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CREDIT RISK DEPENDENCE MODELING FOR COLLATERALIZED DEBT OBLIGATIONS

Abstract: *The cost of protection on a CDO tranche is driven, to a large extent, by the default dependence among entities composing the underlying portfolio. We propose a CDO valuation model where we independently fit default intensities to CDS market data and then calibrate different types of copulas to match the portfolio default dependence. We have adapted our model to accommodate the after crisis market conditions that changed the way CDO tranches are quoted and traded. The model is calibrated to replicate the up-front fees for the iTraxx Europe tranches. Our approach provides a good approximation of market data and allows for performance comparison of different classes of copulas.*

Keywords**:** CDO, copula, default intensities, dependence structure, multivariate distributions

JEL classification: C13, G12, G13, G21, G24

1 Introduction

Credit derivatives allow investors to trade credit risk in the same way they trade market risk. This is one of the reasons why credit derivatives market has been the most innovative and fastest growing in the past decade. The most popular multi-name product of this market is the collateralized debt obligation (CDO).

The market standard tool for pricing CDOs is the one factor Gaussian copula model introduced by Li (2000). In this setting the univariate distribution of time to default of each asset is derived from market data and then the joint distribution is specified using a one factor Gaussian copula. The main assumption of this methodology is that one parameter is sufficient to model the correlation of times to default among all assets in the portfolio. The literature concerning credit risk dependency modeling has evolved around the copula concept. Many different types of copulas have been proposed such as *t*-copula by O'Kane and Schögl (2005), double *t*-copula by Hull and White (2004), Clayton copula by Friend and Rogge (2005), Gumbel copula by Choroś-Tomczyk (2010) or normal inverse Gaussian by Kalemanova, Schmid and Werner (2007). Hofert and Scherer (2011) provide a comprehensive comparison of different types of Archimedean copulas both in exchangeable and nested form. The method called implied copula proposed by Hull and White (2006) determines the copula parameter implied by market quotes in a manner analogous to that used to derive the implied volatility in the Black-Scholes option pricing model. The implied correlations derived in this way are not equal among tranches depicting the so called correlation smile. We also refer to Burtshell, Gregory and Laurent (2008) for a comparison of selected copula approaches. Alternative valuation methods are the random factor loading model as implemented by Andersen and Sidenius (2004) or intensity based models proposed by Duffie and Gârleanu (2001). The top-down approach for CDO valuation represents another stream of models represented by Schönbucher (2005) and Filipovic, Overbeck and Schmidt (2011).

In this study we propose a CDO valuation model based on default intensities calibrated to market data and credit risk dependence modeled with copula functions. For the empirical study we used iTraxx Europe Series 15 data, retrieved from the Bloomberg database. The analysis was conducted on after the crisis data which led to supplemental challenges due to the changes in quotation styles for CDSs and CDO tranches. Liquidity has become an important factor and therefore we have chosen a period where market quotes were not influenced by lack of liquidity.

The paper is organized as follows. In Section 2 we introduce the intensity based approach used for default modeling and review the computation of CDO payment streams. In Section 3 we present the concept of introducing dependence via copulas and describe the methods we used to sample from different types of copulas. Model calibration to market data is explained in Section 4. Section 5 presents the results and Section 6 concludes.

2 Synthetic CDO Valuation

Portfolios of credit derivatives and CDOs in particular have shown an impressive growth in the period preceding the crisis. This is partly attributable to the optimistic ratings provided by rating agencies and to their appealing characteristics such as low initial investment and explicit choice of risk/return profile.

A *cash* CDO is a specific type of Asset Backed Security where the underlying portfolio is constituted of corporate bonds. In order to set up such a deal the originator would create the portfolio by buying the bonds and then create a waterfall structure to allocate the cash flows generated by the portfolio into different tranches ranked by seniority. Cash CDOs were not popular because of the initial investment required to finance the bond portfolio.

