Martha FLORES-SOSA, PhD
E-mail: martha.flores@uadeo.mx
Universidad Autónoma de Occidente, Culiacán, México
Ernesto LEON-CASTRO, PhD*
E-mail: eleon@ucsc.cl
Faculty of Economics and Administrative Sciences,
Universidad Católica de la Santísima Concepción, Chile
Ezequiel AVILES-OCHOA, PhD
E-mail: ezequiel.aviles@uadeo.mx
Universidad Autónoma de Occidente, Culiacán, México
Jose M. MERIGO
E-mail: jose.merigo@uts.edu.au
School of Systems, Management and Leadership,
Faculty of Engineering and Information Technology
University of Technology Sydney, Australia
*** Corresponding author**

FORECASTING VOLATILITY WITH SIMPLE LINEAR REGRESSION AND ORDERED WEIGHTED AVERAGE OPERATORS

Abstract. *Estimating and forecasting volatility is an important issue for financial decision-makers. Therefore, it is important to build models that adapt to the current characteristics of the time series. The ordered weighted average (OWA) has some extensions that provide interesting ways to adapt to these characteristics. This work proposes a new application that uses the simple linear regression (LR) and OWA operators in the same formulation. We use the heavy ordered weighted average (HOWA), the prioritized ordered weighted average (PrOWA), the probabilistic ordered weighted average (POWA) and their combinations with induced cooperators. The main idea in linear regression with OWA operator is to obtain an estimate and forecast that can be adaptable to situations of uncertainty and information known to the decision maker. The work analyzes the applicability of this new approach in a problem regarding exchange rate volatility forecasting, where the operators that we can use in high or low seasons are located and thus generate ranges.*

Keywords: *linear regression, volatility, ordered weight average operator, ordinary least squares.*

JEL Classification: C20, F31, C50, C53

DOI: 10.24818/18423264/56.3.22.13

1. Introduction

The vulnerability of financial markets can be measured by price volatility (Markowitz, 1952; Jiang & Liu, 2016). Therefore, forecasting and estimating volatility has become the subject of study in financial instruments such as the stock market (Kim & Won, 2018), gold price (Chen & Xu, 2019) and exchange rate (Christou, 2018).

One of the most well-known approaches to generating a simple equation between parameters is simple linear regression. This method can often derive results which match, to a useful approximation, those found in the real world (Roni et al., 2019). A linear regression estimator frequently used for its simplicity and functionality is the ordinary least squares (OLS) (Stone and Brooks, 1990; Huang, 2017).

However, the data in some areas could be large and complex, which hinders the operation of the linear regression and the OLS estimator. Therefore, several proposals have been made to use fuzzy logic to forecast in an uncertain environment, in multiple regression analysis (Jung et al., 2015) or the least-squares method (Hose & Hanss, 2019). To improve the forecast with linear regression and the process of estimation with ordinary least squares, this work proposes the use of the operator of the weighted ordered average (OWA) (Yager, 1988) to create new estimators that contemplate nonlinearity and high degrees of uncertainty.

The OWA operator (Yager, 1988) is a method that considers an aggregation process, providing the maximum, the minimum, and the average. In addition, it has some extensions that provide a better solution for handling complex and uncertain data; the Heavy Ordered Weighted Averaging HOWA (Yager, 2002), the prioritized OWA (PrOWA) (Yager, 2008), The probabilistic ordered weighted averaging POWA operator (Merigó, 2009), the prioritized induced heavy OWA (PIHOWA) PIHOWA (León-Castro et al., 2021) and the prioritized induced probabilistic OWA (PIPOWA) (Perez-Arellano et al., 2021). Since its introduction, OWA operators have been successfully applicated forecasting exchange rate (León-Castro et al., 2016), business administration (Vigier et al., 2017), and portfolio selection (Zhang & Zhang, 2004), among others.

The paper presents a new operator using some extensions of the OWA operator in calculating the parameters in linear regression; these are called: LR-HOWA, LR-PrOWA, LR-POWA, LR-PIHOWA and LR-PIPOWA. The main idea is to obtain a linear model which can be estimated according to the decision maker's degree of optimism or pessimism. Moreover, the new proposals can capture probabilistic and diffuse information in a single formulation, using two or more group decision-making problems with a weighting vector greater than one. This feature allows us to work with different degrees of uncertainty and model complex variables. The proposed estimators forecast the Mexican peso-US dollar exchange rate volatility. With the results, the ranking changes can be analyzed according to additional information provided.

This work is organized as follows: Section 2 reviews some basic preliminaries regarding linear regression, OWA operators and some of its extensions. Section 3 presents different propositions of the simple linear regression with OWA operator extensions. Section 4 develops an application of the new framework in analyzing volatility exchange rate forecasting between USD and MXN. Finally, the conclusions summarize the main findings of the paper.

