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DECISION MAKING WITH INDUCED AGGREGATION OPERATORS AND THE ADEQUACY COEFFICIENT

Abstract. *We present a method for decision making by using induced aggregation operators. This method is very useful for business decision making problems such as product management, investment selection and strategic management. We introduce a new aggregation operator that uses the induced ordered weighted averaging (IOWA) operator and the weighted average in the adequacy coefficient. We call it the induced ordered weighted averaging weighted averaging adequacy coefficient (IOWAWAAC) operator. The main advantage is that it is able to deal with complex attitudinal characters in the aggregation process. Thus, we are able to give a better representation of the problem considering the complex environment that affects the decisions. Moreover, it is able to provide a unified framework between the OWA and the weighted average. We generalize it by using generalized aggregation operators, obtaining the induced generalized OWAWAAC (IGOWAWAAC) operator. We study some of the main properties of this approach. We end the paper with a numerical example of the new approach in a group decision making problem in strategic management.*

Keywords: *Induced aggregation operators,* OWAWA operator, adequacy *coefficient, decision making, strategic management.*

JEL Classification D81, M12, M51

1. INTRODUCTION

The adequacy coefficient is an aggregation technique very useful in a wide range of applications including similarity problems. It is very similar to the Hamming distance (Hamming, 1950) with the difference that it establishes a threshold in the comparison process when one set is higher than the other so the results are equal from this point. Since its appearance, it has been used in a wide range of problems (Gil-Aluja, 1998; Gil-Lafuente, 2002; Karayiannis, 2000; Kaufmann, 1975; Merigó and Gil-Lafuente, 2010; Xu and Chen, 2008; Yager, 2010; Zeng and Su, 2012).

Often, when dealing with the adequacy coefficient, we have to aggregate the information in order to obtain a final result. In the literature, we find a wide range of aggregation operators (Beliakov et al., 2007; Xu and Cai, 2012; Torra and Narukawa, 2007; Xu and Da, 2003) (or aggregation functions). A very well-known aggregation operator often used for decision making is the OWA operator (Yager, 1988). The OWA operator provides a parameterized family of aggregation operators from the minimum to the maximum. Since its introduction, it has been used in a lot of problems (Wei, 2011; Yager, 1993; Yager and Kacprzyk, 1997; Yager et al., 2011; Zhao et al., 2010).

An interesting extension of the OWA operator is the induced OWA (IOWA) operator (Yager, 2003; Yager and Filev, 1999). It is very similar to the OWA with the difference that the reordering step is not developed according to the values of the arguments. In this case, the reordering is carried out with order inducing variables and it includes the OWA operator as a particular case. The IOWA operator has been studied in a lot of situations (Chen and Zhou, 2011; Merigó and Casanovas, 2009; 2011a; 2011b; Merigó and Gil-Lafuente, 2009; Merigó et al., 2012; Wei et al., 2010; Xu and Wang, 2012a; Xu and Xia, 2012).

Further interesting extensions are those ones that use the weighted average and the OWA operator in the same formulation (Merigó, 2010; Torra, 1997, Xu and Da, 2003). It is worth noting the work developed by Merigó (2011) where he introduces the OWA weighted average (OWAWA) operator that unifies these two concepts considering the degree of importance that each concept has in the aggregation. This approach has also been extended for the case when using induced aggregation operators, obtaining the induced OWAWA (IOWAWA) operator.

The OWA operator and its extensions are very useful for decision making. They have also been studied by using the adequacy coefficient, obtaining the OWA adequacy coefficient (OWAAC) and the induced OWAAC (IOWAAC) operators (Merigó and Gil-Lafuente, 2008; 2010; 2011; 2012a; 2012b; Merigó et al., 2011).

In this paper, we present a new type of adequacy coefficient that we believe that provides a more complete formulation because it considers the weighted average and the IOWA operator at the same time. We call it the induced OWAWA adequacy coefficient (IOWAWAAC). The main advantage of this approach is that it includes the weighted adequacy coefficient and the IOWAAC operator in the same formulation. Moreover, it also uses complex reordering processes that represent more complex environments that the usual ones assessed with the OWA operator. We study some of its main properties and we generalize it by using generalized aggregation operators. Thus, we obtain the induced generalized OWAWA adequacy coefficient (IGOWAWAAC). We see that this operator includes a wide range of particular cases such as the IOWAWAAC, the quadratic IOWAWAAC, the induced generalized OWAAC (IGOWAAC) and the usual weighted adequacy coefficient.

We also study the applicability of the IOWAAC operator and we see that it can be used in a lot of problems in decision making, economics and statistics. We focus on a business decision making problem concerning human resource selection where a company is looking for a new worker in its financial department. The main advantage of the IOWAWAAC and the IGOWAWAAC operators is that they provide a more complete representation of the decision problem because they include a wide range of particular cases.