Two important market innovations led to the increasing popularity of multi-name credit derivatives. In a first market development it was recognized that a long position in a bond has a similar risk to a short position in a CDS written on the same entity that issued the bond. This led to an explosion of CDS issuance and trading that facilitated the development of an alternative structure named *synthetic CDO*. The originator of a synthetic CDO chooses a portfolio of companies and sells CDS protection on each of the names. The maturity of the structure is given by the maturity of the CDS contracts and the principal is given by the sum of the notional of the underlying CDSs. Inflows generated by the CDS spreads and outflows generated by defaults constitute the cash flows of the structure that are allocated to form the tranches. Because of the straightforward waterfall structure and minimal initial investment synthetic CDOs have become very popular. The second market development was the *single-tranche trading*. This term is used to describe the prevalent form of synthetic CDOs which involves trading of tranches without the underlying portfolio of short CDSs being created. The portfolio of short CDS positions is used only as a reference to define the cash flows between parties engaged in tranche transactions.

The two most important reference points for synthetic CDO tranches are the CDX NA IG and iTraxx Europe indices. These indices cover 125 investment grate companies in United States and Europe respectively. Market participants have used the portfolios of CDSs underlying these indices to define standardized synthetic CDO tranches. The purpose of this study is the valuation of these CDO tranches.

2.1 The credit curve

The most important quantity in pricing multi-name credit derivatives is the portfolio loss process. In the case of CDOs this stochastic process can be derived from individual times to default of underlying reference entities.

We assume the existence of a filtered probability space where is a pricing measure calibrated to market quotes. The reference portfolio is comprised of companies and their times to default are given by a positive random variable . The default status of each entity is specified via an intensity model, also called hazard rate model. The intensity used to derive the default probabilities is assumed to be a deterministic, nonnegative function denoted by . The term structure of unconditional default probabilities (i.e. the credit curve), is related to by the following function:

(1)

and the corresponding survival probability is given by:

(2)

In the following we adopt the canonical model for constructing as it is implemented by Schönbucher (2003), page 122 for its suitability to simulation. If, the variables are uniformly distributed in and independent of , then:

(3)

The default times , are obtained by taking the inverse of the survival function at points . The greatest challenge in modeling the portfolio loss process is to determine the joint distribution of the times to default. We introduce dependence among the stopping times by making the trigger variables dependent. Multivariate copula functions are a suitable method to specify the dependence structure with given margins. By assuming this setting on which the dependency is specified independently of the univariate margins the distribution function of the loss process is not available in an analytical form. However, the loss process can be simulated, via Monte Carlo, by sampling from different types of copulas.

2.2 Present value of expected cash flows

Generally speaking, CDSs and CDOs provide protection against defaults in exchange for a fee that can take the form of a spread, an up-front fee or both. In a typical CDO contract the protection buyer pays a spread , quarterly in arrears, times the outstanding principal. In case of a default the outstanding principal is reduced and two payment streams occur. First there is an accrual payment to bring the periodic payments up to date and then the protection seller reimburses the buyer for the losses caused by the default.

The credit crisis that started in 2007 caused some changes to the way credit derivatives are quoted and traded; two of them are important for the purpose of this analysis. First, spreads for CDS and CDS indices became very volatile with large swings caused more by market sentiment than true perception of risk. As a result, even though CDSs continued to be quoted as a spread, the actual transaction is carried out as an up-front fee calculated as the present value of the difference between the CDS spread and the bond coupon. The following relation:

(4)

where stands for the duration of the CDS payments, for the quoted spread and for the bond coupon, denotes the required up-front fee that an investor has to pay in order to receive protection. For the remaining maturity the investor would also pay the fixed coupon times the outstanding principal. This arrangement facilitates trading by decoupling future periodic payments from the spread that was prevalent in the market at transaction date. The second important change is the implementation of the same CDS trading arrangement to the CDO tranches. Before the crisis it was typical to assume a running-spread of 500bp only for the most subordinate tranche. This tranche was quoted and traded as an up-front fee, expressed in percentage points, while all the others were quoted as a spread that would be applicable to all future periodic payments. After the crisis all tranches, including the super senior, are quoted as up-front fees expressed in percentage points with different fixed running-spreads expressed in basis points. This follows the same reasoning of making the future payments independent of the spread at the time of the transaction.

