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RISK AND UNCERTAINTY: ANALYZING THE INCOME AND SUBSTITUTION EFFECTS

Abstract.*This paper presents a model of decomposing the price effect into the income and substitution effect when uncertainty is present. There are two distinct approaches in the classical theory of consumer: the first one – the consumption set – based approach is introduced in the first section. The second approach treats consumer's choices using the utility function; in the second section we present the most important concepts like uncompensated and compensated demands, expenditure function and indirect utility function, which we use in the last part of the paper. In the last section we show how the Slutsky equation can be adapted for the case when risk and uncertainty is present.*
Key words: *substitution and income effects, Slutsky equation, risk aversion*.

JEL Classification: D11

We consider a market economy *E* with a finite number of consumers, *H* and where *n* commodities exist. We refer the *consumer* as an individual, family or household who make a decision to select a consumption program, meaning a specification of all his inputs and outputs.

1. THE POSSIBLE CONSUMPTIONS SET - BASED APPROACH

We will use the notation "*h*" to refer to one particular consumer, where *h=1,2,…,H*.

A consumption program or commodity bundle for the *consumer h* is a list of amounts of the different commodities or a commodity vector and can be viewed as a point in R^n ($x_h \in R^n$), the commodity space. The vector x_h is also called the demand of *consumer h*. This consumption bundle could be possible or impossible for the individual *h* to consume.

Definition 1. *The consumption set for the consumer h* is a subset of the commodity space R^n, denoted by X_h, whose elements are all the possible consumptions x_h that the h-individual can consume given the physical constraints imposed by his environment.

Many commodities are not listed in x_h, so X_h is, generally, a subset of a relatively small dimension space R^n.

Definition 2. *The aggregate consumption vector or aggregate demand* is the vector $x = \sum_{h=1}^{H} x_h$.

Definition 3. *The set of aggregate demands* is the set $X = \sum_{h=1}^{H} X_h$.

In order to go further, we need to introduce some assumptions regarding the consumption set.

Assumptions made on the possible consumption set [1]

We will make the following assumptions on X_h:

Assumption 1. X_h is closed.

Let $\left(x_h^q\right)_{q\in N^*}$ be a sequence of consumption vectors. If x_h^q is a possible consumption for the consumer *h*, $\forall q \in N^* \left(x_h^q \in X_h\right)$ and if $x_h^q \to x_h^0$, then x_h^0 is a possible consumption for him $\left(x_h^0 \in X_h\right)$.

Assumption 2. The set X_h has a minorant with respect to the relation "$\leq$", i.e. there is a point $x_h^* \in R^n$ such that $x_h^* \leq x_h$ for all $x_h \in X_h$. This statement is equivalent with the inclusion $X \subset \left\{x_h^*\right\} + \Omega$. The set Ω is defined as: $\Omega = \left\{\omega \middle| \omega \in R^n, \omega \geq 0\right\}$.

This assumption is realistic from an economic perspective. If the commodity *i* is an input, then x_{hi} has the lower bound 0. If the commodity *i* is an output (meaning that the consumer make an offer on the labour market or is willing to work being paid) it is superior bounded (when consider the absolute value). This is because the consumer cannot supply an infinite quantity of working time during a finite period of time.

Assumption 3. There is a possible consumption vector $\overline{\overline{x}}_h \in X_h$, such that $\overline{\overline{x}}_h \leq x_h$ for all $x_h \in X_h$, $h \in \{1,2,...,H\}$.

Assumption 4. The set X_h is connected (i.e., intuitively, X_h is a single "piece").

A set $S \subset R^n$ is connected if it cannot be partitioned into two nonempty, closed and disjointed subsets in S.

Assumption 5. The set X_h is convex, i.e. $\forall x_h^1, x_h^2 \in X_h$, the following holds:

$$tx_h^1 + (1-t)x_h^2 \in X_h, \forall t \in [0,1].$$

To formalize the next assumption, we suppose that the n commodities are all traded in the market at known prices. Formally, these prices are represented by the price vector $p = (p_1, p_2, ..., p_n)$.

In the economy E, society's initial endowments and technological possibilities are owned by the consumer. We suppose that consumer h initially owns the vector $\bar{x}_h$ and he can sell it at the market prices. In addition, the consumer h owns a share d_{hf} of firm f, where d_{hf} is nonnegative and it gives him a claim to fraction d_{hf} of firm f's profits.

