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ANOTHER CATEGORY OF THE STOCHASTIC DEPENDENCE FOR ECONOMETRIC MODELING OF TIME SERIES DATA

***Abstract**. In this paper we propose a modality for conceiving a "data-generating process", from which classical econometric modeling starts, by showing a β-dependence between random variables defined on probability spaces different in general, which create a random phenomenon S that takes place in several "steps" and with evolution in discrete time.*

***Key words**: measurable spaces and mappings, probability spaces, product of probability spaces, stochastic transition function, stochastic processes, time series, economic variables, econometric model.*

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1. Introduction

In classical econometric models, linear regression models or nonlinear regression models, simultaneous equation models, systems of regression equations models, dynamic regression models, time series models and others, economic variables of the real world economy are econometrically modeled by random variables defined on the same probability space [3], [4], [5], [7].

The supposition that all random variables which interfere within econometric models are defined on the same probability space, allows us to highly elaborate precepts which have led to the development of econometric techniques. Here are a few examples of these precepts:

- correlation precept, essential in linear or nonlinear regression [4], [7];
- autocorrelation precept, essential in the autoregressive models [4], [6];
- co- integrated systems and multivariate co integration [2], [6].

Remark 0. The above-mentioned supposition allows us to calculate the correlation coefficient between any two random variables, although there is no economic ground.

In the paper we take into consideration another approach of econometric modeling which means:

a. to replace the above-mentioned hypothesis with random variables hypotheses which interfere with the econometric models are defined on different probability spaces;
b. the economic variables econometrically modeled are part of variables systems which evolve in discrete time, are part of an interdependent chain or circular type.

For example, we may consider the system made of the following three macroeconomic indexes (variables): GDP, consumption and investments. This system represents a subsystem of the whole system of indexes which are used for the entire economy that are part of the National Accounts System.

Making use of the index system evolution during 1990-1995, where the indexes are:

Index	Measurement unit	1990	1991	1992	1993	1994	1995
GDP	Bill RON	857.9	2203.9	6029.2	20035.7	49767.6	72559.7
Consumption(C)	Bill RON	679.5	1672.5	4642.5	15235.8	37417.5	56315.4
Investment(I)	Bill RON	168.410	314.014	888.566	2821.819	8004.621	12995.49

Source*: The Romanian Statistics Yearbook, 1996*

According to b., the 3 indices I_1 (GDP), I_2 (consumption) and I_3 (investments) are components of a random system *S*, with evolution in discrete time. Each index is econometric modeled by a stochastic process defined on a proper probability space.

We consider $I_i(t),\ t \in \mathbf{N}^*$, random variable at *t* moment which models index $I_i,\ i = \overline{1,3}$. Then, $I_i(t+1)$ depends on $I_i(t)$ due to the circular dependency and also on $I_j(t)$ or on $I_j\ (t+1)$ due to the chain dependency.

The elements above mentioned have suggested a definition of another type of dependency called *β-dependence* between random variables defined on spaces of different probability.

The econometric models constructed on the grounds of the hypothesis a. using *β-dependence* are different from the classical econometric models.

The paper consists of the fundaments of econometric modeling of a random phenomenon *S* that takes place in several "steps" and with evolution in discrete time.

In conclusion, *β-dependence* proposes a means of conceiving a "data-generating process". This process for time series data is studied in econometric literature and proposes different methods to derive synthetic time series [5].

In this paper, we use notices and results from [1].

2. Probabilistic modeling to generate a statistical series

Let $m \in \mathbf{N}^*$. For $i = \overline{1,m}$ and $t \in \mathbf{N}^*$ we denote by $\mathrm{V}_i(t)$ the set of the real random variables which are defined on probability space $(\Omega_i(t), \mathcal{K}_i(t), P^i(t))$, where $\Omega_i(t)$ is finite or denumerably infinite.

Let us take the functions $X_i : \mathbf{N}^* \to V_i(t), i = \overline{1,m}$ and let us take *m-tuples* $X(t) = (X_1(t), X_2(t)), \ldots, X_m(t)), t \in \mathbf{N}^*$.

For a fixed *m-tuples* $X(t) = (X_1(t), X_2(t)), \ldots, X_m(t))$, let us consider for $i \in \{1,...,m\}$, $x_i(t) = X_i(t)(\omega)$, with $\omega \in \Omega_i(t)$, where $x_i(t)$ means the value of the random variable $X_i(t)$, at $t \in \mathbf{N}^*$ time.

We consider m conditions bivalent $C_1, C_2, ..., C_m$, that refer to the values $x_i(t)$, $\forall t \in \mathbf{N}^*$, $i = \overline{1,m}$; more accurate, for fixed i, the condition C_i refers to $x_i(t)$ and $x_j(t)$, $j \neq i, \forall t \in \mathbf{N}^*$ is not about.

With those m conditions define a dependence between $X_1, ..., X_m$.