In the following setup we assume an equally weighted CDO portfolio consisting of reference entities, tranches and maturity . The payment schedule denotes the specific dates on which the premium payments are made. Taking account of the changes described above, we define as the expected principal of tranche at time expressed as a percentage of initial tranche principal. The discount factors of the form , where is the continuously compounded interest rate, give the present value of received at time . The spread is the number of basis points paid per year in order to buy protection on tranche .

A reference entity is deemed to default before time , if . Then the loss variable is defined as:

(5)

The portfolio loss process at time is the average of all losses:

(6)

where is the deterministic recovery rate applicable to all companies. Alternatively, can be replaced with LGD (loss given default) as an equivalent measure of loss severity.

The losses are absorbed by tranches in order of seniority. The losses incurred by a particular tranche , at time are determined by the attachment point, , and detachment point . Therefore the remaining principal of tranche at time is given by:

(7)

The defaults may happen anywhere in however, in order to simplify the computation we discretize the time frame according to the payment schedule and defer all defaults that happen on or before time to the middle of the interval . Assuming defaults occur on continuous time may seem a more realistic assumption however it does not materially affect the results and serve only to clutter the notation.

The value of a contract is the present value of the expected cash flows. This is also the case with CDO tranches, and this present value involves three terms. The present value of the expected regular spread payments is given by:

(8)

where .

The regular payments from the buyer of protection to the seller cease in case of a credit event. However, due to the fact that the payments are made in arrears, a final accrual payment is required. The present value of the expected accrual payments is:

(9)

The present value of the expected payoffs caused by defaults is given by:

(10)

A remark about the recovery rate

Provided that the same recovery rate is used both for estimating default probabilities and CDS valuation, the value of a CDS contract is not sensitive to the recovery rate because the default probability is proportional to and the payoffs in case of default are proportional to . However this is not the case with synthetic CDO tranches. By applying the methodology described above, every time a default occurs the total notional of the CDO portfolio on which the payments are made is reduced by whereas the total notional of the CDSs underlying the CDO portfolio reduces by . In practice the notional of the most senior tranche is reduced by every time there is a loss that affects junior tranches and directly by every time there is a default affecting the senior tranche. In this way the outstanding notional of the CDO is synchronized to the sum of the notional of the underlying CDSs.

2.3 CDO Pricing

Pricing a CDO involves determining the breakeven spread or up-front fee that would make the present value of the payments equal to the present value of the payoffs. In case the tranche is quoted as an up-front fee pricing implies determining the up-front fee that would compensate for the arbitrary running-spread. From the perspective of the protection buyer the breakeven up-front payment is given by:

(11)

where the superscript in and denote that has been replaced by the specific running-spread, , according to the specification of the deal.

Assuming deterministic discount factors, Equation (11) only requires the computation of the portfolio loss at each point of the payment schedule. Unfortunately, as can be seen from Equation (7), the loss affecting a certain tranche and therefore the remaining principal is not a linear function of the individual loss variables . As a consequence the expected trance principal cannot be determined analytically and has to be computed via simulation.

In the following we present the Monte Carlo routine for pricing CDO tranches:

Step 1. Provide all inputs that specify the model: maturity , the payment schedule , number of companies , hazard rates , discount factors , recovery rate , the set of attachment and detachment points for each tranche, number of simulation runs , a copula for the default triggers and the vector that parameterizes the copula.

Step 2. Given and compute the survival probabilities as indicated in Equation (2).

Step 3. For each of the iterations sample from the chosen copula as described in Section 3. Then compute the default times according to Equation (3) and based on them compute the portfolio loss for each of the points in the payment schedule as indicated in Equation (6). Having the portfolio loss profile, for each tranche and each compute the remaining tranche principal as described in Equation (7).