We can state now the assumption:

Assumption 6. For each household h, the income R_h - calculated at any price set p and any production possibilities set Y_f, is given by: $R_h = p\bar{x}_h + \sum_{f=1}^{F} d_{hf}(py_f)$, with $d_{hf} \geq 0$ and $\sum_{h=1}^{H} d_{hf} = 1$.

This assumption shows that each firm is a partnership between households and the firm itself: the household could either obtain some profit, either looses some money in case of bankruptcy. But $0 \in Y_f$ and so, the firm f, maximizing his profits, will always choose the vector y_f such that $py_f \geq p \cdot 0 = 0$.

In the same way of choosing the right decision regarding profit maximizing vectors, there is a choice of the consumption vectors that are preferred with respect to the budget constraint and are maximizing utility vectors.

If the consumer h (or a decision maker) has a preference ordering over the commodity bundles in the consumption set X_h, denoted by $\succeq_h$ and satisfying the following properties: transitivity, completeness and convexity, then the preferred vector for the set $B \subset X_h$ is a vector $\tilde{x}_h$ such that $\tilde{x}_h \in B$ and $\tilde{x}_h \succeq_h x_h, \forall x_h \in B$.

It is possible that more than one vector exist, having these properties, i.e. there is a set of vectors satisfying the budget constraint, all contained in the same indifference set, but all at least as good as any other vector from the consumption set.

Another optimization problem, similar with the above problem of choosing the preferred vector satisfying the budget constraint is: find the vector $x_h \in X_h$, which minimize the cost of obtaining a given or fixed utility level. This vector is known in the literature as *the compensated demand* or *Hicksian demand*.

Definition 4. *The Hicksian Demand Correspondence* is the set:

$$X_h(p, x_h^0) = \left\{ x_h^{'} \middle| px_h^{'} = \min_{x_h \succeq x_h^0} px_h, x_h \in X_h \right\}$$

Proposition 1. The correspondence $X_h(p, x_h^0)$ is upper semi-continuous in p, for every fixed vector $x_h^0 \in X_h$.

Proof

Let $\{p^\gamma\}_{\gamma \in N^*}$ a prices sequence and $p^\gamma \to p$. We suppose that $x_h^\gamma \in X_h(p^\gamma, x_h^0)$ and $x_h^\gamma \to x_h$. Then, the vector x_h^γ is preferred to the vector x_h^0 and also the vector x_h is preferred to same vector x_h^0. If $x_h^{'}$ is preferred to x_h^0, then, from Definition 1.4, we have that $p_\gamma x_h^\gamma \le p^\gamma x_h^{'}$ and taking the limit we obtain $px_h \le px_h^{'}$, i.e. $X_h(p, x_h^0)$ is upper semi-continuous in the price vector *p*.

2. THE UTILITY FUNCTION – BASED APPROACH

In the following section we drop the index *h* used to index the consumers from the economy.

If we define a preference relation (denoted by $\succeq$) over the possible consumption set and this ordering has the known properties (transitivity, completeness and convexity), then we can define a *utility function* assigned to the consumer.

Definition 5. A function $U : X \to R$ is a utility function representing preference relation $\succeq$ if, for all $x, y \in X$, $x \succeq y$ if and only if $U(x) \ge U(y)$.

It is possible that the compensated demand correspondence and the uncompensated demand correspondence are singletons (they contain only one element, i.e. the optimal vector is unique). Therefore, the relationships between compensated and uncompensated demands could be used for analysing the income and substitution effects. [1, 2, 4].

Assumption 7. The utility function $U : X \to R$ is: strictly increasing, strictly quasi-concave and belongs to the C^2 class.

Definition 6. The *Uncompensated (Walrasian or Marshallian) Demand*, denoted by $X(p,R) \in R^n$ is the optimal solution of the following *utility maximization problem*:

$$\max_x U(x)$$
$$px \le R \qquad\qquad (P)$$
$$x \ge 0$$

Definition 7. The function $V(p,R)$ is called the *Indirect Utility Function* and represents the objective's optimal value in the program (P), i.e. $V(p,R) = U(X(p,R))$.

