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AN APPROACH BASED ON THE INDUCED UNCERTAIN PURE LINGUISTIC HYBRID HARMONIC AVERAGING OPERATOR TO GROUP DECISION MAKING

Abstract. *In this paper, we propose an induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator. The operator is very suitable to deal with the situation where the input arguments are represented in uncertain pure linguistic variables. And the IUPLHHA operator can reflect the importance degrees of both the given uncertain linguistic variables and their ordered positions. Also, in the situations where the information about all the attribute weights, the attribute values and the expert weights are expressed in the form of linguistic label variables, we develop an approach based on the IUPLHHA operator for multiple attribute group decision making with uncertain pure linguistic environment. Finally, an illustrative example is given to verify the developed approach and to demonstrate its feasibility and practicality.*

Key words: *Group decision making; Operator; Uncertain pure linguistic; Induced; Hybrid harmonic averaging.*

JEL Classification: D81, M12, M51

1. Introduction

The ordered weighted averaging (OWA) operator [28] is one of the most common aggregation operators developed by Yager (1988). It provides a method to aggregate operators including the maximum, the minimum, and the average, as special cases. Later, Xu and Da (2002) [19] introduced the ordered weighted geometric (OWG) operator based on the OWA operator and the geometric mean. Both the operators can be used in situations where the input arguments are the exact numerical values. However, as the increasing complexity of the social and economic environment, in many situations, the decision information is expressed in the form of uncertain linguistic variables rather than numerical ones because of time pressure, people's limited expertise related to the problem domain and so on. Zadeh (1975, 1976) [31-33] introduced the notion of a linguistic variable, whose values are words or sentences in a natural or artificial language. Linguistic variables such as 'good', 'poor', 'fair', 'fast', 'slow', etc. are usually used to express fuzzy qualitative information and involved in the calculation such as [1, 4, 5, 6, 7, 9, 10, 15, 16, 21, 25, 34, 35], etc.

An interesting generalization of the OWA operator called the induced ordered weighted averaging (IOWA) operator developed by Yager and Filev (1999) [29]. It takes OWA pairs as their argument pairs, in which the first component is used to induce an ordering over the second component which is exact numerical values. Xu (2006) [22] developed the uncertain linguistic ordered weighted geometric (ULOWG) operator and induced uncertain linguistic ordered weighted geometric (IULOWG) operator, which based on the uncertain ordered weighted averaging (UOWA) operator [17] and the induced ordered weighted geometric (IOWG) operator [18]. These operators can be used in situations where the aggregated arguments are taken in the form of uncertain linguistic variables. The aggregation operators with induced uncertain information has attracted great attentions, refer to [2, 8, 12, 13, 15, 17, 23, 24, 27, 30, 34, 36].

Recently, a few authors have done some research on the hybrid averaging (HA) operators, which can reflect the importance degrees of both the given variables and their ordered positions. Xu (2004) [14] developed uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator, and proposed a method for multiple attribute group decision making with uncertain linguistic information. Wei (2009) [11] introduced uncertain linguistic hybrid geometric mean (ULHG) operator, and proposed a method for multiple attribute group decision making under uncertain linguistic environment. Moreover, in the real-life world, the decision information takes the form of pure linguistic variables rather than part of the linguistic variables. In the situations where all attribute weights, attribute values and expert weights took the form of linguistic labels. The prominent characteristic of the approach is straightforward and does not produce any loss of information. Therefore, research in this topic has great significance and it is necessary to extend the above linguistic operators to accommodate the uncertain pure linguistic situation, which is also the focus of this paper.

For this purpose, we shall develop an induced uncertain pure linguistic aggregation operator called an induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator, which reflects the importance degrees of both the given uncertain linguistic variables and their ordered position. To do so, we are able to combine the weighted harmonic average and the induced order weighed harmonic average in the same formulation with uncertain pure linguistic information. Moreover, in the situations where the information about all the attribute weights, the attribute values and the expert weights are taken in the form of linguistic variables, we shall develop an approach to multiple attribute group decision making based on the IUPLHHA operator with uncertain pure linguistic information.

