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PRICING VULNERABLE OPTIONS WITH CONSTANT ELASTICITY OF VARIANCE VERSUS STOCHASTIC ELASTICITY OF VARIANCE

Abstract. *In order to handle option writer's credit risk, a different underlying price model is required beyond the well-known Black-Scholes model. This paper adopts a recently developed model, which characterizes the 2007-2009 global financial crisis in a unique way, to determine the no-arbitrage price of European options vulnerable to writer's default possibility. The underlying model is based on the randomization of the elasticity of variance parameter capturing the leverage or inverse leverage effect. We obtain an analytic formula explicitly for the stochastic elasticity of variance correction to the Black-Scholes price of vulnerable options and show how the correction effect is compared with the one given by the constant elasticity of variance model. The result can help to design a dynamic investment strategy reducing option writer's credit risk more effectively.*

Keywords*: Vulnerable option, Default risk, Stochastic elasticity of variance, Ornstein-Uhlenbeck process, Monte-Carlo simulation.*

JEL Classification: G12, C65

1. Introduction

Since the Global Financial Crisis of 2007-2008, the pricing of financial derivatives subject to credit risk has become one of more interesting subjects in mathematical finance. Options subject to credit default risk, called vulnerable options, have been studied by researchers already after Black & Scholes (1973) and Merton (1974) established the well-known seminal work on the classical European options. A brief review of the studies of vulnerable options is as follows. Johnson &Stulz (1987) first proposed an option pricing formula for a vulnerable European option. They assumed that the liability of the counterparty is just the option. Hull & White (1995) considered that the option writer has the other liabilities and the payment is decided by a proportion of the nominal claim for the option if default occurs. However, they did not consider the existing dependence between the asset of the option writer and the asset itself underlying the option and so Jarrow& Turnbull (1995) presented a new approach for the pricing and hedging of derivative securities with credit risk. Rich (1996) considered the timing of default and an uncertain recovery value to price an European option subject to an intertemporal default risk. Klein (1996) extended Johnson &Stulz (1987) by allowing the option writer to have other liabilities and considered the correlation between the option writer's assets and an asset underlying the option. Klein & Yang (2010) extended Johnson &Stultz (1987) and Klein (1996) to price American vulnerable options. Wang & Wang (2010) employed the regime switching Esscher transform to study the European vulnerable options under a Markov-modulated jump-diffusion model. Liu & Liu (2011)studied the pricing of vulnerable options by using the conditional binomial tree algorithm and the binomial pyramid algorithm. Xu*et al.* (2012)showed numerical results for vulnerable options under the assumption that the underlying asset and the counterparty's asset follow jump-diffusion processes. Yang *et al.* (2014) adopts a fast mean-reverting stochastic volatility model proposed in Fouque*et al.* (2000) to price vulnerable options and obtain the stochastic volatility effect on the Black-Scholes price.

This paper adopts an underlying model recently developed by Kim *et al.* (2014). The model extends the well-known and practically popular constant elasticity of variance (CEV) model introduced in Cox & Ross (1976) by randomizing the constant elasticity and it is called the stochastic elasticity of variance (SEV) model. It attempts to capture the stochastic leverage or inverse leverage effect and discovers a singular behavior of the S&P market during the Global Financial Crisis as shown in Kim *et al.* (2015). Also, Yoon *et al.* (2015) shows that the SEV model with stochastic interest rates gives a more accurate fit of implied volatility to a market data set as time-to-maturity becomes shorter. This paper applies the SEV model to the work of Klein (1996) and its CEV extension and investigates the stochastic elasticity of variance correction effect to the price of vulnerable options.

The rest of the paper is organized as follows. Section 2 obtains a singularly perturbed partial differential equation (PDE) problem for the vulnerable option price based on the SEV model. Section 3 derives an analytic formula for the approximation of the PDE problem solution. Section 4 provides a comparison analysis of three different prices given by Klein (1996), the CEV model and the SEV model, respectively. Finally, section 5 concludes.