Definition 8. The *Compensated (Hicksian) Demand*, denoted by $\varphi(p,u) \in R^n$, is the optimal solution of the following *expenditure minimization problem*:

$$\min_x px$$
$$U(x) \ge u \qquad\qquad (D)$$
$$x \ge 0$$

Definition 9. The function $C(p,u)$ is called the *Expenditure Function* and represents the objective's optimal value in the program (D), i.e. $C(p,u) = p\varphi(p,u)$.

Theorem 1. [1,4] Suppose that $U : X \to R$ is a utility function:

i) strictly increasing,
ii) strictly quasi-concave,
iii) from the class C^2

and suppose that $C(p,u) \ge 0$.

Then:

1) $C(p,u)$ is strictly increasing in *u*, increasing in p_i, for any $i = 1,2,...,n$ and concave in *p*;

2) $C(p,u)$ is differentiable in p_i, for any $i = 1,2,...,n$ and satisfies

$$\frac{\partial C(p,u)}{\partial p_i} = \varphi_i(p,u), \ \forall i = 1,2,...,n. \text{ (\textit{Shepard's Lema}).}$$

If the function $C(p,u)$ is from the class C^2 in *p*, then, using the Schwartz's Theorem (the symmetry of second derivatives) we have:

$$\frac{\partial^2 C(p,u)}{\partial p_i \partial p_j} = \frac{\partial^2 C(p,u)}{\partial p_j \partial p_i}, \text{ and so } \frac{\partial \varphi_j(p,u)}{\partial p_i} = \frac{\partial \varphi_i(p,u)}{\partial p_j} \qquad (1)$$

The compensated and uncompensated demand functions satisfy the following relations:

$$\begin{aligned} &\varphi(p, V(p,R)) = X(p,R) \\ &X(p, C(p,u)) = \varphi(p,u) \end{aligned} \qquad (2)$$

Proposition 2. [2,4] The *Slutsky Matrix*, denoted by $\nabla_p \varphi(p,u)$, is symmetric and negative semi definite.

Proof

Differentiating the expression from (2) with respect to p_i (as a compound function) we get:

$$\begin{aligned} \frac{\partial \varphi_j(p,u)}{\partial p_i} &= \frac{\partial X_j(p,R)}{\partial p_i} + \frac{\partial X_j(p,R)}{\partial R} \cdot \frac{\partial C(p,u)}{\partial p_i} = \\ &= \frac{\partial X_j(p,R)}{\partial p_i} + \frac{\partial X_j(p,R)}{\partial R} \cdot \varphi_i(p,u) \end{aligned}$$

or

$$\frac{\partial \varphi_j(p,u)}{\partial p_i} = \frac{\partial X_j(p,R)}{\partial p_i} + X_i(p,R) \cdot \frac{\partial X_j(p,R)}{\partial R}$$

Using the symmetry condition (1), yields:

$$\frac{\partial X_i(p,R)}{\partial p_j} + X_j(p,R) \cdot \frac{\partial X_i(p,R)}{\partial R} = \frac{\partial X_j(p,R)}{\partial p_i} + X_i(p,R) \cdot \frac{\partial X_j(p,R)}{\partial R} \qquad (3)$$

Finally, since the expenditure function $C(p,u)$ is concave in *p* (from Theorem 1), the matrix $\nabla_p \varphi(p,u)$ is negative semi definite.

Note that $\nabla_p \varphi(p,u) = \left(\frac{\partial^2 \varphi(p,u)}{\partial p_i \partial p_j} \right)_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,n}}$.

Another immediate result is that:

$$\frac{\partial X_i(p,R)}{\partial p_i} + X_i(p,R) \cdot \frac{\partial X_i(p,R)}{\partial R} \leq 0, \ \forall i = 1,2,\ldots,n \qquad (4)$$

Proposition 3. [2,4] The Walrasian Demand satisfies:

$$X_i(p,R) = -\frac{\dfrac{\partial V(p,R)}{\partial p_i}}{\dfrac{\partial V(p,R)}{\partial R}},\ \forall i = 1,2,...,n\ .\ (\textit{Roy's Identity})$$