This paper is organized as follows. In Section 2, we briefly review the induced aggregation operators, the OWAWA operator and the adequacy coefficient. Section 3 presents the IOWAWAAC and the IGOWAWAAC operators and analyzes some of its main families. In Section 4 we describe the group decision making process to use when dealing with the IOWAWAAC in strategic management and Section 5 presents a numerical example of the new approach. In Section 6, we present the main conclusions of the paper.

2. PRELIMINARIES

INDUCED AGGREGATION OPERATORS

The IOWA operator was introduced by Yager and Filev (1999) and it represents an extension of the OWA operator. The main difference is that the reordering step of the IOWA is carried out with order-inducing variables, rather than depending on the values of the arguments a_i. The IOWA operator also includes the maximum, the minimum and the average operators, as special cases. It can be defined as follows.

Definition 1. An IOWA operator of dimension n is a mapping $IOWA$: $R^n \times R^n \rightarrow R$ defined by an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, and a set of order-inducing variables u_i, by a formula of the following form:

$$IOWA\ (\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where $(b_{1, \ldots,} b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of the u_i, u_i is the order-inducing variable and a_i is the argument variable.

The IOWA operator can be generalized by using generalized means, obtaining the induced generalized OWA (IGOWA) operator (Merigó and Gil-Lafuente, 2009). It is defined as follows.

Definition 2. An IGOWA operator of dimension n is a mapping IGOWA: $R^n \times R^n \to R$, which has an associated weighting vector W with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$IGOWA\ (\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (2)$$

where b_j is the a_i value of the IGOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, a_i is the argument variable and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

THE OWAWA OPERATOR

The ordered weighted averaging – weighted averaging (OWAWA) operator is an aggregation operator that unifies the WA and the OWA operator in the same formulation considering the degree of importance that each concept has in the analysis (Merigó, 2011; Merigó and Wei, 2011). It can be defined as follows.

Definition 3. An OWAWA operator of dimension n is a mapping $OWAWA$: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (3)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

THE ADEQUACY COEFFICIENT

The normalized adequacy coefficient (Kaufmann and Gil-Aluja, 1986) is an index used for calculating the differences between two sets or two fuzzy sets. For two sets $A = \{\mu_1, \ldots, \mu_n\}$ and $B = \{\mu_1^{(k)}, \ldots, \mu_n^{(k)}\}$, it is defined as follows.

Definition 4. A weighted adequacy coefficient (WAC) of dimension n is a mapping WAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$WAC\ (A\ ,B) = \sum_{i=1}^{n} w_i [1 \wedge (1 - \mu_i + \mu_i^{(k)}] \tag{4}$$

where μ_i and $\mu_i^{(k)}$ are the ith arguments of the sets A and B respectively.

Merigó and Gil-Lafuente (2008; 2010) proposed a new version of the adequacy coefficient that uses the OWA operator in the aggregation. They called it the OWAAC operator. It can be defined as follows for two sets P and P_k.

Definition 5. An OWAAC operator of dimension n is a mapping OWAAC: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$OWAAC\ (\langle \mu_1, \mu_1^{(k)} \rangle, \ldots, \langle \mu_n, \mu_n^{(k)} \rangle) = \sum_{j=1}^{n} w_j K_j \tag{5}$$

where K_j represents the jth largest of $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal P, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative under consideration and $k = 1, 2, \ldots, m$.

3. INDUCED AND OWAWA AGGREGATION OPERATORS IN THE ADEQUACY COEFFICIENT

In this section, we present the IOWAWAAC operator. It is a new aggregation operator that uses induced aggregation operators and the adequacy coefficient in the OWAWA operator. The main advantage is that it is able to present an adequacy coefficient that uses IOWAs and WAs in the same formulation. It can be defined as follows.

Definition 6. An IOWAWAAC operator of dimension n is a mapping $IOWAWAAC$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$f(\langle u_1, \mu_1, \mu_1^{(k)} \rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)} \rangle) = \sum_{j=1}^{n} \hat{v}_j K_j \tag{6}$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the IOWAWAAC triplet $\langle u_i, \mu_i, \mu_i^{(k)}\rangle$ having the jth largest u_i, u_i is the order inducing variable, each argument $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta)v_j$ with $\beta \in [0, 1]$, v_j is the weight (WA) v_i ordered according to K_j, that is, according to the jth largest of the u_i, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative, $k = 1, 2, \ldots, m$.

The IOWAWAAC operator can be generalized by using generalized means in a similar way as it was done in Merigó and Gil-Lafuente (2008) and Yager (2004). Thus, we obtain the induced generalized OWAWA adequacy coefficient (IGOWAWAAC). It can be defined as follows.

Definition 7. An IGOWAWAAC operator of dimension n is a mapping $IGOWAWAAC$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$IGOWAWAAC\,(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} \hat{v}_j K_j^{\lambda} \right)^{1/\lambda} \tag{7}$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the IGOWAWAAC triplet $\langle u_i, \mu_i, \mu_i^{(k)}\rangle$ having the jth largest u_i, u_i is the order inducing variable, each argument $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta)v_j$ with $\beta \in [0, 1]$, v_j is the weight (WA) v_i ordered according to K_j, that is, according to the jth largest of the u_i, $\mu_i \in [0, 1]$, for the ith characteristic of the ideal, $\mu_i^{(k)} \in [0, 1]$, for the ith characteristic of the kth alternative, $k = 1, 2, \ldots, m$, and λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$.