This paper is structured as follows. In Section 2, we present some basic concepts and review some of the most common aggregation operators. In Section 3, we develop the induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator, and study some its various properties. In Section 4, we present an approach based on the developed operator for uncertain pure linguistic multiple attribute group decision making. Section 5 gives an illustrative example, and finally, the main conclusions of the paper are summarized in Section 6.

2. Preliminaries

In this section, we firstly introduce the uncertain linguistic variables and present their operational laws. And then, we briefly review some aggregation operators including the OWA operator, the OWG operator, and the IOWA operator.

2.1 Uncertain linguistic variables and their operational laws

Suppose that $S=\left\{s_\alpha \,\middle|\, \alpha=\frac{1}{t},\cdots,\frac{1}{2},1,2,\cdots,t\right\}$ is a finite and totally ordered discrete term set, where s_α represents a possible value for a linguistic variable.

And it is usually required that there exist the following characteristics [26]:

1) $s_\alpha > s_\beta$, if $\alpha > \beta$;

2) There is the interaction operator: $rec(s_\alpha)=s_{\frac{1}{\alpha}}$. Especially, $rec(s_1)=s_1$;

3) Max operator: $\max(s_\alpha, s_\beta)=s_\alpha$ if $s_\alpha > s_\beta$;

4) Min operator: $\min(s_\alpha, s_\beta) = s_\alpha$ if $s_\alpha < s_\beta$.

In the process of given information aggregating, some decision results may do not match any linguistic labels exactly. To preserve all the given information, Xu (2004) [26] extended the discrete label set S to a continuous label set $S = \left\{ s_\alpha \middle| \alpha \in [\frac{1}{q}, q] \right\}$, where $q(|q| > t)$ is a sufficiently large real number. If $s_\alpha \in S$, then we call s_α original linguistic label, otherwise, we call s_α the virtual linguistic label. In general, the decision maker uses the original linguistic labels to evaluate attributes and alternatives, and the virtual linguistic labels can only appear in calculation. Xu and Da (2002) [17] developed the uncertain linguistic variable and the degree of possibility of the uncertain linguistic variable, which can be defined as following:

Definition 1. Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in S$, s_α and s_β are the lower and the upper limits, respectively, then we call $\tilde{s}$ the uncertain linguistic variable.

Definition 2. Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$, $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}] \in S$ be two uncertain linguistic variables, and let $l_{\tilde{s}_1} = \beta_1 - \alpha_1$, $l_{\tilde{s}_2} = \beta_2 - \alpha_2$, then, the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as following:

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \max\left\{1 - \max\left(\frac{\beta_2 - \alpha_1}{l_{\tilde{s}_1} + l_{\tilde{s}_2}}, 0\right), 0\right\} \tag{1}$$

From definition above, we can get the following useful result easily:

(1) $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1$, $0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$;

(2) $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = 1/2$.

Let S be the set of all uncertain linguistic variables. Consider any two uncertain linguistic variables $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$, $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}] \in S$, $\lambda, \lambda_1, \lambda_2 > 0$, then, their operational laws are defined as following:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}]$;

(2) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}]$;

(3) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta] = [s_{\lambda\alpha}, s_{\lambda\beta}]$;

(4) $\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda\tilde{s}_1 \oplus \lambda\tilde{s}_2$;

(5) $(\lambda_1 + \lambda_2)\tilde{s} = \lambda_1\tilde{s} \oplus \lambda_2\tilde{s}$;

(6) If $s_\alpha = s_\beta \in S$, then, the uncertain linguistic variable $\tilde{s} = [s_\alpha, s_\beta]$ is reduced to the certain linguistic variable $s = s_\alpha$ or $s = s_\beta$;

(7) $[s_\beta, s_\alpha] = -[s_\alpha, s_\beta]$;

(8) $\tilde{s}_1 / \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] / [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1/\alpha_2}, s_{\beta_1/\beta_2}]$.