Proof

We use the relations:

$$\varphi_i(p,V(p,R)) = X_i(p,R)$$
$$R = C(p,V(p,R))$$

and differentiating with respect to R and p_i we get:

$$\frac{\partial C(p,V(p,R))}{\partial u}\cdot\frac{\partial V(p,R)}{\partial R} = 1 \text{ and}$$
$$\frac{\partial C(p,V(p,R))}{\partial p_i} + \frac{\partial C(p,V(p,R))}{\partial u}\cdot\frac{\partial V(p,R)}{\partial p_i} = 0$$

These yield:

$$-\frac{\dfrac{\partial V(p,R)}{\partial p_i}}{\dfrac{\partial V(p,R)}{\partial R}} = \frac{\partial C(p,V(p,R))}{\partial p_i} = \varphi_i(p,V(p,R)) = X_i(p,R)$$

Example

Let $X \subset R^2$ be the possible consumption set and consider the consumer's utility function $U : X \to R$, $U(x_1,x_2) = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$.

Then, the utility maximization problem is:

$$\max_{x_1,x_2}\left\{\frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2\right\}$$
$$p_1x_1 + p_2x_2 \le R$$
$$x_1 \ge 0, x_2 \ge 0$$

The Uncompensated Demand, $X(p,R)=\begin{pmatrix} R/2p_1 \\ R/2p_2 \end{pmatrix}$ is the optimal solution of this program and the indirect utility function is $V(p,R)=\frac{1}{2}\ln\frac{R^2}{4p_1p_2}$.

Solving the expenditure minimization problem:

$$\min_{x_1,x_2}\{p_1x_1+p_2x_2\}$$
$$\frac{1}{2}\ln x_1+\frac{1}{2}\ln x_2\geq u$$
$$x_1\geq 0, x_2\geq 0$$

we obtain the Hicksian Demand $\varphi(p,u)=\begin{pmatrix} e^u\sqrt{p_2/p_1} \\ e^u\sqrt{p_1/p_2} \end{pmatrix}$ and the expenditure function $C(p,u)=2e^u\sqrt{p_1p_2}$.

It is easy to verify that the expenditure function satisfies all the properties listed above in Theorem 1: increasing in *u*, p_1 and p_2, strictly concave in *p* and belonging to the class C^2. More than this, we have:

$$\frac{\partial C(p,u)}{\partial p_i}=\frac{2\sqrt{p_j}}{2\sqrt{p_i}}e^u=e^u\sqrt{\frac{p_j}{p_i}}=\varphi_i(p,u), \forall i=1,2; j=1,2, i\neq j$$

We can now verify the relationships between the two types of demands (compensated and uncompensated demand) – the relations (2):

$$\varphi(p,V(p,R))=\begin{pmatrix} e^{V(p,R)}\sqrt{p_2/p_1} \\ e^{V(p,R)}\sqrt{p_1/p_2} \end{pmatrix}=\begin{pmatrix} e^{\ln\frac{R}{2\sqrt{p_1p_2}}}\sqrt{p_2/p_1} \\ e^{\ln\frac{R}{2\sqrt{p_1p_2}}}\sqrt{p_1/p_2} \end{pmatrix}=\begin{pmatrix} R/2p_1 \\ R/2p_2 \end{pmatrix}=X(p,R)$$

and

$$X(p,C(p,u))=\begin{pmatrix} C(p,u)/2p_1 \\ C(p,u)/2p_2 \end{pmatrix}=\begin{pmatrix} 2e^u\sqrt{p_1p_2}/2p_1 \\ 2e^u\sqrt{p_1p_2}/2p_2 \end{pmatrix}=\begin{pmatrix} e^u\sqrt{p_2/p_1} \\ e^u\sqrt{p_1/p_2} \end{pmatrix}=\varphi(p,u)$$

We have also:

$$C(p,V(p,R))=2e^{V(p,R)}\sqrt{p_1p_2}=2e^{\ln\frac{R}{2\sqrt{p_1p_2}}}\sqrt{p_1p_2}=\frac{2\sqrt{p_1p_2}}{2\sqrt{p_1p_2}}\cdot R=R.$$

By differentiating the indirect utility function with respect to *R* and p_i we obtain:

$$\frac{\partial V(p,R)}{\partial R} = \frac{1}{R} \text{ and } \frac{\partial V(p,R)}{\partial p_i} = -\frac{1}{2p_i}$$

Hence, $-\dfrac{\dfrac{\partial V(p,R)}{\partial p_i}}{\dfrac{\partial V(p,R)}{\partial R}} = \dfrac{R}{2p_i} = X_i(p,R)$, which is exactly The Roy's Identity.