The IGOWAWAAC operator can also be formulated separating the part that affects the OWA and the WA. In this case, it is worth noting that the parameter λ may be different for the OWA and the WA, Therefore, we use λ for the OWA and δ for the WA. We get the following definition.

Definition 8. An IGOWAWAAC operator of dimension n is a mapping f: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and a weighting vector V of dimension n with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$ =

$$= \beta\left(\sum_{j=1}^{n} w_j K_j^{\lambda}\right)^{1/\lambda} + (1-\beta)\left(\sum_{i=1}^{n} v_i [1 \wedge (1-\mu_i + \mu_i^{(k)})]^{\delta}\right)^{1/\delta} \tag{8}$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the IGOWAWAAC triplet $\langle u_i, \mu_i, \mu_i^{(k)}\rangle$ having the jth largest u_i, u_i is the order inducing variable, $\beta \in [0, 1]$, and λ and δ are parameters such that $\lambda, \delta \in (-\infty, \infty) - \{0\}$.

Note that it is possible to distinguish the descending IGOWAWAAC operator and the ascending IGOWAWAAC operator by using $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DIGOWAWAAC operator and w^*_{n+1-j} the jth weight of the AIGOWAWAAC operator.

The IGOWAWAAC operator includes a wide range of particular cases by using different types of weighting vectors and values in the parameter λ and δ. In Table 1, we present some of the main particular cases.

Table 1. Families of IGOWAWAAC operators

Particular type	**IGOWAWAAC**
$\beta = 1$	IGOWA adequacy coefficient (IGOWAAC)
$\beta = 0$	Generalized weighted adequacy coefficient (GWAC)
$w_i = 1/n, \forall i$	Generalized arithmetic WA adequacy coefficient
$v_i = 1/n, \forall i$	Generalized arithmetic IOWA adequacy coefficient
$w_i = 1/n, v_i = 1/n, \forall i$	Generalized adequacy coefficient (GAC)
Ordering: $u_i = j$	GOWAWAAC
Ordering: $u_i = i$	GWAC
$\lambda = \delta = 1$	Induced OWAWA adequacy coefficient (IOWAWAAC)
$\lambda = \delta = 2$	Quadratic (IOWAWAAC)
$\lambda = \delta \to 0$	Geometric (IOWAWAAC)
$\lambda = \delta = -1$	Harmonic (IOWAWAAC)
$\lambda = \delta = 3$	Cubic (IOWAWAAC)
$\lambda = \delta \to \infty$	Maximum adequacy coefficient
$\lambda = \delta \to -\infty$	Minimum adequacy coefficient
$\lambda = 1, \delta = 2$	IOWA weighted quadratic averaging adequacy coefficient
$\lambda = 2, \delta = 1$	Quadratic IOWA weighted averaging adequacy coefficient
$\lambda = 2, \delta = 3$	Quadratic IOWA cubic WA adequacy coefficient
$\lambda = 1, \delta \to 0$	IOWA weighted geometric averaging adequacy coefficient
Etc.	

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, the IGOWAWAAC operator can be expressed as:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\frac{1}{\hat{V}}\sum_{j=1}^{n}\hat{v}_j K_j^{\lambda}\right)^{1/\lambda} \quad (9)$$

If B is the vector consisting of the ordered arguments K_j^{λ}, and W^T is the transpose of the weighting vector, then the IGOWAWAAC operator can be expressed as:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = W^T B \quad (10)$$

The IGOWAWAAC operator is monotonic, bounded and idempotent. It is monotonic because if $[1 \wedge (1 - \mu_i + \mu_i^{(k)})] \geq [1 \wedge (1 - r_i + r_i^{(k)})]$, for all $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$, then, $f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \geq g(\langle u_1, r_1, r_1^{(k)}\rangle, \ldots, \langle u_n, r_n, r_n^{(k)}\rangle)$. It is bounded because $\text{Min}\{[1 \wedge (1 - \mu_i + \mu_i^{(k)})]\} \leq f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \leq \text{Max}\{[1 \wedge (1 - \mu_i + \mu_i^{(k)})]\}$. It is idempotent because if $\{[1 \wedge (1 - \mu_i + \mu_i^{(k)})] = \{[1 \wedge (1 - \mu + \mu^{(k)})]$, for all $\{[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$, then, $f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$.

Analogously to the IGOWAWAAC operator, we can suggest a removal index that is the dual of the IGOWAWAAC operator, because $Q(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = 1 - K(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$. We will call it the IGOWAWADAC operator.