Especially, $[s_1, s_1]/[s_\alpha, s_\beta] = s_1/[s_\alpha, s_\beta] = [s_{1/\alpha}, s_{1/\beta}]$.

2.2 **Some aggregation operators**

In the following, we briefly review some aggregation operators including the OWA operator, the OWG operator, and the IOWA operator.

Definition 3. An OWA operator of dimension *n* is a mapping OWA: $R^n \to R$ that has an associated *n* vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$OWA_w(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j is the *j*th largest of the a_i.

Motivated by the idea of the OWA operator, Xu and Da (2002) [19] developed an ordered weighted geometric averaging (OWG) operator, which can be defined as following:

Definition 4. An OWG operator of dimension *n* is a mapping OWG: $R^{+^n} \to R^+$ that has an associated *n* vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$OWG_w(a_1, a_2 \cdots, a_n) = \prod_{j=1}^{n} b_j^{\,w_j} \tag{3}$$

where b_j is the *j*th largest of the a_i.

Definition 5. An IOWA operator of dimension *n* is a mapping IOWA: $R^n \times R^n \to R$ that has an associated *n* vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$IOWA_w(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \cdots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

where b_j is the a_i value of the OWA pair $\langle u_i, a_i\rangle$ having the *j*th largest u_i, and u_i in $\langle u_i, a_i\rangle$ is referred to as the order inducing variable and a_i as the argument variable.

The IOWA operator takes OWA pairs as its argument pairs, in which the first component is used to induce an ordering over the second components which are then aggregated.

3. Induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator

From **Definition** 3 and 4 above, we can see the prominent characteristic of the OWA and the OWG operators are the reordering step, and the weights associated with the OWA and the OWG operators are the weights of the ordered positions of the input data rather than the weights of the input data. However, both the OWA and the OWG operators can only be used in situations where the input arguments are the exact numerical values. Xu (2004) [14] developed an uncertain linguistic ordered weighted averaging (ULOWA) operator, which can be defined as following:

Definition 6. An uncertain linguistic ordered weighted averaging (ULOWA) operator of dimension *n* is a mapping ULOWA: $S^n \to S$ that has an associated *n* vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$ULOWA_w(\tilde{s}_1,\tilde{s}_2,\cdots,\tilde{s}_n) = w_1\tilde{r}_1 \oplus w_2\tilde{r}_2 \oplus \cdots \oplus w_n\tilde{r}_n \qquad (5)$$

where $\tilde{r}_j$ is the *j*th largest of the $\tilde{s}_i, \tilde{s}_i \in S(i=1,2,\cdots,n)$.

In many situations, however, due to the increasing complexity of the social and economic environment and the lack of knowledge or data about the practical problem domain, the input arguments may be uncertain pure linguistic variables in nature, in which all attribute weights, attribute values and expert weights took the form of linguistic labels. As we know, the harmonic mean is an important statistic, which can be used to describe the statistical distribution characteristics of the data. So, we can consider using the form of the harmonic mean when other forms of mean involved in the calculation are unavailable due to data. Therefore, in the following we shall develop the uncertain pure linguistic weighted harmonic averaging (UPLWHA) operator, the induced uncertain pure linguistic ordered weighted harmonic averaging (IUPLOWHA) operator and the induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator.

Definition 7. Let UPLWHA : $S^n \to S$, if

$$UPLWHA_{s_\omega}(\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n) = \frac{1}{\sum_{i=1}^{n} \frac{s_{\omega_i}}{\tilde{s}_i}} \tag{6}$$

where $s_{\omega_i} = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T$ is the weighting vector of uncertain linguistic variables $(\frac{1}{\tilde{s}_1}, \frac{1}{\tilde{s}_2}, \cdots, \frac{1}{\tilde{s}_n})$, and $s_{\omega_i} \in S$, then UPLWHA is called the uncertain pure linguistic weighted harmonic averaging (UPLWHA) operator.