Definition 1. Called $C_1, C_2, ..., C_m$*– dependence* or *β–dependence* between $X_1, ..., X_m$, the dependence defined by the following two procedures.

a) <u>X_{i+1} *depends on* X_i,</u> , $i = \overline{1, m-1}$, so:

At $t \in \mathbf{N}^* \backslash \{1\}$ time, the space of probability that the real random variable $X_{i+1}(tI)$ is defined depends on the value $x_i(t)$ as:

- if $x_i(t)$ meet the condition imposed by the C_i, then: $\Omega_{i+1}(t) \supseteq \Omega_{i+1}(t-1)$, $\mathcal{K}_{i+1}(t) = \mathcal{P}(\Omega_{i+1}(t))$, $P^{i+1}(t) \neq P^{i+1}(t-1)$ and $P^{i+1}(t)$ is part of the same class of probability laws, as well as $P^{i+1}(t-1)$, $X_{i+1}(t) \neq X_{i+1}(t-1)$, $X_{i+1}(t) : \Omega_{i+1}(t) \to \mathbf{R}$ there is the following restriction $X_{i+1}(t)\big|_{\Omega_{i+1}(t-1)} = X_{i+1}(t-1)$.
- if $x_i(t)$ does not meet the condition imposed by the C_i, then: $\Omega_{i+1}(t) = \Omega_{i+1}(t-1)$, $\mathcal{K}_{i+1}(t) = \mathcal{K}_{i+1}(t-1)$, $P^{i+1}(t) = P^{i+1}(t-1)$ and $X_{i+1}(t) = X_{i+1}(t-1)$.

Note that if m=1 the a) procedure does not make sense.

b) <u>X_1 *depends on* X_m</u> so:

At t+1, $t \in \mathbf{N}^*$ time, the probability space on which $X_1(t+1)$ random variable is defined depends on the value of $x_m(t)$ as follows:

• if $x_m(t)$ meets the condition imposed by the C_m , then:

$\Omega_1(t+1) \supseteq \Omega_1(t)$, $\mathcal{K}_1(t+1) = \mathcal{P}(\Omega_1(t+1))$, $P^1(t+1) \neq P^1(t)$, $P^1(t+1)$ being part of the same class of probability distributions, such as $P^1(t)$, $X_1(t+1) \neq X_1(t), X_1(t+1) : \Omega_1(t+1) \to \mathbf{R}$ there is the following restriction $X_1(t+1)\big|_{\Omega_1(t)} = X_1(t)$.

• if $x_m(t)$ does not meet the condition imposed by the C_m , then:

$\Omega_1(t+1) = \Omega_1(t)$, $\mathcal{K}_1(t+1) = \mathcal{K}_1(t)$, $P^1(t+1) = P^1(t)$ and $X_1(t+1) = X_1(t)$.

We use *β– dependence* for the probabilistic modelled of a random phenomenon *S*, with an evolution in *m* "steps", on which *m* bivalent conditions $C_1, C_2, ..., C_{m-1}$ and C_m respectively acts. Each *i* step is econometric modeled at *t* time by the random variable $X_i(t)$ which is defined on the probability space $(\Omega_i(t)$, $\mathcal{K}_i(t)$, $P^i(t))$, where $\Omega_i(t)$ is finite or denumerably infinite $\forall i = \overline{1,m}, \forall t \in \mathbf{N}^*$.

According to the *β–dependence* makes sense relation
$S(t) \mapsto (X_1(t), X_2(t), \ldots, X_m(t))$, $\forall\ t \in \mathbf{N}^*$, which means that the state of the phenomenon S at *t* time is characterized by *m-tuples* $(X_1(t), X_2(t)), \ldots, X_m(t))$. In this case, the values of *m* random variables make *m* time series data whose generation is described by the evolution of the phenomenon *S*. Note that *β–dependence* induces ordering of the random variables.

Remark 1.

At *t*+1 time the phenomenon *S* has the state
$S(t+1) \mapsto (X_1(t+1), X_2(t+1), \ldots, X_m(t+1))$, which depends on the state *S*(*t*) with $t \in \mathbf{N}^*$, because $X_1(t+1)$ depends on $X_m(t)$ according to b) procedure of the *β-dependence*, and $X_{i+1}(t+1)$ depends on $X_{i+1}(t)$, $i = \overline{1, m-1}$, according to a) procedure.

Remark 2.

At *t* time, $t \in \mathbf{N}^*$, for $i = \overline{1, m-1}$ the effects of $x_i(t)$'s values upon the random variables $X_{i+1}(t)$ are instantaneous. In this way, the *simultaneity relations* between the *m* random variables is postulate.

Remark 3.

The b) procedure can be regarded as a variant of feedback between the state of phenomenon *S*.

Remark 4.

It can be assumed that $\exists t_0 \in \mathbf{N}^*$, so that at $t=t_0$ time, whatever $i = \overline{1,m}$ there is the random variable $X_i(t_0)$, defined on probability space

$(\Omega_i(t_0),\ \mathcal{K}_i(t_0),\ P^i(t_0)),\ \mathcal{K}_i(t_0) = \mathcal{P}(\Omega_i(t_0))$, at the start of the procedures a) and b).

Remark 5.

The *m* conditions refer to the values of the random variables, ensuring the compatibility with the phenomenon *S*. Each condition is bivalent, i.e. *yes* (*satisfied*) / *no* (*omitted*).

Remark 6.

The probability spaces $(\Omega_i(t),\ \mathcal{K}_i(t),\ P^i(t)),\ i = \overline{1,m}$ and $t \in \mathbf{N}^*$, are standard Borel spaces and we can apply the disintegration theorem.

Remark 7.

$x(t) = (x_1(t), x_2(t)), \ldots, x_m(t)),\ \forall\, t \in \mathbf{N}^*,$ is the random vector that numerically characterizes the state *S*(*t*).

Theorem 1.