2. Formulation

Economically markets are assumed to be frictionless, i.e., there are no transaction cost, no taxes, and no constraining regulations and all assets are perfectly divisible and marketable. Let S_t be the value of asset underlying a European option and Y_t a mean-reverting Ornstein-Uhlenbeck (OU) process whose distribution is Gaussian with mean m and variance u^2 such that the dynamics of S_t and Y_t are given by the stochastic differential equations (SDEs)

$$dS_t = \mu_s S_t dt + \sigma_s S_t^{\gamma_t} dW_t^s, \quad \gamma_t := \sqrt{\delta} f(Y_t), \tag{1}$$

$$dY_t = \frac{1}{\varepsilon}(m - Y_t)dt + \frac{u\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}_t, \tag{2}$$

respectively. Here, μ_s denotes the instantaneous expected return of the asset, σ_s denotes the instantaneous standard deviation of the return of the asset, m and $u\ (> 0)$ are constants, and W_t^s and $\hat{Z}_t$ are standard Brownian motions with correlation coefficient ρ_{sy} so that $\hat{Z}_t = \rho_{sy} W_t^s + \sqrt{1 - {\rho_{sy}}^2} Z_t$ for some Brownian motion Z_t independent of W_t^s. Based on the real market data traded in finance, the results of empirical performance give that the elasticity parameter γ_t is close to 1. Refer to Chen et al. (2009). So, f is chosen to be a function satisfying $-\infty < c_1 \le f \le c_2 < \infty$ for some constants c_1 and c_2. The parameters ε and δ are assumed to satisfy $0 < \varepsilon, \delta \ll 1$.

On the other hand, let V_t denote the market value of the assets of the option writer which is given by the SDE

$$dV_t = \mu_v V_t dt + \sigma_v V_t dW_t^v, \tag{3}$$

where W_t^v is the standard Brownian motion satisfying $d < W_t^s, W_t^v >_t = \rho_{sv} dt$ and $d < \hat{Z}_t, W_t^v >_t = \rho_{vy} dt$. Here, the instantaneous expected return μ_v and the instantaneous variance σ_v^2 of the return are assumed to be constants. Then, by the Girsanov theorem (cf. Fouque et al., 2011), under the equivalent martingale measure $\mathbb{Q}^*$, the above SDE (1)-(3) become

$$dS_t = rS_t dt + \sigma_s S_t^{1+\sqrt{\delta} f(Y_t)} dW_t^{s*},$$
$$dY_t = \left(\frac{1}{\varepsilon}(m - Y_t)dt + \frac{u\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(S_t, Y_t)\right) dt + \frac{u\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}_t^{\ *},$$
$$dV_t = rV_t dt + \sigma_v V_t dW_t^{v*},$$

where W_t^{s*}, $\hat{Z}_t^{\,*}$ and W_t^{v*} are the standard Brownian motions satisfying $d < W_t^{s*}, W_t^{v*} >_t = \rho_{sv} dt$, d $< W_t^{s*}, \hat{Z}_t^{\,*} >_t = \rho_{sy} dt$ and d $< W_t^{v*}, \hat{Z}_t^{\,*} >_t = \rho_{vy} dt$, and Λ is the risk premium of the elasticity risk.

At time $t = T$, a payoff function of a vulnerable European call option depending upon the financial distress circumstance can be defined by

$$h(S_T, V_T) = (S_T - K)^+ \left(1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D}\right),$$

where $K$is the option's strike price and the value$D^*$is the other liabilities of the option writer. If the asset value of the option writer at the maturity$T$is below the default level D^*, default occurs and the option writer pays out the proportion $\frac{(1-\alpha)V_T}{D}$of $(S_T - K)^+$, whereas if the assets of the option writer are above the default level, the entire nominal claim is paid out. Here, α denotes the deadweight bankruptcy/reorganization cost of default event and is given as a percentage of the asset value of the option writer. $D$is the amount of D^*plus an additional liability produced by the possibility of the option writer's continuation in operation even while $V_T$is less than D^*.

Using the Markov property of the joint process (S_t, V_t, Y_t) the value of the vulnerable call option is given by

$$P^{\varepsilon,\delta}(t, s, v, y)$$
$$:= \mathrm{E}^* \left[e^{-r(T-t)} (S_T - K)^+ \left(1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D}\right) \middle| S_t = s, V_t = v, Y_t = y \right]$$

under the risk neutral measure$\mathbb{Q}^*$. Then the Feynman-Kac theorem (cf. Fouque et al., 2011) allows $P^{\varepsilon,\delta}(t, x, v, y)$ to satisfy the singularly perturbed partial differential equation (PDE) given by