Definition 8. Let IUPLOWHA : $S^n \times S^n \to S$, if

$$IUPLOWHA_w(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \frac{1}{w_1 \tilde{r}_1 \oplus w_2 \tilde{r}_2 \oplus \cdots \oplus w_n \tilde{r}_n} \tag{7}$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is a weighting vector, such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\tilde{r}_i$ is the $\frac{1}{\tilde{s}_i}$ value of the uncertain pure linguistic ordered weighted harmonic averaging (UPLOWHA) pair $\langle s_{u_i}, \tilde{s}_i \rangle$ having the *j*th largest s_{u_i}, and s_{u_i} in $\langle s_{u_i}, \tilde{s}_i \rangle$ is referred to as the order inducing linguistic variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

If there is a tie between the UPLOWHA pairs $\langle s_{u_i}, \tilde{s}_i \rangle$ and $\langle s_{u_j}, \tilde{s}_j \rangle$ with respect to order inducing variables, in this case, we replace each argument of the tied UPLOWHA pairs by their harmonic mean. If *k* items are tied, then we can replace these by *k* replicas of their harmonic mean.

Note that in many situations, however, the order inducing linguistic variables $s_{u_i} (i = 1, 2, \cdots, n)$ take the form of uncertain linguistic variables $\tilde{s}_{u_i} (i = 1, 2, \cdots, n)$.

In order to rank these uncertain linguistic variables, at first, we can compare each variable $\tilde{s}_i \in S$ with all arguments $\tilde{s}_j \in S(i, j = 1, 2, \cdots, n)$ by using Eq. (1), and let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$. Then, we construct a complementary matrix [3, 17] $P = \left(p_{ij}\right)_{n\times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \cdots, n$. Sum all elements in each line of matrix $P = \left(p_{ij}\right)_{n\times n}$, and we have $p_i = \sum_{j=1}^{n} p_{ij} (i = 1, 2, \cdots, n)$. Finally, we can rank the arguments $\tilde{s}_i (i = 1, 2, \cdots, n)$ in descending order in accordance with the values of p_i .

From Definitions 7 and 8 above, we know that the UPLWHA operator weights the uncertain linguistic variables while the IUPLOWHA operator weights the ordered positions of the uncertain linguistic variables instead of weighting the variables themselves. Therefore, weights represent different aspects in both the UPLWHA and the IUPLOWHA operators. To overcome the drawback, we develop an induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator, which is defined as follows:

Definition 9. An induced uncertain pure linguistic hybrid harmonic averaging (IUPLHHA) operator of dimension *n* is a mapping IUPLHHA : $S^n \to S$ that has an associated *n* vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$IUPLHHA_{s_\omega, w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \frac{1}{w_1 \gamma_1 \oplus w_2 \gamma_2 \oplus \cdots \oplus w_n \gamma_n} \quad (8)$$

where γ_i is the $\frac{1}{\tilde{s}_i'}$ value $(\frac{1}{\tilde{s}_i'} = n \frac{s_{\omega_i}}{\tilde{s}_i}, i = 1, 2, \cdots, n)$ of the UPLOWHA pair

$\langle s_{u_i}, \tilde{s}_i \rangle$ having the *j*th largest s_{u_i}, and s_{u_i} in $\langle s_{u_i}, \tilde{s}_i \rangle$ is referred to as the order inducing linguistic variable. $s_{\omega_i} = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T \in S$ is the weighting vector of uncertain linguistic variables $(\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n)$, n is the balancing coefficient.

Remark 1. If there is a tie between the UPLOWHA pairs $\langle s_{u_i}, \tilde{s}_i \rangle$ and $\langle s_{u_j}, \tilde{s}_j \rangle$ with respect to order inducing variables, in this case, we replace each argument of the tied UPLOWHA pairs by their harmonic mean. If *k* items are tied, then we can replace these by *k* replicas of their harmonic mean.

Remark 2. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then IUPLHAA is reduced to the UPLWHA operator; if $\omega = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, we can have $\tilde{s}_i' = \tilde{s}_i$, then IUPLHAA is reduced to the IUPLOWHA operator.