Step 4. Having for each of the iterations compute the present value of the regular spread payments, the accrual payment and the payoffs in case of default as in Equations (8), (9), (10) and take the expectation as their sample means.

Step 5. Compute the up-front fees as in Equation (11) but using the sample means generated at *Step 4*. The results would have the following form for the breakeven up-front fee:

(12)

Step 6. Repeat *Steps 3-5* and search for copula parameters that minimize the objective function .

2.4 Objective function

The purpose of the model is to calibrate the copula so that the computed breakeven up-front fees would reproduce as accurately as possible the values observed on the market. The main goal of the calibration is to minimize de cumulative absolute deviations of the computed up-front fees from the market up-front fees with respect to the copula parameter :

(13)

3 Specifying the dependence structure

We specify the dependence structure among default triggers using Gauss, Student's *t* and 5 families of Archimedean copulas. In this section we will point out the relevant characteristics of these copulas and present efficient sampling algorithms.

3.1 Elliptical and Archimedean Copulas

Elliptical copulas are derived from the elliptical distributions using Sklar's theorem (1959). The most prominent versions are Gauss and *t*-copula. These copulas have been extensively studied in the literature and we will not continue with their description however, we point out that Gauss copula has evolved as the market standard for CDO valuation and despite its critiques we will use it as a benchmark.

Archimedean copulas are related to the Laplace transforms of univariate distribution functions. According to Joe (1997) if we denote by the class of Laplace transforms that consist of strictly decreasing differentiable functions than the function defined as:

(14)

is a *d*-dimensional exchangeable Archimedean copula where is called the generator function and is the copula parameter. For a comprehensive description of copulas we refer to Nelsen (2006). Table 1 presents the selected Archimedean copulas and parameters relevant to this study.

Table 1: Parameter space, generator functions and Kendall's coefficients.

Family
Gumbel
Clayton
Frank
Joe
A-M-H

Nested Archimedean copulas provide an efficient way to recursively define the dependence structure. However, fitting a fully nested structure to a large data set is unfeasible. As an alternative to the fully nested model, we can consider copula functions with arbitrary combinations at each level. The high dimensionality of our data set and the knowledge about sectorial repartition of the companies made us particularly interested in nested structures given by the following form:

(15)

where is the generator function, , is the dimension of the outer copula and with is the dimension of the inner copulas. This copula has margins and it is easier to model because the number of parameters is much smaller than . As demonstrated by McNeil (2008) a sufficient condition for this structure to be a copula function is that the parameters satisfy for any . Okhrin, Okhrin and Schmid (2012) provide several methods for determining the optimal structure.

3.2 Sampling algorithms

Algorithm 1 - Sampling from Gauss copula

(1) Perform a Cholesky decomposition on the correlation matrix to obtain the factor that satisfies

(2) Sample

(3) Compute

(4) Return

Algorithm 2 – Sampling from copula
(1) Sample , as was generated in *Algorithm 1*
(2) Sample independent of and compute
(3) Compute
(4) Return

Archimedean copulas are convenient to work with as they are fully specified by some generator function. For the purpose of simulating from an exchangeable Archimedean copula Marshall and Olkin (1988) proposed a method that is extremely efficient for large samples. The idea behind this approach is based on the fact that Archimedean copulas are derived from Laplace transforms.

Algorithm 3 – Sampling from exchangeable Archimedean copula
(1) Sample from inverse Laplace transform of i.e .
(2) Sample independent
(3) Return where

As long as the distribution function can be sampled the Marshal – Olkin algorithm becomes very fast even for large dimensions. This major advantage comes from the fact that it suffices to know the inverse Laplace – Stieltjes transform of the copula generator. For the specific case of the Gumbel copula we generate a positive stable variable where and . For the Clayton case we generate a variable with .

In particular the contributions of McNeil (2008), Hofert (2008) and Hofert (2011) provide the theoretical foundations to efficiently sample random vectors from nested Archimedean copulas.