$$\begin{aligned}
&\frac{\partial P^{\varepsilon,\delta}}{\partial t} + rs \frac{\partial P^{\varepsilon,\delta}}{\partial s} + \frac{1}{2}\sigma_s^2 s^{2+2\sqrt{\delta}f(y)} \frac{\partial^2 P^{\varepsilon,\delta}}{\partial s^2} + \left(\frac{1}{\varepsilon}(m-y) - \frac{u\sqrt{2}}{\sqrt{\varepsilon}}\Lambda\right)\frac{\partial P^{\varepsilon,\delta}}{\partial y} \\
&+ \frac{u^2}{\varepsilon}\frac{\partial^2 P^{\varepsilon,\delta}}{\partial y^2} + rv \frac{\partial P^{\varepsilon,\delta}}{\partial v} + \frac{1}{2}\sigma_v^2 v^2 \frac{\partial^2 P^{\varepsilon,\delta}}{\partial v^2} + \rho_{sy}\sigma_s s^{1+\sqrt{\delta}f(y)} \frac{u\sqrt{2}}{\sqrt{\varepsilon}} \frac{\partial^2 P^{\varepsilon,\delta}}{\partial s \partial y} \\
&+ \rho_{sv}\sigma_s\sigma_v v s^{1+\sqrt{\delta}f(y)} \frac{\partial^2 P^{\varepsilon,\delta}}{\partial s \partial v} + \rho_{vy}\sigma_v v \frac{u\sqrt{2}}{\sqrt{\varepsilon}} \frac{\partial^2 P^{\varepsilon,\delta}}{\partial y \partial v} - rP^{\varepsilon,\delta} = 0 \qquad (4)
\end{aligned}$$

with the terminal condition

$$P^{\varepsilon,\delta}(T, s, v, y) = (s - K)^+ \left(1_{\{v \geq D^*\}} + 1_{\{v < D^*\}} \frac{(1-\alpha)v}{D}\right).$$

3. Analysis

In this section, we derive a pricing formula for the vulnerable option by using the asymptotic analysis of Fouque*et al.* (2011). First, we arrange the PDE (4) as

$$\frac{1}{\varepsilon}\left(u^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}\right) P^{\varepsilon,\delta}$$

$$+\frac{1}{\sqrt{\varepsilon}}\left(\sqrt{2}\rho_{sy}u\sigma_s s^{1+\sqrt{\delta}f(y)}\frac{\partial^2}{\partial s\partial y}+\sqrt{2}\rho_{vy}uv\sigma_v\frac{\partial^2}{\partial y\partial v}(m-y)-\sqrt{2}u\Lambda\frac{\partial}{\partial y}\right)P^{\varepsilon,\delta}$$
$$+\left[\frac{\partial}{\partial t}+\frac{1}{2}\sigma_s^2 s^{2+2\sqrt{\delta}f(y)}\frac{\partial^2}{\partial s^2}+\frac{1}{2}\sigma_v^2v^2\frac{\partial^2}{\partial v^2}+\rho_{sy}\sigma_s\sigma_v vs^{1+\sqrt{\delta}f(y)}\frac{u\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial^2}{\partial s\partial v}\right.$$
$$\left.+r(s\frac{\partial}{\partial s}+v\frac{\partial}{\partial s}-\cdot)\right]P^{\varepsilon,\delta}=0 \tag{5}$$

and then from the Taylor series expansion in the small parameter δ, the PDE (5) becomes

$$\frac{1}{\varepsilon}\mathcal{L}_{00}P^{\varepsilon,\delta}+\frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_{10}+\sqrt{\delta}\mathcal{L}_{11}+\ \delta\mathcal{L}_{12}+\delta\sqrt{\delta}\mathcal{L}_{13}+\cdots\right)P^{\varepsilon,\delta}$$
$$+\left(\mathcal{L}_{20}+\sqrt{\delta}\mathcal{L}_{21}+\delta\ \mathcal{L}_{22}+\delta\sqrt{\delta}\mathcal{L}_{23}+\cdots\right)P^{\varepsilon,\delta}=0, \tag{6}$$

where

$\mathcal{L}_{00}=u^2\frac{\partial^2}{\partial y}+(m-y)\frac{\partial}{\partial y},$

$\mathcal{L}_{10}=\sqrt{2}\rho_{sy}\ u\sigma_s\ s\frac{\partial^2}{\partial s\,\partial y}+\sqrt{2}\rho_{vy}\ uv\sigma_v\ \frac{\partial^2}{\partial y\,\partial v}-\sqrt{2}\ u\Lambda\frac{\partial}{\partial y},$

$$\mathcal{L}_{1i}=\sqrt{2}\rho_{sy}\ u\sigma_s\ s\frac{(f(y)\ln s)^i}{i!}\frac{\partial^2}{\partial s\,\partial y},\ \ i=1,2,\cdots,$$