Let us consider $t \in \mathbf{N}^*$ as the time at which the *S*(*t*+1) state of the phenomenon *S* is determined by the *S*(*t*) state, according to *β– dependence*. Then:

a) there is a stochastic transition function $P^1(t+1)$ relative to $(\Omega_m(t),\ \mathcal{K}_1(t+1))$ and whatever $i = \overline{1, m-1}$, there is a stochastic transition function $Q^i(t+1)$ relative to $(\prod_{j=1}^{i} \Omega_j(t+1),\ \mathcal{K}_{i+1}(t+1))$.

b) $\forall k \in \mathbf{N}^*$ there is a stochastic transition function $P^1(t+k+1)$ relative to $(\prod_{j=1}^{k} \Omega_{1,m}(t+j),\ \mathcal{K}_1(t+k+1))$, where $\Omega_{1,m}(t) = \Omega_1(t) \times \Omega_2(t) \times \ldots \times \Omega_m(t)$.

Proof.

a) As $x_m(t)$ meets or does not meet the condition imposed by the C_m are two possible cases.

Case 1. $x_m(t)$ meets the condition imposed by the C_m.

According to b) procedure is defined $P^1(t+1)$ as:

$P^1(t+1) : \mathcal{K}_1(t+1) \to [0,1]$, $P^1(t+1) \neq P^1(t)$, but $P^1(t+1)$ is part of the same class of probability laws, as well as $P^1(t)$, where $\Omega_1(t+1) \supseteq \Omega_1(t)$,

$\mathcal{K}_1(t+1) = \mathcal{P}(\Omega_1(t+1))$, $X_1(t+1) : \Omega_1(t+1) \to \mathbf{R}$ and $X_1(t+1)\big|_{\Omega_1(t)} = X_1(t)$.

Suppose *i*=1 and we define a stochastic transition function $Q^1(t+1)$ from $(\Omega_1(t+1),\ \mathcal{K}_1(t+1))$ in Borel space $(\Omega_2(t+1),\ \mathcal{K}_2(t+1))$, if $x_1(t+1)$ meets the condition imposed by the C_1.

Then, according to a) procedure, we have:

$\Omega_2(t+1) \supseteq \Omega_2(t),\ \mathcal{K}_2(t+1) = \mathcal{P}(\Omega_2(t+1))$.

In this case, we define two probabilities denoted by $P^{2,a}(t+1)$ and $\overline{P^2}(t+1)$ respectively.

$P^{2,a}(t+1)$ is part of the same class of probability laws, as well as $P^2(t)$ and $P^{2,a}(t+1) \neq P^2(t)$. $\overline{P^2}(t+1)$ is an extension of $P^2(t)$ to $(\Omega_2(t+1),\ \mathcal{K}_2(t+1))$, where:

$$\overline{P^2}(t+1)(\omega) = \begin{cases} P^2(t)(\omega), \omega \in \Omega_2(t) \\ 0 \qquad\qquad , \omega \in \Omega_2(t+1) \setminus \Omega_2(t) \end{cases}.$$

Note that $X_2(t+1) : \Omega_2(t+1) \to \mathbf{R}$, so we have $X_2(t+1)\big|_{\Omega_2(t)} = X_2(t)$.

Let us $\Omega_1^a(t+1) = \{\omega \in \Omega_1(t+1) \big| X_1(t+1)(\omega) = x_1(t+1)$ meets the condition imposed by the $C_1\}$, and $Q^1(t+1)$: $(\Omega_1(t+1),\ \mathcal{K}_2(t+1)) \to [0,1]$, where:

$$Q^1(t+1)(\omega, \cdot\,) = \begin{cases} P^{2,a}(t+1), \omega \in \Omega_1^a(t+1) \\ \overline{P^2}(t+1) \ \ , \omega \in \Omega_1(t+1) \setminus \Omega_1^a(t+1) \end{cases}.$$

Because $\forall A \in \mathcal{K}_2(t+1)$, $Q^1(t+1)(\,\cdot\,,A)$ is $\mathcal{K}_1(t+1)$-*measurable*, it follows that $Q^1(t+1)$ is a stochastic transition function relative to $(\Omega_1(t+1),\ \mathcal{K}_2(t+1))$.

In addition, $P^1(t+1) \otimes Q^1(t+1)$ is a product probability on the Borel space $(\Omega_1(t+1) \times \Omega_2(t+1),\ \mathcal{K}_1(t+1) \otimes \mathcal{K}_2(t+1))$ and $(\Omega_1(t+1) \times \Omega_2(t+1),\ \mathcal{K}_1(t+1) \otimes \mathcal{K}_2(t+1),\ P^1(t+1) \otimes Q^1(t+1))$ is a probability space.

If $x_1(t+1)$ does not meet the condition imposed by the C_1 , then according to a) procedure , we have: $\Omega_2(t+1) = \Omega_2(t)$, $\mathcal{K}_2(t+1) = \mathcal{K}_2(t)$ and $X_2(t+1) = X_2(t)$. It follows that there is defined on the Borel space $(\Omega_2(t+1),\ \mathcal{K}_2(t+1))$ only the probability $P^2(t+1)$ and $\overline{P^2}(t+1) = P^2(t+1)$.

In this case, let us take $Q^1(t+1)$: $(\Omega_1(t+1),\ \mathcal{K}_2(t+1)) \to [0,1]$, $Q^1(t+1)(\omega, \cdot\,) = \overline{P^2}(t+1), \forall \omega \in \Omega_1(t+1)$. Immediately follows that $Q^1(t+1)$ is a stochastic transition function relative to $(\Omega_1(t+1),\ \mathcal{K}_2(t+1))$ and $(\Omega_1(t+1) \times \Omega_2(t+1),\ \mathcal{K}_1(t+1) \otimes \mathcal{K}_2(t+1),\ P^1(t+1) \otimes Q^1(t+1))$ is a probability space.