Algorithm 4 – Sampling from nested Archimedean copula (Hofert and Mächler (2011)). Let be a nested Archimedean copula with root copula generated by and let be a vector of the same dimension as .
(1) Sample from inverse Laplace transform of i.e .
(2) For all components of that are nested Archimedean copula repeat:
a) set with generator to the nested Archimedean copula
b) sample
c) set , and
(3) For all other components of repeat:
a) sample
b) set the component of corresponding to to
(4) Return

Sampling from nested Archimedean copulas when all generators belong to the same parametric family only require to know:

, (16)

as all distribution functions have the same form as , but with different parameters. The algorithm goes through the recursively determined structure of the nested copula and samples from and . Hofert and Mächler (2011) provide the theoretical foundation for this algorithm and tables with analytical forms for the inverse Laplace transforms of different classes of copulas.

4 Implementation of the model and calibration

We calibrate our model following a two steps process. First we construct a credit curve by fitting a step function of hazard rates to match market quotes of CDS spreads under the risk neutral measure. Second, the dependence structure is determined by calibrating the parameters of the copula from which the default triggers are sampled. We also calibrate so as to determine the market implied LGD. An important advantage of this method is that default probabilities are specified and modeled independently of the dependence structure.

4.1 Data

The empirical part of this study was carried out on the iTraxx Europe index which is comprised of the most liquid 125 CDSs referencing European investment grade credits. The constituents of the index are changed every six months in a process known as rolling the index that is meant to replace the companies that are no longer investment grade. Every time the index in rolled a series is created and for the time until the next roll it is called the on-the-run series. The index trades with maturities of 3,5,7 and 10 years. The market has used the portfolios underlying this index to define standardized tranches that cover losses in the ranges of and . For this analysis we have used Series 15, that initiated on March 20th, 2011, with a maturity of 5 years because it had the highest liquidity among all traded tranches on series issued after the crisis. For the reasons outlined in the remark about recovery rate in Section 2.2 we calibrate our model to match the up-front fees of the first 5 tranches. Consequently, the number of companies , the number of tranches and the attachment and detachment points are set according to the loss ranges provided by the first 5 tranches. The recovery rate is , a commonly accepted assumption. The iTraxx series impose quarterly payments in arrears and therefore we set the payment schedule to match the regular payment dates with and . In order to test the consistency in time we have calibrated the model on the following 5 days: 2011-06-01, 2011-06-22, 2011-07-21, 2011-08-10, 2011-09-30. For each of these days we took the observed market up-front fees for each of the tranches and the zero interest rates needed to compute the discount factors . As described in Section 2.2 tranche quotes imply an up-front fee and a running-spread. For the particular case of Series 15 the first 5 tranche running-spreads expressed in basis points are

. For the calculation of discount factors we have used the standard ISDA Curve for Euro which is comprised of EONIA and EURIBOR rates on the short end and the average of interest rate swap fixings for maturities greater than one year. For all calculations requiring present values we used the time convention. The model evolves around the Monte Carlo routine presented in Section 2.3 and up to this point we have defined all the variables required by *Step 1* except for the term structure of the hazard rates and copula specification that will be presented in the next sections. All data was retrieved from the Bloomberg Database.

4.2 Fitting the default intensities

In order to model the loss process we consider an inhomogenous portfolio by letting default intensities vary both across companies and time. To determine the step function of hazard rates we have used the term structure of CDS spreads up to 5 years. The idea behind this procedure is that given the term structure of the hazard rate that is complete up to time , find the hazard rate at time that is consistent with de CDS spread at time . To implement this procedure we have used a numerical root-finding algorithm based on Newton-Raphson method. The spread is an increasing function of hazard rate and we also assumed that this relation is linear. In general the algorithm is very fast because the convergence is quadratic, however the execution time is significantly influenced by the initial guess from which the algorithm starts. We mentioned this aspect because there is significant variability in CDS spreads across companies.

To determine the unconditional default probabilities (PD) at each we numerically integrated over the term structure of the hazard rates. Figures 1 and 2 present the results and give a clear indication that in order for a copula to accurately describe the dependence structure it should provide enough tail dependency to catch the extreme co-movement of the variables.