Note that if the real set $(\mu_1^{(k)}, \ldots, \mu_n^{(k)})$ is empty, then, the IGOWAWADAC operator becomes the IGOWAWA operator. Thus, we can see that the IGOWAWAAC operator includes the IGOWAWA operator as a particular case. Therefore, all the families of IGOWAWA operators are also included in this approach. This idea can be proved with the following theorem.

Theorem 1. *Assume* $f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$ *is the IGOWAWADAC operator*. *If* $\mu_i^{(k)} = 0$ *for all i, then*:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = f(\langle u_1, \mu_1\rangle, \ldots, \langle u_n, \mu_n\rangle) \quad (11)$$

Proof. Let

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n}\hat{v}_j K_j^{\lambda}\right)^{1/\lambda} \quad (12)$$

Since $\mu_i^{(k)} = 0$ for all i, $K_j = [0 \vee (\mu_i - \mu_i^{(k)})] = \mu_i$ for all i, then:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = f(\langle u_1, \mu_1\rangle, \ldots, \langle u_n, \mu_n\rangle) \quad \blacksquare$$

Another interesting issue to consider is that the IGOWAWAAC operator becomes the induced generalized OWAWA distance (IGOWAWAD) operator

(Gil-Lafuente and Merigó, 2010) under certain conditions. As it is explained by Merigó and Gil-Lafuente (2007; 2008), the adequacy coefficient and the Minkowski distance (and also further generalizations) become the same measure when the adequacy coefficient fulfils the following theorem.

Theorem 2. *Assume* $f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$ *is the IGOWAWAD operator, and* $g(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle)$ *is the IGOWAWADAC operator. If* $\mu_i \geq \mu_i^{(k)}$ *for all i, then*:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = g(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \tag{13}$$

Proof. Let

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j \mid \mu_i - \mu_i^{(k)} \mid^{\lambda}\right)^{1/\lambda} \tag{14}$$

$$g(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = \left(\sum_{j=1}^{n} w_j [0 \vee (\mu_i - \mu_i^{(k)})]^{\lambda}\right)^{1/\lambda} \tag{15}$$

Since $\mu_i \geq \mu_i^{(k)}$ for all i, $[0 \vee (\mu_i - \mu_i^{(k)})] = (\mu_i - \mu_i^{(k)})$ for all i, then:

$$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) = g(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) \quad \blacksquare$$

Another interesting issue is to analyze is the different measures used to characterize the weighting vector of the IGOWAWAAC operator. For example, we could consider the degree of orness, the entropy of dispersion, the balance operator and the divergence of W (Merigó, 2011; Yager, 1988).

A further interesting issue is the problem of ties in the reordering step. To solve this problem, we recommend following the method developed by Yager and Filev (1999) where they replace each argument of the tied IOWA pair by its average. For the IGOWAWAAC operator, we will use the generalized normalized adequacy coefficient.

Furthermore, the IGOWAWAAC operator can be generalized by using quasi-arithmetic means forming the quasi-arithmetic IOWAWAAC (Quasi-IOWAWAAC) operator. It can be defined as follows:

Definition 9. A Quasi-IOWAWAAC operator of dimension n is a mapping f: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and a weighting vector V of dimension n with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$f(\langle u_1, \mu_1, \mu_1^{(k)}\rangle, \ldots, \langle u_n, \mu_n, \mu_n^{(k)}\rangle) =$

$$= \beta g^{-1}\left(\sum_{j=1}^{n} w_j g\left(K_j\right)\right) + (1-\beta)h^{-1}\left(\sum_{i=1}^{n} p_i h\left([1 \wedge (1-\mu_i + \mu_i^{(k)})]\right)\right) \quad (16)$$

where K_j is the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$ value of the Quasi-IOWAWAAC triplet $\langle u_i, \mu_i, \mu_i^{(k)}\rangle$ having the jth largest u_i, u_i is the order inducing variable, $\beta \in [0, 1]$, and g and h are strictly continuous monotonic functions.

The IGOWAWAAC and the Quasi-IOWAWAAC operators are extensions of the adequacy coefficient and the OWA operator. Therefore, they are applicable in a wide range of situations already considered with these two methods. For example, it is possible to extend them to the use of Choquet integrals in a similar way as it has been developed by Merigó and Casanovas (2011a; 2011b) and Yager (2004b). Moreover, they are also applicable to other situations such as different problems in statistics, mathematics, and economics.

4. GROUP DECISION MAKING WITH IGOWAWAAC OPERATORS

The IGOWAWAAC operator is applicable in a wide range of situations such as in decision making, statistics and engineering. In this paper, we will consider a decision making application in the selection of human resources. The main reason for using the IGOWAWAAC operator in business decision making problems such as the selection of production strategies is because the decision maker wants to take the decision according to a complex attitudinal character that is represented with order inducing variables. This can be useful in a lot of situations, for example, when the board of directors of a company wants to take a decision. Obviously, the attitudinal character of the board of directors is very complex because it involves the decision of different persons and their interests may be different.