Suppose that the a) statement of the Theorem is true for any $k \in \mathbf{N}^*$, $k \leq m$ and we will demonstrate for a phenomenon S with an evolution in $m+1$ "steps" and $m+1$ bivalent conditions. It follows that $\exists\ Q^i(t+1)$, $i = \overline{1, m-1}$, is a stochastic transition function relative to $(\prod_{j=1}^{i} \Omega_j(t+1),\ \mathcal{K}_{i+1}(t+1))$ and particularly, there is the probability space $(\prod_{i=1}^{m}\Omega_i(t+1), \bigotimes_{i=1}^{m}\mathcal{K}_i(t+1), R^m(t+1))$, where

$R^m(t+1) = P^1(t+1)\otimes Q^1(t+1)\otimes\ldots\otimes Q^{m-1}(t+1)$.

Let us take the function $Q^m(t+1)$: $(\prod_{i=1}^{m}\Omega_i(t+1), \mathcal{K}_{m+1}(t+1)) \to [0,1]$, which is defined by $x_m(t+1)$, as follows:

If $x_m(t+1)$ meets the condition imposed by the C_m, then according to the a) procedure extended to the $m+1$ components, we have:

$$\Omega_{m+1}(t+1) \supseteq \Omega_{m+1}(t),\ \mathcal{K}_{m+1}(t+1) = \mathcal{P}(\Omega_{m+1}(t+1)).$$

In this case, we define two probabilities on $(\Omega_{m+1}(t+1), \mathcal{K}_{m+1}(t+1))$, denoted by $P^{m+1,a}(t+1)$ and $\overline{P^{m+1}}(t+1)$ respectively:

$P^{m+1,a}(t+1) : \mathcal{K}_{m+1}(t+1) \to [0,1]$, $P^{m+1,a}(t+1) \neq P^{m+1}(t)$, $P^{m+1,a}(t+1)$ is part of the same class of probability laws, as well as $P^{m+1}(t)$;

$\overline{P^{m+1}}(t+1) : \mathcal{K}_{m+1}(t+1) \to [0,1]$, where:

$$\overline{P^{m+1}}(t+1)(\omega) = \begin{cases} P^{m+1}(t)(\omega), & \omega \in \Omega_{m+1}(t) \\ 0 & , \omega \in \Omega_{m+1}(t+1) \setminus \Omega_{m+1}(t) \end{cases}.$$

Note that we can define at least one random variable $X_{m+1}(t+1) : \Omega_{m+1}(t+1) \to \mathbf{R}$, so that $X_{m+1}(t+1)\big|_{\Omega_{m+1}(t)} = X_{m+1}(t)$.

Let us take $Q^m(t+1)$: $(\prod_{i=1}^{m}\Omega_i(t+1), \mathcal{K}_{m+1}(t+1)) \to [0,1]$, where:

$$Q^m(t+1)(\omega,\cdot) = \begin{cases} P^{m+1,a}(t+1), & pr_{\Omega_m(t+1)}(\omega) \in \Omega_m^a(t+1) \\ \overline{P^{m+1}}(t+1), & pr_{\Omega_m(t+1)}(\omega) \in \Omega_m(t+1) \setminus \Omega_m^a(t+1) \end{cases},$$

where $\Omega_m^a(t+1) = \{\omega \in \prod_{i=1}^{m}\Omega_i(t+1) \big| X_m(t+1)(pr_{\Omega_m(t+1)}(\omega)) = x_m(t+1)$ meets the condition imposed by the $C_m\}$.

Because $P^{m+1,a}(t+1)$ and $\overline{P^{m+1}}(t+1)$ are probabilities, and $Q^m(t+1)(.,A)$ is $\mathcal{K}_{m+1}(t+1)$–measurable, $\forall A \in \mathcal{K}_{m+1}(t+1)$, it follows that $Q^m(t+1)$ is a stochastic transition function relative to $(\prod_{i=1}^{m}\Omega_i(t+1),$

$\mathcal{K}_{m+1}(t+1))$, $R^m(t+1)\otimes Q^m(t+1)$ is a product probability and $(\prod_{i=1}^{m}\Omega_i(t+1)$, $\mathcal{K}_{m+1}(t+1)$, $R^{m+1}(t+1)$) is a probability space, where $R^{m+1}(t+1)=R^m(t+1)\otimes Q^m(t+1)$, on which the random variable $X_{m+1}(t+1)$ is defined.

In conclusion, the statement a) is true for any $m\in\mathbf{N}^*$.

If $x_m(t+1)$ does not meet the condition imposed by the C_m , then according to the a) procedure extended to the $m+1$ "steps" and according to the definition of *β-dependence* we have:

$\Omega_{m+1}(t+1)=\Omega_{m+1}(t)$, $\mathcal{K}_{m+1}(t+1)=\mathcal{K}_{m+1}(t)$, $P^{m+1}(t+1)=P^{m+1}(t)$ and $X_{m+1}(t+1)=X_{m+1}(t)$.

In this case, let us take $Q^m(t+1)$: $(\prod_{i=1}^{m}\Omega_i(t+1),\ \mathcal{K}_{m+1}(t+1))\rightarrow[0,1]$, where $Q^m(t+1)(\omega,.)=P^{m+1}(t+1),\forall\omega\in\prod_{i+1}^{m}\Omega_i(t+1)$. Immediately it follows that $Q^m(t+1)$ is a stochastic transition function relative to $(\prod_{i=1}^{m}\Omega_i(t+1)$, $\mathcal{K}_{m+1}(t+1))$, $R^m(t+1)\otimes Q^m(t+1)$ is a product probability $(\prod_{i=1}^{m}\Omega_i(t+1)$, $\mathcal{K}_{m+1}(t+1)$, $R^{m+1}(t+1)$) is a probability space, where
$R^{m+1}(t+1)=R^m(t+1)\otimes Q^m(t+1)$, on which the random variable $X_{m+1}(t+1)$ is defined.