2. Preliminaries

This section presents the main definitions that will be used to generate the LR-HOWA, LR-PrOWA, LR-POWA, LR-PIHOWA and LR-PIPOWA. The main idea is to include different weighting vectors in the traditional Linear Regression formulations; this will allow further analysis of the historical data by having the weighting vector and the reordering step that are the main components of the OWA operator. Furthermore, with these additional steps in the LR formulation, the decision maker's expectation, knowledge and aptitude will be included in the result.

2.1 Linear regression

The purpose of linear regression is to explain the variation of a dependent variable in terms of the interpretation of explanatory variables in a linear function (Seber & Lee, 2003). To perform it, a functional relationship between the variables must be postulated. It is defined as follows:

Definition 1. Is a simple regression when given a set of variables (x_k, y_k), where a function $f_\theta: R^n \rightarrow R$, parameterized by a parameter vector $\theta = \alpha + \beta$. Simple linear regression model is developed as:

$$y_j = \alpha + \beta x_j, \quad (1)$$

where y_j is the dependent variable and x_j is the independent variable. Some assumptions about the given data bound the linear regression; the values of the unobserved error term are mutually independent and identically distributed. It is also assumed that the relationship between variables is crisp (Peters, 1994)

The most popular method to estimate the vector θ is Ordinary least squares (OLS) (Gujarati & Porter, 2009). The OLS seeks to minimize $F(\theta) = \sum_{K=1}^{K} w_k \left(r_k(\theta)\right)^2$. The values α and β that minimizes the previous expression are calculated as follows:

$$\beta = \frac{Cov(x,y)}{var(x)} = \frac{\sum_{k=1}^{k}(x_k - \bar{x})(y_k - \bar{y})}{\sum_{k=1}^{k}(x_k - \bar{x})^2}, \quad (2)$$

$$\alpha = \bar{y} - \beta\bar{x}, \quad (3)$$

where $\bar{x}$ and $\bar{y}$ are the averages in the sets x_k, y_k Respectively. Note that β could be estimated with variance and covariance.

As can be seen, the LR formulation uses averages in both the variance and the covariance; in this sense, it is assumed that from the whole set of data, all the values/attributes has the same importance in the average (1/n). Still, sometimes it would be possible to assume that some information impacts the results more than others. Because of that, it is possible to generate a variance using a weighted average or more complex results by utilizing a reordering step. These ideas are important because it is possible to create new scenarios that usually cannot be obtained by the traditional formulations. With these, the decision maker's expectations, knowledge and aptitude for the results can be included.

2.2 OWA operator and its extensions

The OWA operator (Yager, 1988) belongs to a parameterized family of aggregation operators. This offers values as the maximum, minimum, and arithmetic mean. The main idea of the reordering step between the arguments and the weighting vector allows It is defined as follows:

Definition 2. An OWA operator with dimension n is a model $OWA: R^n \rightarrow R$ characterized by an n dimensional vector $W = [w_1, w_2, .., w_n]^T$, called the weighting vector, such that its components lie in the unit interval and sum to one, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = w_1 + \cdots + w_n = 1$, then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \quad (4)$$

where b_j is the jth largest in $a_1, a_2, .., a_n$.

Also, some of the most used extensions of the OWA operator will be described. The first is the induced OWA (IOWA) operator (Yager & Filev, 1999). The main advantage of IOWA operator is the reordering step with order-inducing variables using tuples (v_i, a_i). The IOWA operator can work with complex variables in modelling. It can be defined as follows:

Definition 3. An IOWA operator of dimension n is a mapping $IOWA: R^n \rightarrow R$ with an associated weights vector $W = [w_1, w_2, .., w_n]^T$ where $w_i = \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then:

$$IOWA(\langle u_1, a_1\rangle\langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \quad (5)$$

where b_j is the a_i value of the OWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i. The variable u_i is the order inducing variable, and a_i is the argument variable.

The second extension will be presented based on the idea that the weighting vector is not bounded to 1. In this sense, the heavy OWA (HOWA) operator (Yager, 2002) lets the decision maker over or underestimate the results based on a weighting vector that theoretically can range from $-\infty$ to ∞. The definition proposed is the following.

Definition 4. A HOWA operator is a mapping $HOWA: R^n \rightarrow R$, associated with a weight vector $W = [w_1, w_2, .., w_n]^T$such that $0 \leq w_j \leq 1$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, so that:

$$HOWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \quad (6)$$

where b_j is the jth largest value of the collection a_i. An important element to consider with the HOWA operator presented by Yager (2002) is the beta value of the vector W. The beta value is defined as $\beta(W) = (|W| - 1)/(n - 1)$. Since $|W| \in [1, n]$, it follows that $\beta \in [0,1]$. That is why if $\beta = 1$ the total operator is obtained and if $\beta = 0$ the usual moving average is obtained.

The third extension analyzed in the paper is the prioritized OWA (PrOWA) operator (Yager, 2008), an operator for group decision-making. The main idea of PrOWA is that in group decision making not all the members have the same importance in the results. Because of that, their opinions should not be considered equally. The formulation can be as follows.