The IUPLHHA operator has many desirable properties, which can be proved with the following theorems:

Theorem 1. (Commutativity). $IUPLHHA_{s_\omega, w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle)$

$$= IUPLHHA_{s_\omega, w}(\langle s_{u_1}', \tilde{s}_1' \rangle, \langle s_{u_2}', \tilde{s}_2' \rangle, \cdots, \langle s_{u_n}', \tilde{s}_n' \rangle),$$

where $(\langle s_{u_1}', \tilde{s}_1' \rangle, \langle s_{u_2}', \tilde{s}_2' \rangle, \cdots, \langle s_{u_n}', \tilde{s}_n' \rangle)$ is any permutation of

$(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle)$.

Proof. Let

$$IUPLHHA_{s_\omega, w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \frac{1}{w_1 \gamma_1 \oplus w_2 \gamma_2 \oplus \cdots \oplus w_n \gamma_n},$$

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1}{}', \tilde{s}_1{}' \rangle, \langle s_{u_2}{}', \tilde{s}_2{}' \rangle, \cdots, \langle s_{u_n}{}', \tilde{s}_n{}' \rangle) = \frac{1}{w_1\gamma_1{}' \oplus w_2\gamma_2{}' \oplus \cdots \oplus w_n\gamma_n{}'}$$

Since $(\langle s_{u_1}{}', \tilde{s}_1{}' \rangle, \langle s_{u_2}{}', \tilde{s}_2{}' \rangle, \cdots, \langle s_{u_n}{}', \tilde{s}_n{}' \rangle)$ is any permutation of

$(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle)$, we have $\gamma_j = \gamma_j{}'$, $j = 1, 2, \cdots, n$

This completes the proof of Theorem 1.

Theorem 2. (Idempotency). If $\tilde{s}_j = \tilde{s}$ for all *j*, then

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \tilde{s}.$$

Proof. Since $\tilde{s}_j = \tilde{s}$ for all *j*, we have

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \frac{1}{w_1\gamma_1 \oplus w_2\gamma_2 \oplus \cdots \oplus w_n\gamma_n}$$

$$= \frac{1}{\frac{w_1}{\tilde{s}} \oplus \frac{w_2}{\tilde{s}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}}} = \tilde{s}$$

This completes the proof of Theorem 2.

Theorem 3. (Monotonicity). If $\tilde{s}_j \le \tilde{s}_j{}'$ for all *j*, then

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle)$$

$$\le IUPLHHA_{s_\omega,w}(\langle s_{u_1}, \tilde{s}_1{}' \rangle, \langle s_{u_2}, \tilde{s}_2{}' \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n{}' \rangle).$$

Proof. Let

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \cdots, \langle s_{u_n}, \tilde{s}_n \rangle) = \frac{1}{w_1\gamma_1 \oplus w_2\gamma_2 \oplus \cdots \oplus w_n\gamma_n},$$

$$IUPLHHA_{s_\omega,w}(\langle s_{u_1},\tilde{s}_1'\rangle,\langle s_{u_2},\tilde{s}_2'\rangle,\cdots,\langle s_{u_n},\tilde{s}_n'\rangle)=\frac{1}{w_1\gamma_1'\oplus w_2\gamma_2'\oplus\cdots\oplus w_n\gamma_n'}$$

Since $\tilde{s}_j \le \tilde{s}_j'$ for all *j*, we can have $\gamma_j \ge \gamma_j'$, $j=1,2,\cdots,n$.

This completes the proof of Theorem 3.

Theorem 4 (Boundary).

$$\min_j(\tilde{s}_j') \le IUPLHHA_{s_\omega,w}(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\cdots,\langle s_{u_n},\tilde{s}_n\rangle) \le \max_j(\tilde{s}_j').$$

Proof. It is straightforward and thus omitted.

