market, it is an important inducing factor for systemic risks in the financial market. Following the changes in the financial market, especially the stock market, a more precise pricing of the American-style warrant will be an important demand from the practical and theoretical circles in the future. How can we further improve the effectiveness of the control equation on the basis of a fixed boundary? Does the stock value have a more specific range that is not less than the exercise value? This will be an important direction for us to continue to explore the American-style warrant in the future.

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