Definition 5. A PrOWA operator is a mapping $PrOWA: [0,1]^n \rightarrow [0,1]$, associated with a weight vector W such that $w_i = \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Additionally, assume a collection of criteria partitioned into q categories, $H_1, H_2, \dots, H_q$ then $H_i = \{C_{i1}, C_{i2}, \dots, C_{in}\}$ denotes the criteria of the ith category. The formula is as follows:

$$PrOWA((a_{11}, \dots, a_{1n_1}), \dots, (a_{q1}, \dots, a_{qn_q})) = \sum_{i=1}^{q} \left(\sum_{h=1}^{n_i} w_{ij} C_{ij} b_j(x) \right), \tag{7}$$

where w_{ij} is the corresponding weight of the jth criteria in the ith category, $i = 1, \dots, q, j = 1, \dots, i_i$. $X = \{x_1, \dots, x_m\}$ is the set of alternatives.

Finally, the probabilistic OWA (POWA) operator was introduced by Merigó (2009) as an extension of the OWA operator for situations where exist probability information. It provides a complete representation of reality because it unifies the probabilities and the OWA operators, considering the degree of importance of each case in the aggregation process. The definition is as follows:

Definition 6. A POWA operator of dimension n is a mapping $POWA: R^n \rightarrow R$ that has an associated weights vector $W = [w_1, w_2, \dots, w_n]^T$ where $0 \leq w_i \leq 0$ and $\sum_{i=1}^{n} w_i = 1$. Additionally, consider an associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, then:

$$POWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \widehat{p_j}\, b_j \tag{8}$$

where b_j is the jth element of the largest of the collection $a_1 a_2, \dots, a_n$, where each argument a_i is associated with a probability p_i, where $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$,$\hat{p}_j = \beta w_j + (1 - \beta) p_j$ with $\beta \in [0,1]$, and p_j is the probability of p_i ordered according to b_j, according to the jth largest element of a_i. Additionally, if $\beta = 0$, the PA operator is obtained, and if $\beta = 1$, the OWA operator is obtained.

Also, more complex extensions of the OWA operator combining different characteristics of different formulations into one have been done. In this paper, there is an emphasis on the following. An extension OWA that combines PrOWA, IOWA and HOWA operators is the prioritized induced heavy OWA (PIHOWA) proposed by León-Castro et al. (2021). The main characteristic of the PIHOWA operator is that it captures in one operator the prioritization of criteria and order with complex variables where the vector of weights can be greater than one. The formula can be defined as follows:

Definition 7. A PIHOWA operator of dimension n is a mapping $PIHOWA: R^n \rightarrow R$ with an associated weight vector w, where $0 \leq w_j \leq 1$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, then:

$$PIHOWA\left(\left(\langle u_{11}a_{11}\rangle, \dots, \langle u_{1n_1}a_{1n_1}\rangle\right), \dots, \left(\langle u_{q1}a_{q1}\rangle, \dots, \langle u_{qn_q}a_{qn_q}\rangle\right)\right) = \sum_{i=1}^{q}\sum_{h=1}^{n_i} b_j \hat{v}_{ij} C_{ij}(x), \tag{9}$$

where b_j is the jth element that has the largest value of u_i, u_i is the induced order of variables, $\hat{v}_{ij}$ is the corresponding weight of the jth criteria in the ith category, $i = 1, \dots, q, j = 1, \dots, i_i$ and $C_{ij}(x)$ measures the satisfaction of the jth criteria in the ith group by alternative $x \in X$, for each $i = 1, \dots, q, j = 1, \dots, i_i$.

Another extension is the prioritized induced probabilistic OWA (PIPOWA) operator (Perez-Arellano et al. 2019) combines IOWA, POWA and PrOWA operators in the same formulation. The PIPOWA operator uses prioritized criteria, inducing variables u_i and probabilistic data. The definition is as follows:

Definition 8. A PIPOWA operator of dimension n is a mapping $IPOWA: R^n \text{x} R^n \rightarrow R$ that has an associated weight vector w of dimension n where $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, so that

$$PIPOWA(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \sum_{i=1}^{q}\sum_{h=1}^{n_i} b_j \hat{v}_{ij} C_{ij}(x), \tag{10}$$

where b_j is the jth element that has the largest value of u_t. u_t is the induced order of variables, $\hat{v}_{ij}$ is the corresponding weight of the jth criteria in the ith category, $i = 1, \dots, q, j = 1, \dots, i_i$, and $C_{ij}(x)$ measures the satisfaction of the jth criteria in the ith group by alternative $x \in X$, for each $i = 1, \dots, q, j = 1, \dots, i_i$. Additionally, each element has an associated probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) p_j$, where $\beta \in [0,1]$ and p_j is the probability of p_i.