4. Multiple-attribute group decision making with the IUPLHHA operator under uncertain pure linguistic information

In this section, we shall purpose an approach based on the developed operator to multiple-attribute group decision making under uncertain pure linguistic information. Let $d_k \in D(k=1,2,\cdots,m)$ be the set of decision makers (DMs), and $s_\nu=(s_{\nu_1},s_{\nu_2},\cdots,s_{\nu_m})^T \in S$ be the weight vector of the DMs, let $G=\{g_1,g_2,\cdots,g_l\}$ be the set of attributes, and $s_\omega=(s_{\omega_1},s_{\omega_2},\cdots,s_{\omega_l})^T \in S$ be the weight vector. $X=\{x_1,x_2,\cdots,x_n\}$ be a discrete set of alternatives. Suppose that $A^{(k)}=(a_{ij}^{(k)})_{l\times n}$ is the decision matrix, where $a_{ij}^{(k)} \in S$ is a preference value, which takes the form of uncertain pure linguistic variable, given by the DM, for alternative $x_i \in X$ with respect to attribute $g_j \in G$. We shall utilize the IUPLHHA operator to develop an approach to multiple-attribute group decision making with uncertain pure linguistic information, which involves the following steps:

Step 1: Calculate the $\tilde{s}_i'$ value by using $\frac{1}{\tilde{s}_i'}=n\frac{s_{\omega_i}}{\tilde{s}_i}, i=1,2,\cdots,m$ to aggregate all the decision matrices $A^{(k)}=(a_{ij}^{(k)})_{l\times n}$, where $\tilde{s}_i$ is the uncertain linguistic variables of the UPLOWHA pair $\left\langle s_{u_i}, \tilde{s}_i \right\rangle$ and s_ω is the weighting vector of $\tilde{s}_i$.

Step 2： Utilize the IUPLHHA operator

$$a_j = IUPLHHA_{s_\omega, w}(\left\langle s_{u_1}, \tilde{s}_1 \right\rangle, \left\langle s_{u_2}, \tilde{s}_2 \right\rangle, \cdots, \left\langle s_{u_m}, \tilde{s}_m \right\rangle) = \frac{1}{w_1\gamma_{1j} \oplus w_2\gamma_{2j} \oplus \cdots \oplus w_m\gamma_{mj}}$$

to derive the collective overall preference value a_j of alternative $x_j (j = 1, 2, \cdots, n)$ given by all the DMs, where $w = (w_1, w_2, \cdots, w_m)^T$ is the weight vector associated with the IUPLHHA operator, with $w_i \in [0,1]$, $\sum_{i=1}^{m} w_i = 1$.

Step 3 : Compare each a_j with all a_i, $i, j = 1, 2, \cdots, n$ by using Eq. (1) ,and we let $p_{ij} = p\left\{a_i \geq a_j\right\}$, then we construct a complementary matrix $P = \left(p_{ij}\right)_{n \times n}$, where

$$p_{ij} \geq 0 \ , \ p_{ij} + p_{ji} = 1 \ , \ p_{ii} = \frac{1}{2} \ , \ i, j = 1, 2, \cdots, n .$$

Step 4： Sum all elements in each line of matrix $P = \left(p_{ij}\right)_{n \times n}$, and we have $p_i = \sum_{j=1}^{n} p_{ij} (i = 1, 2, \cdots, n)$. Then, we can rank the arguments $a_j (j = 1, 2, \cdots, n)$ in descending order in accordance with the values of $p_i (i = 1, 2, \cdots, n)$.

Step 5： Rank all the alternatives $x_j (j = 1, 2, \cdots, n)$ and select the best one(s) in accordance with $a_j (j = 1, 2, \cdots, n)$.

5. Illustrative example

In this section, we consider a numerical example of the developed approach regarding the selection of investments. Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from F. Herrera and E. Herrera-Viedma (2000) [4]). There is a panel with five possible alternatives in which to invest the money:

(1) x_1 is a travel company; (2) x_2 is an insurance company;

(3) x_3 is a computer company; (4) x_4 is an educational institution;

(5) x_5 is an advertising company.

The investment company must make a decision according to the following four attributes:

(1) g_1 is the risk analysis; (2) g_2 is the growth analysis;

(3) g_3 is the social-political impact analysis; (4) g_4 is the other factors.