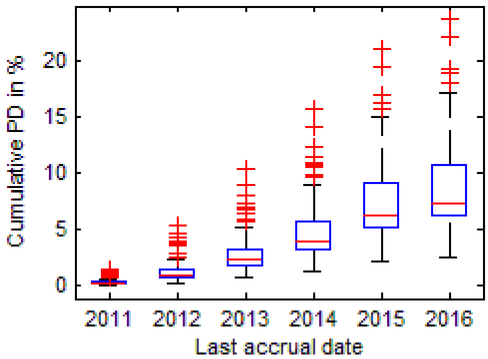

Figure 1. Distribution of cumulative PDs for all companies in the portfolio as seen from 2011-06-01

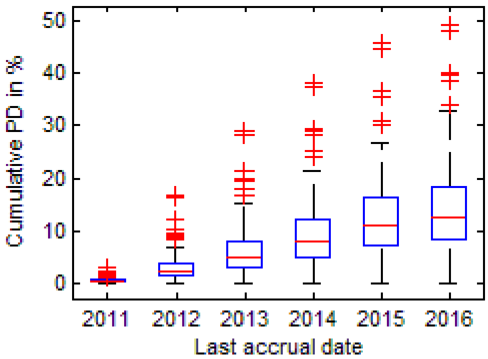

Figure 2. Distribution of cumulative PDs for all companies in the portfolio as seen from 2011-09-30

4.3 Calibration of the dependency structure

The choice of the copula in of crucial importance for the final result because in our model the dependence inherits the structure of the copula from which default triggers are drawn. The model can contain up to $1/2$ parameters if we assume a Gauss or *t*-copula and -1 parameters in the case of Archimedean copulas. For such high dimensionality calibration is unfeasible and therefore we introduce the following dimension reduction technique.

Empirical evidence has shown that companies in the same industry sector tend to have correlated performances. We have assigned the 125 companies in the index in the following sectors: Consumer Non-Cyclical (25), Financial (25), Consumer Cyclical (12), Basic Materials (11), Industrial (13), Communications (20), Energy (4) and Utilities (17).

In the case of elliptical copulas the simplest way to reduce dimensionality is to assume that all credits equally influence each other. In this case the copula would have only one parameter. However, having knowledge of sectorial composition of the index we introduced a second parameter to differentiate between intra and inter sector correlations. Therefore, in a first stage we calibrated the Gauss and *t*-copula with one correlation parameter. In a second stage we parameterized so that companies in the same sector would have correlation and companies in different sectors correlation . In both cases we assumed 4 degrees of freedom for the *t*-copula as in Hull and White (2004).

In the case of Archimedean copulas the information about sectorial composition can be used via nested copulas of the form in Equation (15). Using a partially nested copula we describe the dependency within the sector with a copula with parameter and then joined all the sectors with an outer copula with parameter .Therefore, we calibrated all the copulas listed in Table 1 both in their exchangeable (one parameter) and nested forms (two parameters).

Model calibration was performed with respect to the Kendall's tau parameter because it was efficient to minimize over a bounded parameter space. In case of the nested copulas we calibrated over 2 parameters when LGD was considered fixed and over 3 parameters when the LGD was implied by market quotes. Minimization of the objective function with respect to one parameter was performed using the Brent algorithm, described Nocedal and Wright (2006), that uses a combination of golden section and parabolic interpolation. For the two and three parameter copulas the multi-dimensional minimum of the objective function was computed using a box constraint optimization algorithm based on BFGS methodology and implemented by Nocedal and Wright (2006). Our choice for the method is supported by the fact that it is a direct-search algorithm that uses only function values, does not require calculation of derivatives and admits linear constraints on parameters. At each iteration we performed simulations.