3. Propositions of simple linear regression with different OWA operator extensions

Different extensions of the linear regression formulation can be done using OWA operator and its extensions. This paper uses new formulations based on the operators defined in section 2. The first one will be the linear regression HOWA (LR-HOWA) operator. The main characteristic of the LR-HOWA operator is that the weighting vector can be unbounded and provide an under or overestimated result based on the decision maker's expectations. It can be defined as follows:

Proposition 1. A LR-HOWA operator of dimension n is a mapping $LR-HOWA: R^n \rightarrow R$ given the variables $x_k \in U$ and $y_k \in U$ such that they have an associated weighting vector $W = [w_1, w_2, .., w_n]^T$ where $w_i \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, so.

$$y_{HOWA} = \alpha_{HOWA} + \beta_{HOWA} x_j. \tag{11}$$

The calculation of the β parameter is as follows:

$$\beta_{HOWA} = \frac{\sum_{i=1}^{j} w_j(x_j-\mu)(y_j-\upsilon)}{\sum_{i=1}^{j} w_j(x_j-\mu)^2} = \frac{Cov_{HOWA}(x,y)}{var_{HOWA}(x)}, \quad (12)$$

where x_j y y_j is the jth largest arguments in the variables x and y respectively. μ and υ are the HOWA operator means in *x* and *y*. The components $(a_i-\mu)^2$ and $(x_j-\mu)(y_j-\upsilon)$ have some associated weights (WA) w_j with $1 \leq \sum_{j=1}^{n} w_j \leq n$ where $w_j \in [0,1]$.

Note that it is possible to obtain the HOWA variance and HOWA covariance $(var_{HOWA}; cov_{HOWA})$ as special cases. This is because it is possible to divide the elements of the linear regression formulation and analyze the covariance and variance as different elements. This applied to all the formulations that will be defined in the document. Finally, some special cases of β_{HOWA} are the following.

a) $max - Cov_{HOWA}(x,y) = \sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)$ when minimum HOWA in μ and υ and the $w_j = 1$ is used.

b) $max - var_{HOWA}(x) = \sum_{k=1}^{k} w_j(x_j-\mu)^2$ with a maximum HOWA in μ and the $w_j = 1$.

c) $min - Cov_{HOWA}(x,y) = \sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)$. It is an extreme case where μ and υ are the maximum HOWA and the minimum $w_j = 0$, then it has that $min - Cov_{HOWA}(x,y) = 0$.

d) $min - var_{HOWA}(x) = \sum_{k=1}^{k} w_j(x_j-\mu)^2$. It is a similar case as the previous one where μ is the maximum HOWA and $w_j = 0$, then $max - var_{HOWA}(x) = 0$.

e) $max - \beta_{HOWA}$ is calculated when $max - Cov_{HOWA}(x,y)$ and $max - var_{HOWA}(x)$ are used.

f) $Min - \beta_{HOWA}$ can not be calculated because $\beta_{HOWA} = \frac{0}{0} = IND$.

In the case of α parameter the calculation is developed as follows:

$$\alpha_{HOWA} = \upsilon - \beta_{HOWA}\mu, \quad (13)$$

where μ and υ are the means in vectors *x* and *y* calculated with the HOWA operator. One can see that $min - \beta_{HOWA}$ and maximum and minimum HOWA (μ ,υ) can be used.

The second formulation is based on the idea of a group decision-making process through the PrOWA operator obtaining the simple linear regression PrOWA (LR-PrOWA) operator. The definition is as follows:

Proposition 2. A LR-PrOWA operator of dimension n is a mapping $LR - PrOWA: R^n \rightarrow R$ with two variables $x_k \in U$ and $y_k \in U$ such that they have an associated weighting vector W where the weighs have some characteristics as $0 \leq w_i \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, then the formulation is:

$$y_{PrOWA} = \alpha_{PrOWA} + \beta_{PrOWA}x_j. \quad (14)$$

To estimate β and α parameters the following expressions are used:

$$\beta_{PrOWA} = \frac{\sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)}{\sum_{k=1}^{k} w_j(x_j-\mu)^2} = \frac{Cov_{PrOWA}(x,y)}{var_{PrOWA}(x)}, \quad (15)$$

$$\alpha_{PrOWA} = \upsilon - \beta_{PrOWA}\mu, \quad (16)$$

where μ and υ are the PrOWA operator in *x* and *y*, respectively. w_j is an associated weight $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$. Note that it is possible to obtain three different forms to calculate the LR-PrOWA considering using the PrOWA operator in estimating parameters.

The third proposition is based on the POWA operator. The main contribution of the simple linear regression POWA (LR-POWA) operator is that in the results is possible to combine a weighting vector and a probabilistic vector, with these is possible to add the expectations and knowledge of the decision maker (weighting vector) and the probabilities of obtaining the different results (probability vector) within the simple linear regression model. The formula is as follows.

Proposition 3. A LR-POWA operator of dimension n is a mapping $LR-POWA: R^n \rightarrow R$ with a dependent variable $x_k \in U$ and an independent variable $y_k \in U$, such that they have an associated weighting vector $W = [w_1, w_2, .., w_n]^T$, $0 \leq w_i \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, Additionally, the variables have an associated probability vector v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, so that:

$$y_{POWA} = \alpha_{POWA} + \beta_{POWA} x_j. \quad (17)$$

where the parameters β and α are estimated as follows:

$$\beta_{POWA} = \frac{\sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)}{\sum_{k=1}^{k} w_j(x_j-\mu)^2} = \frac{Cov_{POWA}(x,y)}{var_{POWA}(x)}, \quad (18)$$

$$\alpha_{POWA} = \upsilon - \beta_{POWA}\mu, \quad (19)$$

where μ and υ are the means (POWA operator). The components $(x_j - \mu)(y_j - \upsilon)$ and $(x_j - \mu)^2$ have an associated weight w_j, such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$.