In conclusion, the a) statement is true for any $m\in\mathbf{N}^*$ in this case.

Case 2. $x_m(t)$ does not meet the condition imposed by the C_m.

According to b) procedure we have : $\Omega_1(t+1)=\Omega_1(t)$, $\mathcal{K}_1(t+1)=\mathcal{K}_1(t)$, $P^1(t+1)=P^1(t)$, $X_1(t+1):\Omega_1(t+1)\rightarrow\mathbf{R}$ and $X_1(t+1)=X_1(t)$.

Suppose that $i=1$ and let us take $Q^1(t+1)$ as a stochastic transition function from the $(\Omega_1(t+1),\ \mathcal{K}_1(t+1))$ to the Borel space $(\Omega_2(t+1),\ \mathcal{K}_2(t+1))$ as in case 1 and obtain that $(\Omega_1(t+1)\times\Omega_2(t+1),\ \mathcal{K}_1(t+1)\otimes\mathcal{K}_2(t+1),\ P^1(t+1)\otimes Q^1(t+1)$) is a probability space.

Assuming the statement true for any $k\in\mathbf{N}^*$, $k\leq m$, it follows as in Case 1, it can be shown that for a random phenomenon S with $m+1$ "steps" and $m+1$ bivalent conditions. In this case that statement is true for any m.

b) An inference that is similar to a) demonstration.

Consequence 1.

In terms of Theorem 1 there is the probability space

$(\prod_{i=1}^{m}\Omega_i(t+1), \bigotimes_{i=1}^{m}\mathcal{K}_i(t+1), R^m(t+1))$, where:

$R^m(t+1) = P^1(t+1)\otimes Q^1(t+1)\otimes\ldots\otimes Q^{m-1}(t+1)$.

Demonstration follows from a) statement of Theorem 1.

3. Cases

Next we consider two interesting cases.

<u>Case *m*=1</u>.

In this case, the phenomenon *S* is reduced to a single "step" and to a bivalent condition *C*. The state *S*(*t*) of the phenomenon at *t* time is econometric modeled by the real random variable *X*(*t*), which is defined on the probability space $(\Omega(t), \mathcal{K}(t), P(t))$, $t\in\mathbf{N}^*$, where $\Omega(t)$ is finite or denumerably infinite .

For any $t\in\mathbf{N}^*$ there is a dependence between *X*(*t*) and *X*(*t*+1), determined by the b) procedure of the *β– dependence* and expressed by means of the bivalent condition *C* which refers to the values $x(t) = X(t)(\omega), \omega\in\Omega(t)$.

Remark 8.

If *x*(*t*) does not meet the condition imposed by the *C*, it not follows *x*(*t*)=*x*(*t*+1), because *ω* at *t* time may be different from *ω* at *t*+1 time.

Remark 9.

C-dependence models dependence of the random variable *X* itself at different times, which we call *autodependence*.

Autodependence not the same autocorrelation because generally different probability spaces occur at different times.

Theorem 2.

Let $t\in\mathbf{N}^*$ be the time at which the state *S*(*t*+1) of the phenomenon *S* is determined by the state *S*(*t*) under *C-dependence*. Then:

a) there is *Q*(*t*+1) a stochastic transition function relative to $(\Omega(t), \mathcal{K}(t+1))$ and for any $k\in\mathbf{N}^*$, there is *Q*(*t*+*k*) a stochastic transition function relative to $(\prod_{i=1}^{k}\Omega(t+i-1), \mathcal{K}(t+k))$.

b) Let us take $k\in\mathbf{N}^*$ and $X_{t,t+k} = (X(t), X(t+1),\ldots, X(t+k))$. Then $X_{t,t+k}$ is a measurable function defined on a probability space $(\Omega(t)\times\Omega(t+1)\times\ldots\times\Omega(t+k), \mathcal{K}(t)\otimes\mathcal{K}(t+1)\otimes\ldots\otimes\mathcal{K}(t+k), R_{t,t+k})$ with values in $\mathbf{R}^k$, where $R_{k,t+k} = P(t)\otimes Q(t+1)\otimes\ldots\otimes Q(t+k)$.

Proof.

a) As *x*(*t*) meets the condition imposed by the *C* or not two cases are possible:

Case 1. $x(t)$ meets the condition imposed by the C . Then according to the Definition 2 can be defined $\Omega(t+1) \supseteq \Omega(t)$, $\mathcal{K}(t+1)=\mathcal{P}(\Omega(t+1))$, $P(t+1) \neq P(t)$, $P(t+1)$ being part of the same class of probability laws such as $P(t)$, $X(t+1): \Omega(t+1) \to \mathbf{R}$ there is the following restriction $X(t+1)\big|_{\Omega(t)} = X(t)$.

Let us take the function $\overline{P}(t+1): \mathcal{K}(t+1) \to [0,1]$, so

$$\overline{P}(t+1)(\omega) = \begin{cases} P(t)(\omega), \omega \in \Omega(t) \\ 0 \qquad , \omega \in \Omega(t+1) \setminus \Omega(t) \end{cases}.$$

It follows that $\overline{P}(t+1)$ is a probability on $(\Omega(t+1), \mathcal{K}(t+1))$ and let us take the function

$$Q(t+1)(\omega, \cdot) = \begin{cases} P(t+1), \omega \in \Omega^a(t) \\ \overline{P}(t+1) \ , \omega \in \Omega(t+1) \setminus \Omega^a(t) \end{cases},$$

where $\Omega^a(t) = \{\omega \in \Omega(t) | X(t)(\omega) = x(t)$ meets the condition imposed by the $C\}$.