The process to follow in the selection of strategies with the IGOWAWAAC operator is similar to the process developed by Gil-Lafuente (2005), Kaufmann and Gil-Aluja (1986) and Merigó and Gil-Lafuente (2010; 2011) with the difference that now we are considering a problem of human resource management. The 5 steps of the decision process can be summarized as follows:

Step 1: Analysis and determination of the significant characteristics of the available strategies for the company. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of finite alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$, a set of finite characteristics (or attributes), forming the matrix $(\mu^{(k)}{}_{hi})_{m\times n}$. Let $E = \{E_1, E_2, \ldots, E_p\}$ be a finite set of decision makers. Let $V = (v_1, v_2, \ldots, v_p)$ be the weighting vector of the weighted average of the decision makers such that $\sum_{q=1}^{p} v_q = 1$ and $v_q \in [0, 1]$ and $U = (u_1, u_2, \ldots, u_p)$ be the weighting vector of the decision makers that $\sum_{q=1}^{p} u_q = 1$ and $u_q \in [0, 1]$. Each decision maker provides their own payoff matrix $(\mu^{(kq)}{}_{hi})_{m\times n}$.

Step 2: Fixation of the ideal levels of each characteristic in order to form the ideal strategy.

Table 2. Ideal strategy

	C_1	C_2	…	C_i	…	C_n
S =	μ_1	μ_2	…	μ_i	…	μ_n

where S is the ideal strategy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i \in [0, 1]$; $i = 1, 2, \ldots, n$, is a number between 0 and 1 for the ith characteristic.

Step 3: Fixation of the real level of each characteristic for all the strategies considered.

Table 3. Available alternatives for each expert *q*

	C_1	C_2	…	C_i	…	C_n
S_k =	$\mu_1^{(k)}$	$\mu_2^{(k)}$	…	$\mu_i^{(k)}$	…	$\mu_n^{(k)}$

with $k = 1, 2, \ldots, m$; where S_k is the kth strategy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i^{(k)} \in [0, 1]$; $i = 1, \ldots, n$, is a number between 0 and 1 for the ith characteristic of the kth strategy.

Step 4: Use the weighted average (WA) to aggregate the information of the decision makers E by using the weighting vector U. The result is the collective payoff matrix $(\mu^{(k)}{}_{hi})_{m\times n}$. Thus, $\mu_{hi} = \sum_{q=1}^{p} u_q \ \mu_{hi}^{(kq)}$.

Step 5: Comparison between the ideal strategy and the different alternatives considered using the IGOWAWAAC operator. In this step, the objective is to express numerically the removal between the ideal strategy and the different alternatives considered. Note that it is possible to consider a wide range of IGOWAWAAC operators such as those described in Section 3.

Step 6: Adoption of decisions according to the results found in the previous steps. Finally, we should take the decision about which strategy select. Obviously, our decision is to select the strategy with the best results according to the type of IGOWAWAAC operator used.

Note that in the literature we find a wide range of other approaches for dealing with decision making problems (Canós and Liern, 2008; Figueira et al., 2005; Xu and Wang, 2012b; Zavadskas et al., 2011; Zhou and Chen, 2010).

5. APPLICATION IN STRATEGIC MANAGEMENT

In the following, we present a numerical example of the new approach in a decision making problem. We study a problem in strategic management where a decision maker wants to invest money in a new market and is looking for the optimal investment. Note that other decision applications could be developed such as in production management and human resource selection.

We analyze different particular cases of the IGOWAWAAC operator such as the NAC, the WAC, the OWAAC, the IOWAAC, the arithmetic-WAC (A-WAC), the arithmetic-IOWAAC (A-IOWAAC) and the IOWAWAAC operator. Note that with this analysis, we obtain "optimal" choices that depend on the aggregation operator used. The main advantage of the IGOWAWAAC is that it includes a wide range of particular cases, reflecting different potential factors to be considered in the decision-making problem. Thus, the decision maker is able to consider a lot of possibilities and select the alternative in closest accordance with his interests.

Assume that a company wants to invest money in a new market and considers five possible alternatives.

- A_1 = Invest in South America.
- A_2 = Invest in Asia.
- A_3 = Invest in Africa.
- A_4 = Invest in the three continents.
- A_5 = Do not make any investment.

In order to evaluate these strategies, the decision maker has brought together a group of experts. This group considers that each strategy can be described with the following characteristics:

- C_1 = Benefits in the short term.
- C_2 = Benefits in the midterm.
- C_3 = Benefits in the long term.
- C_4 = Risk of the strategy.
- C_5 = Other variables.

The experts establish the ideal values between [0, 1] that the strategy should have. The results are shown in Table 4.

Table 4. Ideal strategy

	C_1	C_2	C_3	C_4	C_5
I	0.8	0.8	0.9	1	0.9

The results of the available strategies, depending on the characteristic C_i and the alternative A_k that the decision makers choose, are shown in Tables 5, 6 and 7. Note that in this analysis we assume three experts that give their opinion concerning the available strategies.