The fourth proposition is based on the idea that simple linear regression can also be estimated with hybrid operators consisting of two or more OWA operators. For example, the PIHOWA operator in linear regression (LR-PIHOWA) combines Prioritized, Induced, Heavy and OWA operators with OLS estimator. The LR-PIHOWA can estimate parameters in a simple linear regression where more important data come out than others, the variables used are influenced by other variables and where the weighting vector is not bounded to a sum equal to one. It can be defined as follows:

Proposition 4. An LR-PIHOWA operator of dimension n is a mapping $LR-PIHOWA: R^n \rightarrow R$ with two variables x_k, y_k and an associated weight vector W, $1 \leq w_i \leq n$ and $\sum_{j=1}^{n} w_j = 1$, with an inducing variable u_i for the reordering, and corresponding weights according to priority $C(x)$, the formulation is:

$$y_{PIHOWA} = \alpha_{PIHOWA} + \beta_{PIHOWA} x_j. \tag{20}$$

It is possible to develop β and α as follows:

$$\beta_{PIHOWA} = \frac{\sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)}{\sum_{k=1}^{k} w_j(x_j-\mu)^2} = \frac{Cov_{PIHOWA}(x,y)}{var_{PIHOWA}(x)}, \tag{21}$$

$$\alpha_{PIHOWA} = \upsilon - \beta_{PIHOWA}\mu, \tag{22}$$

where the means μ and υ are calculated with PIHOWA operator. The components $(x_j-\mu)(y_j-\upsilon)$ and $(x_j-\mu)^2$ have a weight w_j, such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [1, n]$.

The last proposition will be done with the POWA, IOWA and PrOWA operators. The simple linear regression PIPOWA operator (LR-PIPOWA) is an aggregation operator in group decision-making that will include in the simple linear regression formula an additional probability vector and a reordering step based on induced variables instead of the maximum or minimum value of the attribute. The formulation is as follows.

Proposition 5. An LR-PIPOWA operator of dimension n is a mapping $LR-PIPOWA: R^n \rightarrow R$ with a dependent variable x_k and an independent variable y_k such that they have an associated weighting vector $W = [w_1, w_2, .., w_n]^T$, $0 \leq w_i \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, Additionally, the variables have an associated probability vector v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, corresponding weights according to priority $C(x)$ so that:

$$y_{PIPOWA} = \alpha_{PIPOWA} + \beta_{PIPOWA} x_j. \tag{23}$$

It is possible to develop β and α as follows:

$$\beta_{PIPOWA} = \frac{\sum_{k=1}^{k} w_j(x_j-\mu)(y_j-\upsilon)}{\sum_{k=1}^{k} w_j(x_j-\mu)^2} = \frac{Cov_{PIPOWA}(x,y)}{var_{PIPOWA}(x)}, \tag{24}$$

$$\alpha_{PIPOWA} = \upsilon - \beta_{PIPOWA}\mu, \tag{25}$$

where the means μ and υ are calculated with PIPOWA operator. The associated weight w_j, in variance and covariance has a characteristic $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$.

The LR-PIPOWA and all the propositions presented in this section have the same proprieties the OWA operator; monotonic, idempotent and boundary (Wei et al. 2012); these are explained as follows.

Monotonic: if exist a second attribute vector (b_i) forming a second LR-PIPOWA, if $a_j \geq b_j$ then:

$$F(y_{PIPOWA}(a_i), \ldots, (a_n)) \geq F(y_{PIPOWA}(b_i), \ldots (b_n)) \tag{26}$$

Commutativity: Let $f(y_i)$ be an application of LR-PIPOWA and where $f\big((a_1), \ldots, (a_n)\big)$ is a permutation of the arguments $f\big((e_i), \ldots, (e_n)\big)$. Then.

$$F(y_{PIPOWA}(a_i), \ldots, (a_n)) = F(y_{PIPOWA}(b_i), \ldots (b_n)) \tag{27}$$

Boundary condition: Assume that the ordered argument vector B can be ordered as $min_i\big[a_j\big]$ or $max_j\big[a_j\big]$, then:

$$y_{PIPOWA}(min_i\big[a_j\big]) \leq F(y_{PIPOWA}(a_i)) \leq y_{PIPOWA}(max_j\big[a_j\big]) \tag{28}$$

These properties demonstrate the consistency of the OWA operators and how they are conserved in conjunction with the linear regression.