The utility function homogeneous of degree one

Consider a utility function having the following form $U: X \to R$, $U(x_1,x_2) = x_1 + x_2$, a function homogeneous of degree one.

Given this function, how can we derive the expenditure function? We must solve the following *linear program*:

$$\min_{x_1,x_2}\{p_1x_1 + p_2x_2\}$$

$$x_1 + x_2 \geq u$$

$$x_1 \geq 0, x_2 \geq 0$$

Because the constraint is binding, the problem can be rewritten as:

$$\min_{x_1,x_2}\left[p_1x_1 + p_2(u - x_1)\right] = c(p)u$$

Therefore, the objective function is linear in *u*.

Proposition 4. Suppose that a utility function, $U: X \to R$, is homogeneous of degree one. Then:

a) The expenditure function $C(p,u)$ is linear in *u*;

b) The Engel's curves are linear and go through the origin.

Proof

a) By definition, $C(p,u) = \min\{px | U(x) = u, x \geq 0, x \in X\}$.

The utility function being homogeneous of degree one, the utility level λu is obtained for the vector λx, therefore the associated cost is equal to the initial cost multiplied by λ. Hence, $C(p,u) = c(p)u$ holds.

b) Using the relation $R = C(p, V(p,R)) = c(p)V(p,R)$, we get $V(p,R) = \dfrac{R}{c(p)}$.

From *Theorem 1*, we have: $\varphi_i(p,u) = \dfrac{\partial C(p,u)}{\partial p_i} = \dfrac{\partial c(p)}{\partial p_i} \cdot u, \forall i = 1,2,...,n$

or

$$X_i(p,R) = \varphi_i(p,V(p,R)) = \frac{\partial c(p)}{\partial p_i} V(p,R) = \frac{\partial c(p)}{\partial p_i} \cdot \frac{R}{c(p)} = \alpha_i(p)R, \forall i = 1,2,...,n$$

This result shows that the Engel's curves are linear and go through the origin. This property is valid for any homothetic utility function (an increasing transformation of a utility function homogeneous of degree one).

3. RISK AND UNCERTAINTY IN CHOOSING THE OPTIMAL PORTFOLIO

In the previous sections, we studied choices that result in perfectly certain outcomes. In reality, many important economic decisions involve an element of risk. In this section, we focus on the special case in which the outcome of a risky choice is an amount of money.[5]

We will consider an economic agent representing the private investors that has an initial income w_0 (the initial endowment). He has two choices: the first one-to invest it in one active without risk, where *r* denotes the interest rate; the second-to invest in a risky active. We will consider that the interest rate for the risky active is a random variable $\tilde{e}$ with mean and variance finite.

We will denote with *a* (%) the amount invested in risky active.

The agent's income at the end of the first period will be:

i) The agent invests aw_0 in risky active and at the end of the first period he will have $aw_0(1+e)$ (for a certain value *e* for $\tilde{e}$)

ii) The agent invests $(1-a)w_0$ in active without risk and he will have at the end of the first period $(1-a)w_0(1+r)$.

So, at the end of the first period he will obtain:

$$w(e) = w_0 a(1+e) + w_0(1-a)(1+r) = w_0[a + ae + 1 + r - a - ar] = w_0[1 + ae + (1-a)r]$$

We will analyze the income and substitution effects using the Slutsky Ecuation.

Suppose that $\bar{e}$ is the mean value of $\tilde{e}$ and we denote by σ^2 the variance, negligible with respect to higher moments. We consider:

$$E[\tilde{x}_1] = E\{aw_0(1+\tilde{e})\} = \bar{x}_1 \text{ and } \sigma^2_{\tilde{x}_1} = (w_0^R)\sigma^2$$

The economic agent has a utility function $B(\tilde{x}_1, x_2) = EU(\tilde{x}_1 + x_2)$ approximated by a function dependent on $\bar{x}_1$ and x_2. For this, we use the concept of *certainty equivalent*, denoted by $E_C(\bar{x}_1 + x_2)$. [3,4]

The certainty equivalent satisfies the equation:

$$B(\tilde{x}_1, x_2) = EU(\tilde{x}_1 + x_2) = U(\bar{x}_1 + x_2 - E_C(\bar{x}_1 + x_2)) \quad (5)$$

We will also use the notations $z = \tilde{e} - \bar{e}$, $\sigma_z = \sigma$. Hence $E(z) = 0$ holds.