5 Results of calibration

Our objective was to replicate all tranche prices simultaneously by calibrating over the parameter space of the copula in order to minimize the measure D. Results generated after calibrating for all types of copulas indicated that the model is consistent across time despite the significant changes in the credit market. Credit conditions worsened during 2011 on the background of stagnating European economy and deteriorated perception on sovereign creditworthiness. While Series 15 was on the run the CDS index for 5 year maturity increased from 102.96 to 202.45 bp. Tables 2 and 3 present the estimated tranche up-front fees and measure D for 2011-06-01 and 2011-09-30 respectively. We chose to present these dates in detail as they clearly reflect model consistency through different market conditions. Conclusions were drawn based on results generated for all calibrated days.

A crucial aspect regarding calibration is that up-front payments for junior tranches are inversely correlated with the dependence among firms. However, this is exactly the opposite with senior tranche as their up-front fees are positively correlated with the dependence among firms. This induces a tendency for the model to overprice senior tranches for which we found several justifications. First, we calibrate the copula so as to reflect the overall portfolio credit dependency. Since the highest up-front fee is paid for tranches 0-3% and 3-6% it is normal that they would have the highest influence in determining the credit dependence. This overestimation of default dependence induced by the first two tranches extrapolates by increasing the risk, and therefore the price, for the senior tranches. Second, the periodic payments for all tranches are arbitrarily determined by the running-spread. Since superior tranches have significantly lower running-spreads than junior tranches it is normal for the model to compensate by increasing the up-front fee.

Table 2. Calibrated tranche up-front fees and measure for selected copula classes. The indicators listed after the names of the copulas indicate whether the calibration was performed over 1 or 2 parameters.

2011-06-01	Kendall's tau		**Trance up-front fees (%)**					
			0-3%	3-6%	6-9%	9-12%	12-22%	
Market			**38.25**	**4.60**	**1.87**	**1.94**	**0.91**	
Gauss 1	0.28		39.79	4.73	6.05	7.59	2.42	13.00
Gumbel 1	0.31		37.68	3.00	3.42	3.26	2.04	6.15
Gumbel 2	0.24	0.39	37.69	4.91	2.45	3.38	1.75	3.72
Gumbel 2 0.58 LGD	0.22	0.41	38.05	4.19	2.43	3.22	0.98	2.52
Rotated Clayton 1	0.14		36.98	9.86	6.68	6.08	2.09	16.65
Rotated Clayton 2	0.12	0.21	36.25	9.01	6.32	5.53	2.22	15.75

Frank 1	0.43	37.67	8.04	5.95	8.83	2.61	16.68
Frank 2	0.42 0.45	38.09	8.25	6.01	8.84	2.36	16.28
AMH 1	0.33	47.04	18.92	7.83	6.46	1.69	34.36
AMH 2	0.30 0.33	48.16	17.6	7.91	5.85	1.90	33.84
Joe 1	0.24	39.35	4.35	2.76	5.16	2.16	6.71
Joe 2	0.17 0.32	37.91	4.03	1.20	4.42	1.16	4.31

Table 3. Calibrated tranche up-front fees and measure for selected copula classes.

2011-09-30	Kendall's tau	**Trance up-front fees (%)**					
		0-3%	3-6%	6-9%	9-12%	12-22%	
Market		**61.44**	**27.65**	**19.14**	**4.91**	**2.34**	
Gauss 1	0.40	59.63	28.41	24.07	14.01	9.62	23.87
Gumbel 1	0.39	62.20	27.03	21.92	6.70	5.29	8.89
Gumbel 2	0.34 0.48	61.24	27.83	19.73	6.24	4.72	4.67
Gumbel 2 0.64 LGD	0.33 0.45	61.08	27.62	19.33	6.43	4.77	4.52
Rotated Clayton 1	0.26	54.54	27.69	22.81	20.32	13.73	37.40
Rotated Clayton 2	0.22 0.28	58.27	31.16	24.70	16.99	11.5	33.47
Frank 1	0.43	62.76	30.56	24.22	23.98	13.65	39.68
Frank 2	0.42 0.5	58.80	30.01	24.36	24.31	10.77	38.04
AMH 1	0.30	77.14	33.53	24.51	17.93	9.78	47.40
AMH 2	0.30 0.33	76.99	32.50	25.42	19.06	10.03	48.51
Joe 1	0.35	61.30	28.53	21.51	10.51	5.35	11.99
Joe 2	0.24 0.52	61.55	27.05	20.17	7.74	5.97	8.19

Third, the market itself may be inefficient in pricing superior tranches. This belief is based on the fact that investors are more concerned with correctly pricing junior tranches as these have a higher probability of being hit by defaults and don't manifest the same diligence when pricing superior tranches. Therefore, we may conclude that the model performs consistently across tranches.