3. Estimating the exchange rate volatility using linear regression with OWA operators

Volatility influences decision-making in the corporate and macroeconomic variables (Lahmiri, 2017). Movements in the exchange rate are caused for undefined and ambiguous factors generated by economic and political events (Cazorzi et al., 2017). Therefore, exchange rate volatility series present difficult statistical behaviours such as trends, abrupt changes, and volatility clustering.

In forecasting and estimating exchange rates, the random walk models are frequently used to generate better exchange rate forecasts (Cazorzi et al., 2017). In the case of volatility, the autoregressive (AR) models have been used in different contexts with satisfactory results (Zhang & Ling, 2015). However, the statistical behaviour of the financial series in a time of significant instability has led several authors to propose using combined methodologies to improve forecasting and estimation volatility (Lahmiri, 2017).

Moreover, very often, the variables and data of volatility models cannot be determined in a precise sense. Therefore, fuzzy logic provides an appropriate tool for modelling imprecise models (Singh, 2018). The ordered weighted average (OWA) operator is an aggregation operator used in uncertain environments. In the OWA operator the information can be represented so that it can be under- or overestimated, considering the decision-maker's attitudinal character.

This section develops the proposed formulation previously seen. The Mexican peso exchange rate in 2018 is used to calculate the volatility and estimate and forecast.

4.1 Numerical example

Volatility is the variable that will be calculated. A frequent proposal in scientific research is the volatility measure through logarithmic variances of returns log-volatility (Kim et al. 1988). One characteristic of log-volatility is eliminating the trend by creating a stationary series, which can be adapted much better to volatility models. It can be defined as follows:

Definition 9. Log-volatility of dimension n is a mapping $log-volatility: R^n \rightarrow R$ such that it has logarithmic returns as $R = \ln(\frac{p_t}{p_{t-1}})$, where p_t is the current price. Then,

$$h_t^2 = (R_i - \bar{R})^2, \tag{29}$$

where $\bar{R}$ is the returns mean.

The estimation and forecast of volatility are carried out with an autoregressive AR (1) model as follows:

$$h_t = \alpha + \beta_1 h_{t-1}, \tag{30}$$

where h_{t-1} is the volatility FIX exchange rate with one lag. The proposed models are the followings.

1. $y_{HOWA} = \alpha_{HOWA} + \beta_{1_{HOWA}} x_j$
2. $y_{PrOWA} = \alpha_{PrOWA} + \beta_{1_{PrOWA}} x_j$
3. $y_{POWA} = \alpha_{POWA} + \beta_{1_{POWA}} x_j$
4. $y_{PIHOWA} = \alpha_{PIHOWA} + \beta_{1_{PIHOWA}} x_j$
5. $y_{PIPOWA} = \alpha_{PIPOWA} + \beta_{1_{PIPOWA}} x_j$

The parameters α and β_1 are estimated performing the following steps:

Step 1. The elements and components in attribute vector *B* are defined. The sequence is n = 12 because the decision maker believes these periods influence the volatility. Then, two vectors are presented: one h_t for the current exchange rate volatility and other h_{t-1} for the volatility with a delay period (See Table 1).

Step 2. Determine the weights vector, the probability vector induced vector and the prioritized components. The weight vector is the same for both variables assigning the highest weights to the months closest to the month to estimate and forecast. The probability vector is developed according to the probability of occurrence in 95 previous periods using a probability of importance of 40% to the weight and 60% to the probability. The induced vector is according to the nearest months (See Table 1).

Table 1. Application data

Date	h_t	h_{t-1}	w	w_{HOWA}	p	u_i
January	0.00036453	8.503E-05	0.2	0.2	0.01	3
February	0.00034939	0.00036453	0.1	0.2	0.05	4
March	2.9921E-05	0.00034939	0.05	0.1	0.04	12

April	0.00031963	2.9921E-05	0.05	0.05	0.08	11
May	0.00344596	0.00031963	0.025	0.05	0.10	10
June	0.00096067	0.00344596	0.025	0.05	0.06	9
July	0.0061078	0.00096067	0.05	0.05	0.06	8
August	2.6445E-05	0.0061078	0.05	0.05	0.10	7
September	1.3146E-05	2.6445E-05	0.05	0.1	0.10	6
October	1.7718E-05	1.3146E-05	0.1	0.2	0.10	5
November	0.0024819	1.7718E-05	0.15	0.2	0.15	2
December	0.00014748	0.0024819	0.15	0.25	0.15	1

The prioritized components are developed with three different vector weights chosen by area experts. The weights are the following

$w_1 = 0.099,0.095,0.082,0.082,0.079,0.076,0.075,0.076,0.077,0.078,0.076,0.105$

$w_2 = 0.3,0.2,0,0,0,0,0,0,0.1,0.1,0.1,0.2$

$w_3 = 0.1,0.05,0,0,0.05,0.05,0.05,0.05,0.05,0.1,0.2,0.3$

The importance of each expert for the use of prioritized operators is 40% for expert 1, 40% for expert 2 and 20% for expert 3.

Step 3. In this step, the use of the HOWA, PrOWA, POWA, PIHOWA and PIPOWA operators to obtain υ and μ is done (See Table 2).