Then:

$$B(\tilde{x}_1, x_2) = E[U(\bar{x}_1(1+z) + x_2)] \quad (6)$$

We approximate the function from (6) by a series expansion:

$$U(\bar{x}_1 + x_2 + \bar{x}_1 z) \approx U(\bar{x}_1 + x_2) + U'(\bar{x}_1 + x_2) \cdot \frac{\bar{x}_1 z}{1!} + \frac{U''(\bar{x}_1 + x_2)}{2!} \cdot \bar{x}_1^2 z^2 \quad (7)$$

The right side of (7), $U(\bar{x}_1 + x_2 - E_C(\bar{x}_1 + x_2))$, could be approximated (from Lagrange's Theorem) by:

$$U(\bar{x}_1 + x_2 - E_C(\bar{x}_1 + x_2)) = U(\bar{x}_1 + x_2) - U'(\bar{x}_1 + x_2) E_C(\bar{x}_1 + x_2) \quad (8)$$

From (7), taking the "expected value" of both sides (applying the operator "E") we get:

$$EU(\tilde{x}_1 + x_2) = U(\bar{x}_1 + x_2) + 0 + \frac{U''(\bar{x}_1 + x_2)}{2!} \bar{x}_1^2 \sigma^2 \quad (9)$$

The relations (8) and (9) combined yield:

$$E_C(\bar{x}_1 + x_2) = -\frac{U''(\bar{x}_1 + x_2)}{U'(\bar{x}_1 + x_2)} \frac{\bar{x}_1^2 \sigma^2}{2} = \frac{\bar{x}_1^2 \sigma^2}{2} r_a(\bar{x}_1 + x_2),$$

where $r_a(\cdot)$ represent *the absolute index of risk aversion*.[3,4]

We approximate $B(\tilde{x}_1, x_2)$ by:

$$B(\bar{x}_1, x_2) = U\left(\bar{x}_1 + x_2 - \frac{\bar{x}_1^2 \sigma^2}{2} r_a(\bar{x}_1 + x_2)\right) \quad (10)$$

From the definition of $\bar{x}_1$ and x_2, we can write the following relation:

$$w_0 = \frac{\bar{x}_1}{1+e} + \frac{x_2}{1+r} \quad (11)$$

or

$$\bar{x}_1 = w_0(1+e) - \frac{1+e}{1+r} x_2.$$

The equation (11) is called the *income equation* or *welfare equation*.

If we define $p_e = \dfrac{1}{1+e}$ and $p_r = \dfrac{1}{1+r}$, the equation could be written as: $w_0 = p_e \bar{x}_1 + p_r x_2$.

Therefore, the *problem of choosing the optimal portfolio* is:

$$\begin{aligned} &\max_{\bar{x}_1, x_2} \widetilde{B}(\widetilde{x}_1, x_2) \\ &s.r.\ \ p_e \bar{x}_1 + p_r x_2 = w_0 \end{aligned} \tag{12}$$

or

$$\begin{aligned} &\max_{\bar{x}_1, x_2} U\left(\bar{x}_1 + x_2 - \frac{\sigma^2 \bar{x}_1^2}{2} r_a (\bar{x}_1 + x_2) \right) \\ &s.r.\ \ p_e \bar{x}_1 + p_r x_2 = w_0 \end{aligned}$$

From Gossen Law, the optimal point must satisfy the first order condition:

$$\frac{\dfrac{\partial U}{\partial \bar{x}_1}}{\dfrac{\partial U}{\partial x_2}} = \frac{1+r}{1+e} = \frac{p_e}{p_r} \tag{13}$$

The welfare equation and the condition (13) form a system of equations. Solving this system we obtain the optimal solutions $\bar{x}_1^*$ and x_2^*.