Immediately it follows that $Q(t+1)$ is a stochastic transition function relative to $(\Omega(t), \mathcal{K}(t+1))$ and $(\Omega(t) \times \Omega(t+1)$ $\mathcal{K}(t) \otimes \mathcal{K}(t+1)$, $R_{t,t+k})$ is a probability space, where $R_{k,t+1} = P(t) \otimes Q(t+1)$ is the product of probabilities $P(t)$ and $Q(t+1)$.

Case 2. $x(t)$ does not meet the condition imposed by the C. Then according to the Definition 2 we have $\Omega(t+1) = \Omega(t)$, $\mathcal{K}(t+1)= \mathcal{K}(t)$, $P(t+1) = P(t)$ and $X(t+1) = X(t)$.

It follows that on the Borel space $(\Omega(t+1), \mathcal{K}(t+1))$ is defined only the probability $P(t+1)$ and $\overline{P}(t+1) = P(t+1) = P(t)$.

Let us take $Q(t+1)\colon (\Omega(t), \mathcal{K}(t+1)) \to [0,1]$,

$Q(t+1)(\omega, \cdot) = \overline{P}(t+1)$, $\forall \omega \in \Omega(t+1)$. Immediately it follows that $Q(t+1)$ is a stochastic transition function relative to $(\Omega(t), \mathcal{K}(t+1))$ and $(\Omega(t) \times \Omega(t+1)$, $\mathcal{K}(t) \otimes \mathcal{K}(t+1)$, $R_{t,t+1})$ is a probability space, where $R_{t,t+1} = P(t) \otimes Q(t+1) = P(t) \otimes P(t+1)$.

The existence of the stochastic transition function $Q(t+k)$, $k \in \mathbf{N}^*$, is an inference that is similar to a) demonstration of Theorem 1.

b) Follows immediately according to a).

Remark 10.

In general, $X_{t,t+k}$ is not a random vector because its components are not defined on the same probability space.

Case *m=2*.

In this case the phenomenon S consists of two "steps" and two bivalent conditions C_1 and C_2, and state $S(t)$ of the system at t time is econometric modeled by the *2-tuples* $(X_1(t), X_2(t))$, $t \in \mathbf{N}^*$, where $X_i(t)$ is a random variable which is defined on probability space $(\Omega_i(t), \mathcal{K}_i(t), P^i(t))$, with $\Omega_i(t)$ finite or denumerably infinite, $i = \overline{1,2}$. At each t time let us consider $X_1(t)$ and then $X_2(t)$.

Consequence 2.

If is $t_0 = 1$ the time at which, whatever $t \in \mathbf{N}^*$, $t > t_0$, the state $S(t+1)$ of the phenomenon S is determined by the state $S(t)$ according *β-dependence*, then $\forall n \in \mathbf{N}^*$, $(\prod_{t=1}^{n} \Omega_{1,2}(t), \bigotimes_{t=1}^{n} \mathcal{K}_{1,2}(t), \bigotimes_{t=1}^{n} R(t))$ is a probability space where $\Omega_{1,2}(t) = \Omega_1(t) \times \Omega_2(t)$, $\mathcal{K}_{1,2}(t) = \mathcal{K}_1(t) \otimes \mathcal{K}_2(t)$, $R(t) = P^1(t) \otimes Q^1(t)$, $P^1(n)$ is a stochastic transition function relative to $(\prod_{j=1}^{n-1} \Omega_{1,2}(j), \mathcal{K}_1(n))$, where $n \geq 2$ and $Q^1(1)$ is a stochastic transition function relative to $(\Omega_1(1), \mathcal{K}_2(1))$ and $Q^1(n)$ is a stochastic transition function relative to $\prod_{j=1}^{n-1} \Omega_{1,2}(j) \times \Omega_1(n)$, $\mathcal{K}_2(n))$, where $n \geq 2$.

Follows immediately according to Theorem 1.

4. An example

Let us consider a phenomenon S which takes place in four "steps", which act upon bivalent four conditions C_1, C_2,, C_3 and C_4 respectively, and whose state at t time is $S(t)$, $t \in \mathbf{N}^*$,with evolution in discrete time.

Each step i is econometric modelled at t time by a real random variable $X_i(t)$, which is defined on the probability space $(\Omega_i(t), \mathcal{K}_i(t), P^i(t))$, where $\Omega_i(t) = \{1,2,\ldots,n_i(t)\}$, $n_i(t) \in \mathbf{N}^*$, $\forall t \in \mathbf{N}^*$, $\mathcal{K}_i(t) = \mathcal{P}(\Omega_i(t))$, $P^i(t)(k) = C^k_{n_i(t)} p_i^{n_1(t)-k} \cdot q_i^k$ with $p_i + q_i = 1$, $i = \overline{1,3}$ and $\Omega_4(t) = \mathbf{N}^*$, $\mathcal{K}_4(t) = \mathcal{P}(\mathbf{N}^*)$ and $P^4(t)(k) = \frac{[\lambda(t)]^k}{k!} \cdot e^{-\lambda(t)}$, $\lambda(t) > 0, k \in \mathbf{N}, \forall t \in \mathbf{N}^*$.