The five possible alternatives $x_j\,(j=1,2,3,4,5)$ are to be evaluated using the linguistic label term set

$$S=\{s_{\frac{1}{5}}=extremelypoor, s_{\frac{1}{4}}=verypoor, s_{\frac{1}{3}}=poor, s_{\frac{1}{2}}=slightlypoor,$$

$$s_1=fair, s_2=slightlygood, s_3=good, s_4=verygood$$

by three decision makers $d_k\,(k=1,2,3)$ (whose weight vector $s_v=(s_1, s_2, s_{1/2})^T$) under the attributes above, as listed in Tables 1–3, respectively.

Table 1 Decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
g_1	$[s_{1/3}, s_{1/2}]$	$[s_1, s_2]$	$[s_{1/2}, s_1]$	$[s_{1/3}, s_{1/2}]$	$[s_1, s_2]$
g_2	$[s_{1/2}, s_1]$	$[s_1, s_2]$	$[s_{1/2}, s_1]$	$[s_1, s_2]$	$[s_2, s_3]$
g_3	$[s_3, s_4]$	$[s_{1/3}, s_{1/2}]$	$[s_{1/2}, s_1]$	$[s_{1/3}, s_{1/2}]$	$[s_1, s_2]$
g_4	$[s_{1/2}, s_1]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_{1/2}, s_1]$

Table 2. Decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
g_1	$[s_{1/2}, s_1]$	$[s_{1/2}, s_1]$	$[s_1, s_2]$	$[s_{1/4}, s_{1/3}]$	$[s_{1/2}, s_1]$
g_2	$[s_{1/3}, s_{1/2}]$	$[s_{1/2}, s_1]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_1, s_3]$
g_3	$[s_{1/2}, s_1]$	$[s_2, s_3]$	$[s_{1/4}, s_{1/3}]$	$[s_{1/2}, s_1]$	$[s_{1/3}, s_{1/2}]$
g_4	$[s_2, s_3]$	$[s_{1/4}, s_{1/3}]$	$[s_{1/2}, s_1]$	$[s_3, s_4]$	$[s_{1/2}, s_1]$

Table 3. Decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
g_1	$[s_{1/4}, s_{1/3}]$	$[s_{1/3}, s_{1/2}]$	$[s_{1/3}, s_{1/2}]$	$[s_1, s_2]$	$[s_1, s_2]$
g_2	$[s_1, s_2]$	$[s_1, s_2]$	$[s_2, s_3]$	$[s_2, s_3]$	$[s_{1/4}, s_{1/3}]$
g_3	$[s_{1/2}, s_1]$	$[s_1, s_2]$	$[s_{1/2}, s_1]$	$[s_{1/3}, s_{1/2}]$	$[s_{1/2}, s_1]$
g_4	$[s_{1/3}, s_1]$	$[s_{1/4}, s_{1/3}]$	$[s_1, s_2]$	$[s_{1/3}, s_{1/2}]$	$[s_1, s_2]$

Step 1： Calculate the $\tilde{s}_i'$ value by using $\frac{1}{\tilde{s}_i'} = \frac{s_{\omega_1}}{\tilde{s}_1} \oplus \frac{s_{\omega_2}}{\tilde{s}_2} \oplus \frac{s_{\omega_3}}{\tilde{s}_3} \oplus \frac{s_{\omega_4}}{\tilde{s}_4}, i = 1,2,3$ to aggregate all the decision matrices $A^{(k)} = (a_{ij}^{(k)})_{4\times 5}$, suppose that the weight vector of the four attributes is $s_\omega = (s_2, s_1, s_1, s_2)^T$, then we have the collective decision matrix $\tilde{A}$ as listed in Table 4.