Table 5. Available strategies – Expert 1

	C_1	C_2	C_3	C_4	C_5
A_1	0.5	0.9	0.7	0.8	1
A_2	0.6	0.9	0.8	1	0.7
A_3	0.9	0.3	0.8	0.5	0.7
A_4	0.2	0.8	1	1	0.8
A_5	0.6	0.7	0.3	0.9	0.8

Table 6. Available strategies – Expert 2

	C_1	C_2	C_3	C_4	C_5
A_1	0.8	0.9	0.4	0.8	1
A_2	0.7	0.9	0.6	0.9	0.7
A_3	0.8	0.6	0.8	0.6	0.9
A_4	0.2	0.9	1	1	0.8
A_5	0.5	0.7	0.6	1	0.9

Table 7. Available strategies – Expert 3

	C_1	C_2	C_3	C_4	C_5
A_1	0.8	0.6	0.7	0.8	0.7
A_2	0.8	0.9	0.7	0.5	0.7
A_3	0.7	0.6	0.8	1	0.8
A_4	0.2	0.4	1	1	0.8
A_5	0.7	0.7	0.6	0.8	1

In this example we assume that the three experts are equally important. Therefore, the weighting vector U is: U = (1/3, 1/3, 1/3). Thus, we get the following collective results shown in Table 8.

Table 8. Available strategies – collective result

	C_1	C_2	C_3	C_4	C_5
A_1	0.7	0.8	0.6	0.8	0.9
A_2	0.7	0.9	0.7	0.8	0.7
A_3	0.8	0.5	0.8	0.7	0.8
A_4	0.2	0.7	1	1	0.8
A_5	0.6	0.7	0.5	0.9	0.9

In this problem, the experts assume the following weighting vectors: W = (0.3, 0.2, 0.2, 0.2, 0.1) and V = (0.1. 0.2, 0.3, 0.2, 0.2). Note that the IOWA has a degree of importance of 30% and the WA, 70%. Due to the fact that the attitudinal character is very complex, the experts use order-inducing variables to represent it. The results are shown in Table 9.

Table 9. Order inducing variables

	C_1	C_2	C_3	C_4	C_5
A_1	12	16	20	6	8
A_2	25	20	15	12	10
A_3	16	19	12	8	15
A_4	10	13	17	19	11
A_5	12	15	17	19	22

With this information, we can aggregate the expected results for each characteristic in order to make a decision. In Table 10, we present different results obtained by using different types of IGOWAWAAC operators.

Table 10. Aggregated results

	NAC	WAC	OWAAC	IOWAAC	A-WAC	A-IOWAAC	IOWAWAAC
A_1	0.88	0.87	0.91	0.87	0.866	0.877	0.87
A_2	0.86	0.87	0.88	0.87	0.853	0.863	0.87
A_3	0.84	0.85	0.87	0.84	0.833	0.84	0.847
A_4	0.84	0.79	0.9	0.9	0.882	0.858	0.823
A_5	0.84	0.82	0.88	0.86	0.826	0.846	0.832

If we establish a ranking of the alternatives, a typical situation if we want to consider more than one alternative, we get the results shown in Table 11.

Table 11. Ranking of the strategies

	Ranking		Ranking
NAC	$A_1 \succ A_2 \succ A_3 = A_4 = A_5$	A-WAC	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$
WAC	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$	A-IOWAC	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$
OWAAC	$A_1 \succ A_4 \succ A_2 = A_5 \succ A_3$	IOWAWAAC	$A_1 = A_2 \succ A_3 \succ A_5 \succ A_4$
IOWAAC	$A_4 \succ A_1 = A_2 \succ A_5 \succ A_3$		

As we can see, depending on the aggregation operator used, the ranking of the strategies may be different. Therefore, the decision about which strategy select may be also different.

6. CONCLUSIONS

We have presented a new approach for decision making by using the IGOWAWAAC operator. It is a new aggregation operator very useful for business decision making problems and other aggregation processes. We have seen that it uses order inducing variables in the reordering process of the aggregation with the OWAWA operator in the adequacy coefficient. Thus, it is able to deal with the WAC and the IOWAAC operators in the same formulation and considering the degree of importance that each concept has in the aggregation. We have further extended this approach by using quasi-arithmetic means forming the Quasi-IOWAWAAC operator.

We have proved that this operator includes the IGOWAWA operator as a particular case when the real set is empty. Moreover, we have also seen that the IGOWAWAAC operator becomes the IGOWAWAD operator under certain conditions. Therefore, we have seen that the adequacy coefficient is an extension of the Hamming distance with some changes very useful for some particular aggregation problems such as business decision making.

We have also developed a simple numerical example in order to understand the new approach. We have focussed on a group decision making problem concerning strategic management. We have seen that this operator provides more complete information to the decision maker because it includes a wide range of particular cases.

In future research, we expect to develop further extensions to this approach by using more general formulations such as the use of unified aggregation operators and other selection indexes. We will also use more complete formulations of the OWA operator that includes for example, the probability. We also expect to develop different applications of this approach, especially in business decision making problems such as production and financial management.