For example, suppose that the utility function has a particular form, $U(x) = \ln x$. Then, the system to be solved is:

$$\begin{cases} \dfrac{1 - \dfrac{\sigma^2 \bar{x}_1}{2} r_a (\bar{x}_1 + x_2) - \dfrac{\sigma^2 \bar{x}_1^2}{2} \cdot \dfrac{-1}{(\bar{x}_1 + x_2)^2}}{1 - \dfrac{\sigma^2 \bar{x}_1^2}{2} \cdot \dfrac{-1}{(\bar{x}_1 + x_2)^2}} = \dfrac{1+r}{1+e} \\ p_e \bar{x}_1 + p_r x_2 = w_0 \end{cases}$$

or

$$\begin{cases} \dfrac{1 - \dfrac{\sigma^2 \bar{x}_1}{2(\bar{x}_1 + x_2)}}{1 + \dfrac{\sigma^2 \bar{x}_1}{2(\bar{x}_1 + x_2)^2}} = \dfrac{r-e}{1+e} \\ p_e \bar{x}_1 + p_r x_2 = w_0 \end{cases} \tag{14}$$

In the same way as we did in the previous section (*Definition 6*, *Definition 7*), we denote by $X(p_e, p_r, w_0)$ the uncompensated demand and by $V(p_e, p_r, w_0)$ the indirect utility function, both having the properties from *Proposition 2* and *Proposition 3*.

Consider a fixed utility level for the economic agent, denoted by *b*.

The goal of our analysis is to determine the minimal initial endowment that consumer needs for attaining the utility level *b* (i.e. the utility level associated to one particular social category or for people living in a community).

We can state now the following nonlinear optimization problem:

$$\min_{\bar{x}_1, x_2}\left[p_e \bar{x}_1 + p_r x_2\right] \quad s.t.\ \widetilde{B}(\bar{x}_1, x_2) = b \tag{15}$$

The solution of this problem is $(\overline{\varphi}_1, \varphi_2)$ or the vector $\Phi(p_e, p_r, b) = \begin{pmatrix} \overline{\varphi}_1 \\ \varphi_2 \end{pmatrix}$.

Then, the minimal expenditure function:

$$C(p_e, p_r, b) = p_e \overline{\varphi}_1 + p_r \varphi_2$$

and it has the properties listed in the previous section, i.e. it is concave and increasing in prices, increasing in b and satisfies Shepard's Lemma.

Differentiating the relationships between the functions $X(\cdot)$ and $\Phi(\cdot)$, we get:

$$\frac{\partial \Phi_{\bar{x}_1}}{\partial p_e} = \frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial p_e} + \frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial w} \cdot \frac{\partial w}{\partial p_e} \tag{16}$$

or

$$\frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial p_e} = \frac{\partial \Phi_{\bar{x}_1}}{\partial p_e} - \frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial w} \cdot \bar{x}_1^{\,*} \tag{17}$$

The equation (17) shows how the price effect (the left side of equation) can be decomposed into a substitution effect and an income effect (the first and respectively, the second term of the right side).

Analogous, it follows:

$$\frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial p_r} = \frac{\partial \Phi_{\bar{x}_1}}{\partial p_r} - x_2^{\,*} \frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial w} \tag{18}$$

$$\frac{\partial X_{x_2}(p_e, p_r, w)}{\partial p_r} = \frac{\partial \Phi_{x_2}}{\partial p_r} - x_2^{\,*} \frac{\partial X_{x_2}(p_e, p_r, w)}{\partial w} \tag{19}$$

$$\frac{\partial X_{x_2}(p_e, p_r, w)}{\partial p_e} = \frac{\partial \Phi_{x_2}}{\partial p_e} - \frac{\partial X_{\bar{x}_1}(p_e, p_r, w)}{\partial w} \cdot \bar{x}_1^{\;*} \tag{20}$$

The equations (17), (18), (19) and (20) are combined into the matrix form of Slutsky Equation:

$$\nabla_p X(p,w) = \nabla_p \Phi(p,b) - \nabla_w X(p,w) \cdot \left(\bar{x}_1^{\;*}, x_2^{\;*}\right) \tag{21}$$

The above model could be adapted for analyzing the evolution of physical or chemical process (or any type of process), having a risk when producing. Hence, the goal of the model is minimizing the costs for obtaining a certain result, fixed a priori.

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