Among all tested copula families the two parameter Gumbel copula performed best. This was also our a priori expectation because Gumbel copula exhibits upper tail dependence, that is, it is more suitable to describe outcomes that simultaneously produce upper tail values. Joe copula performed well and close to Gumbel and therefore we consider it suitable for this type of analysis. This consideration is intuitive as they have the same tail characteristics and parameters spaces. Except for the Gumbel copula, all the copula classes do not materially improve the performances of the model by calibrating over 2 parameters instead of one. This provides a solid reason to conclude that it is the structure of (tail) dependence that matters most and not a particular value of the parameter. Clayton copula, even though it was used in the rotated form showed weak performance and therefore sustains the fact that it is not suitable in this modeling context. This is also the only copula that has a tendency to fit in between, that is, to underprice junior tranches and overprice senior ones. AMH and Frank copulas, as expected, performed worst due to their lack of tail dependence. In addition AMH has a restricted parameter space which prevents it from capturing enough dependence. This is the reason why calibrated parameters for the AMH copula came very close to the upper limit of the parameter space.

Even though calibrations for exchangeable and nested copulas were independent the parameter calibrated in the exchangeable form always fell between the parameters calibrated for the nested form. This leads us to conclude that information about sectorial repartition has a significant influence on the dependence structure. Even more so, for every copula and every calibrated day the inter sector parameter was lower than the intra sector one. This reinforces our belief that dependence among companies is clustered according to industry sectors. The naïve Gaussian model does not properly capture the dependence characteristics of the portfolio. What is even more discouraging to using this type of copula is that its performance decreases with the strength of dependence. This finding is supported by the fact that it performed worst in relative terms during the analyzed time period that exhibited worsening credit conditions and increasing dependence.

Since the Gumbel copula provided the best results we calibrated this model over 3 parameters in order to determine a market implied LGD. Our results support the empirical findings namely that the LGD is positively correlated with the dependence among companies. However, we cannot make inferences about whether the LGD is statistically different from the commonly accepted value of 0.6. We may also note that the computational complexity implied by deriving LGD in such a manner undermines the relevancy of the results.

While performing calibration for the 2 and 3 parameter copulas we have encountered various numerical issues. Repeated calibration over the same data and parameter space returned different minimum values for the objective function. This was due to the fact that the optimizer is sensitive to the initial value of the parameters and according to where it starts on the parameter space it might get stuck into a local minimum. In order to overcome this numerical issue we performed the minimization of the objective function in two steps. First we evaluated the function on a loose grid of points spanning the initial parameter space and determined a restrained interval for the parameters. Second, we ran the optimizer over the restricted interval and set up the starting point in the middle of the constraint interval. In some cases unrestricted optimization performed with the Nelder-Mead downhill simplex algorithm starting from the centroid of the restricted interval returned the global minimum.

All programs developed for this study were implemented in Matlab and ran on a shared server with 16 processors and 64GB of RAM. We followed most of the best practices in Matlab programming such as parallel computing, vectorization and memory pre-allocation.

6 Conclusion

We proposed a CDO pricing model that used default intensities calibrated to CDS spreads and 7 copula functions to describe the dependence structure. For each of the copula types we calibrated the model so that it simultaneously reproduces the quoted up-front fees for all tranches. We made use of the fact that default correlation tends to be higher for the companies pertaining to the same industry sector. We showed that, given the changes that affected the credit market, the CDO pricing model for which the dependency structure is given by a nested two parameter Gumbel copula yields the smallest pricing errors.

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