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REFERENCES

[1]**Beliakov, G.; Pradera, A. and Calvo, T. (2007),** ***Aggregation Functions: A Guide for Practitioners***, *Springer-Verlag, Berlin*;

[2]**Canós, L. and Liern, V. (2008),** ***Soft Computing-based Aggregation Methods for Human Resource Management***. *European Journal of Operational Research*, 189, 669–681;

[3]**Chen, H.Y. and Zhou, L.G. (2011),** ***An Approach to Group Decision Making with Interval Fuzzy Preference Relations based on Induced Generalized Continuous Ordered Weighted Averaging Operator***. *Expert Systems with Applications*, 38, 13432–13440;

[4]**Figueira, J., Greco, S. and Ehrgott, M. (2005),** ***Multiple Criteria Decision Analysis: State of the Art Surveys***; *Springer,* Boston;

[5]**Gil-Aluja, J. (1998),** ***The Interactive Management of Human Resources in Uncertainty***; *Kluwer Academic Publishers*, Dordrecht;

[6]**Gil-Lafuente, A.M. (2005),** ***Fuzzy Logic in Financial Analysis***. *Springer, Berlin*;

[7]**Gil-Lafuente, A.M. and Merigó, J.M. (2010),** ***Computational Intelligence in Business and Economics***; *World Scientific*, Singapore;

[8]**Gil-Lafuente, J. (2002),** ***Algorithms for Excellence. Keys for Being Successful in Sport Management*** (in Spanish). *Ed. Milladoiro,* Vigo;

[9]**Hamming, R.W. (1950),** ***Error-detecting and Error-correcting Codes***. *Bell Systems Technical Journal*, 29: 147–160;

[10]**Karayiannis, N. (2000),** ***Soft Learning Vector Quantization and Clustering Algorithms Based on Ordered Weighted Averaging Operators***. *IEEE Transactions on Neural Networks*, 11, 1093–1105;

[11]**Kaufmann, A. (1975),** ***Introduction to the Theory of Fuzzy Subsets***. *Academic Press, New York;*

[12]**Kaufmann, A. and Gil-Aluja, J. (1986),** ***Introduction to the Theory of Fuzzy Subsets in Business Management*** (In Spanish). *Ed. Milladoiro,* Santiago de Compostela;

[13]**Merigó, J.M. (2010),** ***Fuzzy Decision Making with Immediate Probabilities***. *Computers & Industrial Engineering*, 58, 651–657;

[14]**Merigó, J.M. (2011),** ***A Unified Model between the Weighted Average and the Induced OWA Operator***. *Expert Systems with Applications*, 38, 11560–11572;

[15]**Merigó, J.M. and Casanovas, M. (2009),** ***Induced Aggregation Operators in Decision Making with the Dempster-Shafer Belief Structure***. *International Journal of Intelligent Systems*, 24, 934–954;

[16]**Merigó, J.M. and Casanovas, M. (2011a),** ***Decision Making with Distance Measures and Induced Aggregation Operators***. *Computers & Industrial Engineering*, 60, 66–76;

[17]**Merigó, J.M. and Casanovas, M. (2011b),** ***The Uncertain Induced Quasi-arithmetic OWA Operator***. *International Journal of Intelligent Systems*, 26, 1–24;

[18]**Merigó, J.M. and Gil-Lafuente, A.M. (2007),** ***Unification Point in Methods for the Selection of Financial Products***. *Fuzzy Economic Review*, 12, 35–50;

[19]**Merigó, J.M. and Gil-Lafuente, A.M. (2008),** ***The Generalized Adequacy Coefficient and its Application in Strategic Decision Making*.** *Fuzzy Economic Review*, 13, 17–36;

[20]**Merigó, J.M. and Gil-Lafuente, A.M. (2009),** ***The Induced Generalized OWA Operator*****.** *Information Sciences*, 179, 729–741;

[21]**Merigó, J.M. and Gil-Lafuente, A.M. (2010),** ***New Decision Making Techniques and their Application in the Selection of Financial Products*****.** *Information Sciences*, 180, 2085–2094;

[22]**Merigó, J.M. and Gil-Lafuente, A.M. (2011*),*** ***OWA Operators in Human Resource Management****.* *Economic Computation and Economic Cybernetics Studies and Research, ASE Publishing,* 45, 153–168;

[23]**Merigó, J.M. and Gil-Lafuente, A.M. (2012a),** ***A Method for Decision Making with the OWA Operator*** ; *Computer Science and Information Systems* 9, 359–382;

[24]**Merigó, J.M. and Gil-Lafuente, A.M. (2012b),** ***Decision Making Techniques in Business and Economics based on the OWA Operator;*** *SORT – Statistics and Operations Research Transactions*, 36, 81–101;

[25]**Merigó, J.M., Gil-Lafuente, A.M. and Gil-Aluja, J. (2011),** ***Decision Making with the Induced Generalized Adequacy Coefficient*** ; *Applied and Computational Mathematics*, 2, 321–339;