Conditions C_i, $i = \overline{1,4}$ must be compatible with the phenomenon S and the rest can be defined anyway. For example:

a) C_i: $\{x_i(t) > p, p \in \mathbf{N}^*, p \text{ fixed}\}, i = \overline{1,3}, \forall t \in \mathbf{N}^*$, and
C_4: $\{ x_4(t)$ is even number$\}$, $\forall t \in \mathbf{N}^*$;

b) C_i: $\{x_i(t) > M(X_i(t_0)), t_0 \in \mathbf{N}^*, t_0 \text{ fixed}\}, i = \overline{1,4}, \forall t \in \mathbf{N}^*$.

Between random variables $X_i, i=\overline{1,4}$ is defined β - dependence as follows:

a) X_{i+1} depends on $X_i, i=\overline{1,3}$.

At the $t \in \mathbf{N}^* \setminus \{1\}$ time, the probability space on which the real random variable $X_{i+1}(t)$ is defined depends on the value $x_i(t)=X_i(t)(\omega),\ \omega \in \Omega_i(t)$, $i=\overline{1,4}$.

If $x_i(t)$ does not meet the condition imposed by the C_i, then $\Omega_{i+1}(t)=\Omega_{i+1}(t-1)$, $\mathcal{K}_{i+1}(t)=\mathcal{K}_{i+1}(t-1)$, $P^{i+1}(t)=P^{i+1}(t-1)$ and $X_{i+1}(t)=X_{i+1}(t-1)$.

If $x_i(t)$ meets the condition imposed by the C_i, then there are two situations:

a1) for $i=\overline{1,3}$, $\Omega_{i+1}(t)=\{1,2,...,n_i(t-1)+1\}$, $\mathcal{K}_{i+1}(t)=\mathcal{P}(\Omega_{i+1}(t))$, $P^{i+1}(t) \neq P^{i+1}(t-1)$, $P^{i+1}(t)(k)=C^k_{n_i(t-1)+1} p_i^{n_1(t-1)+1-k} \cdot q_i^k$, $X_{i+1}(t) \neq X_{i+1}(t-1)$, $X_{i+1}(t):\Omega_{i+1}(t) \to \mathbf{R}$, there is the following restriction $X_{i+1}(t)\big|_{\Omega_{i+1}(t-1)}=X_{i+1}(t-1)$;

a2) $\Omega_4(t)=\mathbf{N}^*$, $\mathcal{K}_4(t)=\mathcal{P}(\mathbf{N}^*)$, $P^4(t) \neq P^4(t-1)$, $P^4(t)(k)=\dfrac{[\lambda(t)]^k}{k!} \cdot e^{-\lambda(t)}$, where $\lambda(t) \neq \lambda(t-1)$, $k \in \mathbf{N}, \forall t \in \mathbf{N}^*$, $X_4(t):\Omega_4(t) \to \mathbf{R}$, $X_4(t)=X_4(t-1)$, there is the following restriction $X_4(t)\big|_{\Omega_4(t-1)}=X_4(t-1)$.

b) X_1 depends on X_4.

At the $t+1, t \in \mathbf{N}^*$ time, the probability space on which the random variable $X_1(t+1)$ is defined depends on the value of $x_4(t)$.

If $x_4(t)$ does not meet the condition imposed by the C_4, then $\Omega_1(t+1)=\Omega_1(t)$, $\mathcal{K}_1(t+1)=\mathcal{K}_1(t)$, $P^1(t+1)=P^1(t)$ and $X_1(t+1)=X_1(t)$.

If $x_4(t)$ meets the condition imposed by the C_4, then
$\Omega_1(t+1)=\{1, 2, ..., n_1(t)+1\}$, $\mathcal{K}_1(t+1)=\mathcal{P}(\Omega_1(t+1))$, $P^1(t+1) \neq P^1(t)$, $P^1(t+1)(k)=C^k_{n_1(t)+1} p_1^{n_1(t)+1-k} \cdot q_1^k$, $X_1(t+1) \neq X_1(t)$, $X_1(t+1):\Omega_1(t+1) \to \mathbf{R}$, there is the following restriction that $X_1(t+1)\big|_{\Omega_1(t)}=X_1(t)$.

We consider the following situation.

Let us take $t \in \mathbf{N}^*$ as the time at which the state $S(t+1)$ of phenomenon S is determined by the state $S(t)$ according by the *β-dependence* defined above.

Suppose that conditions C_1, C_2 and C_3 are satisfied at the $t+1$ time, and at the t time is satisfied condition C_4.

The stochastic transition function $P^1(t+1)$ relative to $(\Omega_4(t),\ \mathcal{K}_1(t+1))$ is defined as:

$$P^1(t+1)(\omega,\cdot)=\begin{cases}P^{1,a}(t+1),\ \omega\in\Omega_4^a(t)\\ \overline{P}(t)\ \ ,\ \omega\in\Omega_4(t)\setminus\Omega_4^a(t)\end{cases},$$

where $\Omega_4^a(t)=\{\omega\in\Omega_4(t)\big|X_4(t)(\omega)=x_4(t)$ meets the condition imposed by the $C_4\}$, and

$$P^{1,a}(t+1)(\mathrm{k})=\begin{cases}C_{n_1(t)+1}^k p_1^{n_1(t)+1-k}\cdot q_1^k, k\in\Omega_1(t)\\ 0 \qquad\qquad\qquad\quad ,k=n_1(t)+1\end{cases}.$$