Table 2. Results using different OWA extensions

Operator	$\boldsymbol{\upsilon}$	$\boldsymbol{\mu}$
HOWA	0.00228114	0.002278016
PrOWA	0.00170816	0.00170564
POWA	0.00102197	0.00101698
PIHOWA	0.00120145	0.0014803
PIPOWA	0.00111077	0.00109614

Step 4. With the information obtained in Table 2, the variance and covariance calculation are done with the different OWA extensions (See Table 3).

Table 3. Variances and covariances with different OWA extensions

Variable	HOWA	PrOWA	POWA	PIHOWA	PIPOWA
variance	5.96395E-06	2.77502E-06	3.22023E-06	4.43952E-06	3.14874E-06
covariance	2.696E-06	3.1859E-07	-6.17197E-07	-6.90606E-07	-6.1405E-07

Step 5. In this step the estimation of β and α is done with the information provided by the below steps (See Table 4).

Table 4. $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ calculated with OWA operators

Variable	HOWA	PrOWA	POWA	PIHOWA	PIPOWA
β	0.4520491	0.1148064	-0.1916628	-0.1555588	-0.1950154
α	0.0012514	0.0015123	0.0001949	0.0014317	0.0013245

Step 6. The estimation of the volatility using the different OWA extensions is done in this step. The volatility models are the following.

$$h_{HOWA} = 0.001251363 + 0.452049076h_{t-1}$$
$$h_{PrOWA} = 0.001512343 + 0.114806417h_{t-1}$$
$$h_{POWA} = 0.000194916 - 0.19166281h_{t-1}$$
$$h_{PIHOWA} = 0.001431723 - 0.1555588h_{t-1}$$
$$h_{PIPOWA} = 0.001324536 - 0.19501541h_{t-1}$$

Step 7. The final step is to substitute the values of h_{t-1} in the models and compare all the results (See Table 5).

Table 5. Results using different formulations

	LR	HOWA	**PrOWA**	***POWA***	***PIHOWA***	***PIPOWA***
$\boldsymbol{h_t}$	0.00134098	0.00131803	0.00152928	0.00016665	0.00140878	0.00129578

The results show that the simple regression with ordinary least squares has a result similar to the regression with the HOWA operator, then it can use when we use data and linearity is considered. Probability utilization underestimates the result to 0.00016665, this show that we can use LR-POWA in expected scenarios of low volatility. On the contrary, using an estimate with prioritized operators overestimating the result can be applied when there are high volatility expectations.

Linear regression with PIHOWA and PIPOWA operators shows results around the simple OLS, which is different from the use of probability or prioritized the results are closer. This indicates the possibility of using these estimators when there is low or high volatility but not in an extreme sense.

4. Conclusions

This paper proposes to use OWA aggregation operators in the process of estimating ordinary least squares to process complex and uncertain data. The new methodologies are called; the linear regression heavy OWA (LR-HOWA), the linear regression prioritized OWA (LR-PrOWA), the linear regression probabilistic OWA (LR-POWA), the linear regression. Heavy induced prioritized OWA (LR-PIHOWA) and linear regression probabilistic induced prioritized OWA L(R-PIPOWA). The main advantage is estimating or sub-estimating the linear regression parameters, which can adapt to different scenarios ranging from the minimum OWA to its maximum.

The main definitions and properties of the linear regression with OWA operators are presented. Additionally, the OWA families in linear regression are shown. This includes the maximum, minimum, generalized, quasi-arithmetic, quadratic, geometric, harmonic and cubic operators.

The new methodology was applied in exchange rate volatility, where different results are observed depending on the operator used. Using prioritized operators in the estimation process overestimates the results, which is suitable for high volatility scenarios. On the contrary, the result is suggested when the parameters are estimated using probabilistic operators. This information can be used to create volatility ranges according to future scenarios. To continue developing this idea, in future research, the application in other extensions of OWA operators, such as the moving average (OWMA) (Fonseca-Cifuentes et al., 2021) and Bonferroni OWA (BOWA) (Yager, 2009) will be made.

REFERENCES

[1] **Cazorzi M, Kolasa M, Rubaszek M (2017),** ***Exchange Rate Forecasting with DSGE Models***. *Journal of International Economics, 107*:127-146;

[2] **Chen R, Xu J. (2019),** ***Forecasting Volatility and Correlation between Oil and Gold Prices Using a Novel Multivariate GAS Model***. *Energy Economics 78*:379-391;

[3] **Christou C. (2018),** ***The Role of Economic Uncertainty in Forecasting Exchange Rate Returns and Realized Volatility: Evidence from Quantile Predictive Regressions***. *Jounal of Forecasting* 705:719;

[4] **Fonseca-Cifuentes, G., León-Castro, E. & Blanco-Mesa, F. (2021),** ***Predicting the Future Price of a Commodity Using the OWMA Operator: An Approximation of the Interest Rate and Inflation in the Brown Pastusa Potato Price***. *Journal of Intelligent & Fuzzy Systems,* 40 (2), 1971-1981;