[26]**Merigó, J.M., Gil-Lafuente, A.M., Zhou, L.G, Chen, H.Y. (2012),** ***Induced and Linguistic Generalized Aggregation Operators and their Application in Linguistic Group Decision Making***; *Group Decision and Negotiation*, 21, 531-549;

[27]**Merigó, J.M. and Wei, G.W. (2011),** ***Probabilistic Aggregation Operators and their Application in Uncertain Multi-person Decision Making***; *Technological and Economic Development of Economy*, 17, 335–351;

[28]**Torra, V. (1997),** ***The Weighted OWA Operator***; *International Journal of Intelligent Systems*, 12, 153–166;

[29]**Torra, V. and Narukawa, Y. (2007)** ***Modelling Decisions: Information Fusion and Aggregation Operators****. Springer, Berlin*;

[30]**Wei, G.W. (2011), FIOHWM** ***Operator and its Application to Multiple Attribute Group Decision Making*** ; *Expert Systems with Applications*, 38, 2984–2989;

[31]**Wei, G.W., Zhao, X. and Lin, R. (2010),** ***Some Induced Aggregating Operators with Fuzzy Number Intuitionistic Fuzzy Information and their Applications to Group Decision Making***; *International Journal of Computational Intelligence Systems*, 3, 84–95;

[32]**Xu, Y.J. and Wang, H. (2012a),** ***The Induced Generalized Aggregation Operators for Intuitionistic Fuzzy Sets and their Application in Group Decision Making****; Applied Soft Computing*, 12, 1168–1179;

[33]**Xu, Y.J. and Wang, H. (2012b),** ***Power Geometric Operators for Group Decision Making under Multiplicative Linguistic Preference Relations*****;** *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 20, 139–159;

[34]**Xu, Z.S. and Cai, X. (2012),** ***Uncertain Power Average Operators for Aggregating Interval Fuzzy Preference Relations***; *Group Decision and Negotiation*, 21, 381–397;

[35]**Xu, Z.S. and Chen, J. (2008),** ***Ordered Weighted Distance Measures***; *Journal of Systems Science and Systems Engineering*, 17, 432–445;

[36]**Xu, Z.S. and Da, Q.L. (2003),** ***An Overview of Operators for Aggregating Information****; International Journal of Intelligent Systems*, 18, 953-969;

[37]**Xu, Z.S. and Xia, M. (2011),** ***Induced Generalized Intuitionistic Fuzzy Operators****; Knowledge-Based Systems*, 24, 197–209;

[38]**Yager, R.R. (1988),** ***On Ordered Weighted Averaging Aggregation Operators in Multi-criteria Decision Making***; *IEEE Transactions on Systems, Man and Cybernetics B*, 18, 183–190;

[39]**Yager, R.R. (1993),** ***Families of OWA Operators***; *Fuzzy Sets and Systems*, 59, 125–148;

[40]**Yager, R.R. (2003),** ***Induced Aggregation Operators***; *Fuzzy Sets and Systems*, 137, 59–69;

[41]**Yager, R.R. (2004a),** ***Generalized OWA Aggregation Operators***; *Fuzzy Optimization and Decision Making*, 3, 93–107;

[42]**Yager R.R. (2004b),** ***Choquet Aggregation Using Order Inducing Variables*;** *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12, 69–88;

[43]**Yager, R.R. (2010),** ***Norms Induced from OWA Operators***; *IEEE Transactions on Fuzzy Systems*, 18, 57–66;

[44]**Yager, R.R. and Filev, D.P. (1999),** ***Induced Ordered Weighted Averaging Operators****; IEEE Transactions on Systems, Man and Cybernetics B*, 29, 141–150;

[45]**Yager, R.R. and Kacprzyk, J. (1997),** ***The Ordered Weighted Averaging Operators: Theory and Applications***; *Kluwer Academic Publishers*, Norwell, MA;

[46]**Yager, R.R., Kacprzyk, J. and Beliakov, G. (2011),** ***Recent Developments on the Ordered Weighted Averaging Operators: Theory and Practice***; *Springer-Verlag, Berlin;*

[47]**Zavadskas, E.K. and Turskis, Z. (2011),** ***Multiple Criteria Decision Making (MCDM) Methods in Economics: An Overview***; *Technological and Economic Development of Economy*, 17, 397–427;

[48]**Zeng, S.Z. and Su, W. (2012),** ***Intuitionistic Fuzzy Ordered Weighted Distance Operator****; Knowledge-Based Systems*, 24, 1224–1232;

[49]**Zhao, H., Xu, Z.S., Ni, M. and Liu, S. (2010),** ***Generalized Aggregation Operators for Intuitionistic Fuzzy Sets***; *International Journal of Intelligent Systems*, 25, 1–30;

[50]**Zhou, L.G. and Chen, H.Y. (2010),** ***Generalized Ordered Weighted Logarithm Aggregation Operators and their Applications to Group Decision Making*;** *International Journal of Intelligent Systems*, 25, 683–707.