The stochastic transition functions $Q^i(t+1)$, relative to $(\prod_{j=1}^{i}\Omega_j(t),\ \mathcal{K}_{i+1}(t+1))$, where $i=\overline{1,2}$ are defined as:

$$Q^i(t+1)(\omega,\cdot)=\begin{cases}P^{i+1,a}(t+1)\ , & pr_{\Omega_i(t+1)}(\omega)\in\Omega_i^a(t+1)\\ \overline{P^{i+1}}(t+1)\ , & pr_{\Omega_i(t+1)}(\omega)\in\Omega_i(t+1)\setminus\Omega_i^a(t+1)\end{cases},$$

where $\Omega_i^a(t+1)=\{\omega\in\prod_{j=1}^{i}\Omega_j(t+1)\big|X_1(t+1)(pr_{\Omega_i(t+1)}(\omega))=x_i(t)$ meets the condition imposed by the $C_i\}$, $P^{i+1,a}(t+1)(\mathrm{k})=C_{n_{i+1}(t)+1}^k p_i^{n_{i+1}(t)+1-k}\cdot q_i^k$ and

$$\overline{P^{i+1}}(t+1)(k)=\begin{cases}P^{i+1}(t)(k), k\in\Omega_{i+1}(t)\\ 0 \qquad\qquad ,\ k=n_{i+1}(t)+1\end{cases}.$$

The stochastic transition function $Q^3(t+1)$, relative to $(\prod_{j=1}^{3}\Omega_j(t+1),\ \mathcal{K}_4(t+1))$ is defined as:

$$Q^3(t+1)(\omega,\cdot)=\begin{cases}P^{4,a}(t+1)\ , & pr_{\Omega_3(t+1)}(\omega)\in\Omega_3^a(t+1)\\ \overline{P^4}(t+1)\ , & pr_{\Omega_3(t+1)}(\omega)\in\Omega_3(t+1)\setminus\Omega_3^a(t+1)\end{cases},$$

where $\Omega_3^a(t+1)=\{\omega\in\prod_{j=1}^{3}\Omega_j(t+1)\big|X_3(t)(pr_{\Omega_3(t+1)}(\omega))=x_3(t)$ meets the condition imposed by the $C_3\}$, $P^{4,a}\ (t+1)(\mathrm{k})=\dfrac{[\lambda(t+1)]^k}{k!}\cdot e^{-\lambda(t+1)}$, where $\lambda(t+1)\neq\lambda(t),\ k\in\mathbf{N}$, and $\overline{P^4}(t+1)=P^4(t)$.

According to Theorem 1, it follows that:

$(\prod_{i=1}^{4}\Omega_i(t+1), \bigotimes_{i=1}^{4}\mathcal{K}_i(t+1), R^4(t+1))$ is a probability space, where

$R^m(t+1)= P^1(t+1)\otimes Q^1(t+1)\otimes Q^2(t+1)\otimes Q^3(t+1)$.

For $(e,f,g,h)\in\prod_{i=1}^{4}\Omega_i(t+1)$ and $\omega\in\Omega_4(t)$, it follows:

$$R^4(t+1))(\{e\}\times\{f\}\times\{g\}\times\{h\})= P^1(t+1)(\omega,\{e\})\cdot Q^1(t+1)(e,\{f\})\cdot$$
$$\cdot Q^2(t+1)((e,f),\{g\})\cdot Q^3((e,f,g),\{h\})= C^e_{n_1(t)+1}\cdot p_1^{n_1(t)+1-e}\cdot q_1^e\cdot C^f_{n_2(t)+1}\cdot p_2^{n_2(t)+1-f}\cdot$$
$$\cdot q_2^f\cdot C^g_{n_3(t)+1}\cdot p_3^{n_3(t)+1-g}\cdot q_3^g\cdot\frac{[\lambda(t+1)]^h}{h!}\cdot e^{-\lambda(t+1)}.$$

For $A\in\bigotimes_{i=1}^{4}\mathcal{K}_i(t+1)$ and $e\in\Omega_1(t+1)$, let us take

$A_e=\{ f\in\Omega_2(t+1) \mid (e,f)\in \in pr_{\Omega_1(t+1)}A\times pr_{\Omega_2(t+1)}A \}$,

$A_{e,f}=\{ g\in\Omega_3(t+1) \mid (e,f,g)\in pr_{\Omega_1(t+1)}A\times \times pr_{\Omega_2(t+1)}A\times pr_{\Omega_3(t+1)}A \}$ and

$A_{e,f,g}=\{ h\in\Omega_4(t+1) \mid (e,f,g)\in A \}$.

Out of above results:

$$R^4(t+1)(A)=\sum_{e\in\Omega_1(t+1)}\sum_{f\in A_e}\sum_{g\in A_{e,f}}\sum_{h\in A_{e,f,g}}P^1(t+1)(\omega,\{e\})\cdot Q^1(t+1)(e,A_e)\cdot$$
$$\cdot Q^2((e,f),A_{e,f})\cdot Q^3((e,f,g),A_{e,f,g}).$$

Let us take $X(t+1)=(X_1(t+1),X_2(t+1),X_3(t+1),X_4(t+1)),\forall t\in\mathbf{N}^*$ the random function which determines the state $S(t+1)$, where

$$X(t+1):\prod_{i=1}^{4}\Omega_i(t+1)\to\mathbf{R}.$$

In general, because not all probability spaces correspond, there is no random vector .

5. Conclusions

a) According to definition, β - *dependence* allows the modification of the Borel spaces on which the random variables are defined, conditions that may be considering and multivalent. Thus, the evolution of a random phenomenon that takes place in several "steps" may be influenced by external conditions bivalent or multivalent.

b) Also, a finite set consists of several economic variables that are interrelated, can be considered a random phenomenon that takes place in "steps". In this case, a "step" is an economic variable, between "steps" can define a convenient β - *dependence* and values of economic variables are obtained as a result of econometric modeling of the generation process.

c) A goal of future research would be to construct a random algorithm (see [8]) to establish forecasts on future values of economic variables.

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