[5] **Gujarati D., Porter D. (2009),** ***Basic Econometrics***. *McGraw-Hill Irwin,* New York, NY;

[6] **Hose D., Hanss M. (2019),** ***Fuzzy Linear Least Squares for the Identification of Possibilistic Regression Models***. *Fuzzy Sets and Systems 367*:82-95;

[7] **Huang F. (2017),** ***Multilevel Modeling and Ordinary Least Squares Regression: How Comparable Are They?.*** *The Journal of Experimental Education 86*(2):265-281;

[8] **Jiang A., Liu X. (2016),** ***Volatility Forecasting in the Chinese Commodity Futures Market with Intraday Data.*** *Review of Quantitative Finance and Account- ing 48*(4):1-51;

[9] **Jung H., Yoon J., Choi S. (2015)** ***Fuzzy Linear Regression Using Rank Transform Method***. *Fuzzy Sets and Systems 274*:97-108;

[10] **Kim H., Won C. (2018),** ***Forecasting the Volatility of Stock Price Index: A Hybrid Model Integrating LSTM with Multiple GARCH-Type Models.*** *Expert Systems with Applications 103*:25-37;

[11] **Lahmiri S. (2017),** ***Modeling and Predicting Historical Volatility in Exchange Rate Markets.*** *Physica A: Statistical Mechanics and its Applications 471*(1):387-395;

[12] **León-Castro, E., Avilés-Ochoa, E. & Gil Lafuente, A. M. (2016),** ***Exchange Rate USD/MXN Forecast through Econometric Models, Time Series and HOWMA Operators.*** *Economic Computation and Economic Cybernetics Studies and Research, 2016,* 50(4), *p. 135-150*;

[13] **León-Castro, E., Perez-Arellano, L. A., Olazabal-Lugo, M. & Merigó, J. M. (2021),** ***Prioritized Induced Heavy Operators Applied to Political Modelling.*** *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 29*(04), 603-620;

[14] **Markowitz H. (1952),** ***Portfolio Selection*****.** *Journal of Finance 7*(1):77-91;

[15] **Merigó J. (2009),** ***Probabilistic Decision Making with the OWA Operator and its Application in Investment Management****. Proc. of the IFSA-EUSFLAT 2009 Conference* (págs. 1364-1369). Portugal: Lisbon;

[16] **Perez-Arellano L., León-Catro E., Avilés-Ochoa E., Merigó J. (2019),** ***Prioritized Induced Probabilistic Operator and Its Application in Group Decision Making****. International Journal of Machine Learning and Cybernetics 10*(3):451-462;

[17] **Perez-Arellano, L. A., Blanco-Mesa, F., Leon-Castro, E. & Alfaro-Garcia, V. (2021),** ***Bonferroni Prioritized Aggregation Operators Applied to Government Transparency****. Mathematics*, *9*(1), 24;

[18] **Roni S., Merga M., Morris J. (2019),** ***Analysis: Regression in Conducting Quantitative Research in Education****. Springer,* Singapore, pp 133-175;

[19] **Seber G., Lee A. (2003),** ***Linear Regression Analysis****. John Wiley and Sons* INC, New Jersey;

[20] **Singh P. (2018),** ***Rainfall and Financial Forecasting Using Fuzzy Time Series and Neural Networks Based Model****. International Journal of Machine Learning and Cybernetics 9*(3):491-506;

[21] **Stone M., Brooks R. (1990),** ***Continuum Regression: Cross-Validated Sequentially Constructed Prediction Embracing Ordinary Least Squares, Partial Least Squares and Principal Components Regression****. Journal of the Royal Statistical Society: Series B (Methodological) 52*(2):237-258;

[22] **Vigier H., Scherger S., Terceño A. (2017),** ***An Application of OWA Operators in Fuzzy Business Diagnosis.*** *Applied Soft Computing 54*:440-448;

[23] **Yager R. (1988),** ***On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making*****.** *IEEE Tranactions System Man Cybernetics B 18*:183-190;

[24] **Yager R. (2002),** ***Heavy OWA Operators*****.** *Fuzzy Optimization and Decision Making 1*: 379-397;

[25] **Yager R. (2008),** ***Prioritized Aggregation Operators*****.** *International Journal of Approximate Reasoning 48*:263-274;

[26] **Yager R. (2009),** ***On Generalized Bonferroni Mean Operators for Multicriteria Aggregation*****.** *International Journal of Approximate Reasoning 50*(8):1279-1286;

[27] **Yager R., Filev D. (1999),** ***Induced Ordered Weighted Averaging Operators.*** *IEEE Trans Syst Man Cybern B 29*: 141-150;

[28] **Zhang R., Ling S. (2015),** ***Asymptotic Inference for AR Models with Heavy-Tailed G-GARCH Noises.*** *Econometric Theory 31*(4):880-890;

[29] **Zhang Z., Zhang C. (2004), 7** ***Agent-Based Hybrid Intelligent System for Financial Investment Planning*****.** *Agent-Based Hybrid Intelligent Systems* 2938:93-125.