Table 4 . The collective decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
$\tilde{s}_1'$	$[s_{3/37}, s_{4/29}]$	$[s_{1/7}, s_{6/25}]$	$[s_{1/10}, s_{1/5}]$	$[s_{1/12}, s_{2/15}]$	$[s_{2/15}, s_{6/23}]$
$\tilde{s}_2'$	$[s_{1/10}, s_{3/17}]$	$[s_{2/29}, s_{3/28}]$	$[s_{1/11}, s_{2/13}]$	$[s_{3/35}, s_{1/8}]$	$[s_{1/12}, s_{3/19}]$
$\tilde{s}_3'$	$[s_{1/17}, s_{2/19}]$	$[s_{1/16}, s_{1/11}]$	$[s_{2/21}, s_{3/19}]$	$[s_{2/23}, s_{3/22}]$	$[s_{1/10}, s_{1/6}]$

Step 2： Utilize the weight vector of DMs in the form of linguistic labels $s_\nu = (s_1, s_2, s_{1/2})^T \in S$, the weight vector associated with the IUPLHHA operator $w = (0.2429, 0.5142, 0, 2429)^T$, which is derived by the Gaussian distribution based method (see [20] for more details), and the IUPLHHA operator

$$a_j = IUPLHHA_{s_\omega, w}(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \langle s_{u_3}, \tilde{s}_3 \rangle) = \frac{1}{w_1\gamma_{1j} \oplus w_2\gamma_{2j} \oplus w_3\gamma_{3j}}$$

to derive the collective overall preference value a_j of alternative $x_j (j = 1, 2, \cdots, 5)$ given by the three DMs：

$a_1 = [s_{0.07}, s_{0.14}]$, $a_2 = [s_{0.08}, s_{0.12}]$, $a_3 = [s_{0.10}, s_{0.17}]$, $a_4 = [s_{0.09}, s_{0.13}]$, $a_5 = [s_{0.10}, s_{0.18}]$.

Step 3：Comparing each a_j with all a_i , $i, j = 1,2,3,4,5$ by using Eq. (1), we let $p_{ij} = p\left\{a_i \geq a_j\right\}$, then we construct a complementary matrix $P = \left(p_{ij}\right)_{5\times5}$:

$$P = \begin{bmatrix} 0.5 & 0.5455 & 0.2857 & 0.4545 & 0.2667 \\ 0.4545 & 0.5 & 0.1818 & 0.3750 & 0.1667 \\ 0.7143 & 0.8182 & 0.5 & 0.7273 & 0.4667 \\ 0.5455 & 0.6250 & 0.2727 & 0.5 & 0.2500 \\ 0.7333 & 0.8333 & 0.5333 & 0.7500 & 0.5 \end{bmatrix}$$

Step 4：Sum all elements in each line of matrix $P = \left(p_{ij}\right)_{5\times5}$, and we have

$$p_1 = 2.0524 \ , \ p_2 = 1.6780 \ , \ p_3 = 3.2265 \ , \ p_4 = 2.1932 \ , \ p_5 = 3.3499$$

Then we rank the arguments a_j ($j = 1,2,3,4,5$) in descending order in accordance with the values of p_i ($i = 1,2,3,4,5$):

$$a_5 > a_3 > a_4 > a_1 > a_2$$

Step 5：Rank all the alternatives x_j ($j = 1,2,3,4,5$) in accordance with a_j ($j = 1,2,3,4,5$):

$$x_5 \succ x_3 \succ x_4 \succ x_1 \succ x_2$$

and thus the most desirable alternative is x_5 .

6. Concluding remarks

In this paper, we have developed an approach to multiple-attribute group decision making based on a new aggregation operator called the IUPLHHA operator and we have focused on an application in a group decision making problem regarding the selection of investments. We also have studied several desirable properties of the new operator. Moreover, we have developed some uncertain pure linguistic aggregation operators such as the UPLWHA operator and the IUPLOWHA operator.

The IUPLHHA operator can be used in situations where the input arguments are uncertain pure linguistic variables and it reflects the importance degrees of both the given uncertain linguistic variables and their ordered positions. Also, the IUPLHHA operator can alleviate the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. Moreover, based on the IUPLHHA operator, we have given a method for multiple-attribute group decision making with uncertain pure linguistic information.

In future research, we expect to develop further extensions by adding new characteristics in the problem and its applications such as the fields of data mining, information fusion, and pattern recognition, etc.

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