$$\mathcal{L}_{20}=\frac{\partial}{\partial t}+\frac{1}{2}\sigma_s^2s^2\frac{\partial^2}{\partial s^2}+\frac{1}{2}\sigma_v^2v^2\frac{\partial^2}{\partial v^2}+\rho_{sv}\sigma_v\,\sigma_s vs\frac{\partial^2}{\partial s\,\partial v}+r\left(s\frac{\partial}{\partial s}+v\frac{\partial}{\partial v}-\cdot\right),$$

$$\mathcal{L}_{2i}=\frac{1}{2}\sigma_s^2s^2\frac{(2f(y)\ln s)^i}{i!}\frac{\partial^2}{\partial s^2}+\rho_{sv}\sigma_s\,\sigma_v sv\frac{(f(y)\ln s)^i}{i!}\frac{\partial^2}{\partial s\,\partial v},\qquad i=1,2,\cdots.$$

We are interested in a solution of PDE (6) of the form

$$P^{\varepsilon,\delta}(t,s,v,y)=\sum_{i=0,j=0}^{\infty}\varepsilon^{i/2}\delta^{j/2}P_{ij}(t,s,v,y). \tag{7}$$

So, if we substitute the expansion (7) into (6), then we obtain

$$\frac{1}{\varepsilon}\left(\mathcal{L}_{00}P_{00}+\sqrt{\delta}\mathcal{L}_{00}P_{01}+\delta\mathcal{L}_{00}P_{02}+\cdots\right)$$
$$+\frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_{00}P_{10}+\mathcal{L}_{10}P_{00}+\sqrt{\delta}(\mathcal{L}_{00}P_{11}+\mathcal{L}_{10}P_{01}+\mathcal{L}_{11}P_{00})\right.$$
$$\left.+\,\delta(\mathcal{L}_{00}P_{12}+\mathcal{L}_{10}P_{02}+\mathcal{L}_{11}P_{01}+L_{12}P_{00})+\cdots\right)$$
$$+\left(\mathcal{L}_{00}P_{20}+\mathcal{L}_{10}P_{10}+\mathcal{L}_{20}P_{00}+\sqrt{\delta}\,(\mathcal{L}_{00}P_{21}+\mathcal{L}_{10}P_{11}+\mathcal{L}_{11}P_{10}+\mathcal{L}_{20}P_{01}+\mathcal{L}_{21}P_{00})\right.$$
$$+\,\delta(\mathcal{L}_{00}P_{22}+\mathcal{L}_{10}P_{12}+\mathcal{L}_{11}P_{11}+\mathcal{L}_{12}P_{10}+\mathcal{L}_{20}P_{02}+\mathcal{L}_{21}P_{01}$$
$$\left.+\,\mathcal{L}_{22}P_{00})+\cdots\ \right)$$

$$
\begin{aligned}
&+\sqrt{\varepsilon}\Big(\mathcal{L}_{00}P_{30}+\mathcal{L}_{10}P_{20}+\mathcal{L}_{20}P_{10}\\
&\qquad+\sqrt{\delta}\,(\mathcal{L}_{00}P_{31}+\mathcal{L}_{10}P_{21}+\mathcal{L}_{11}P_{20}+\mathcal{L}_{20}P_{11}+\mathcal{L}_{21}P_{10})\\
&\qquad+\delta(\mathcal{L}_{00}P_{32}+\mathcal{L}_{10}P_{22}+\mathcal{L}_{11}P_{21}+\mathcal{L}_{12}P_{20}+\mathcal{L}_{20}P_{12}+\mathcal{L}_{21}P_{11}\\
&\qquad+\mathcal{L}_{22}P_{10})+\cdots\Big)\\
&+\varepsilon\,\Big(\mathcal{L}_{00}P_{40}+\mathcal{L}_{10}P_{30}+\mathcal{L}_{20}P_{20}\\
&\qquad+\sqrt{\delta}\,(\mathcal{L}_{00}P_{41}+\mathcal{L}_{10}P_{31}+\mathcal{L}_{11}P_{30}+\mathcal{L}_{20}P_{21}+\mathcal{L}_{21}P_{20})\\
&\qquad+\delta(\mathcal{L}_{00}P_{42}+\mathcal{L}_{10}P_{32}+\mathcal{L}_{11}P_{31}+\mathcal{L}_{12}P_{30}+\mathcal{L}_{20}\mathcal{L}_{22}+\mathcal{L}_{21}P_{21}\\
&\qquad+\mathcal{L}_{22}P_{20})+\cdots\Big)\\
&=0. \qquad (8)
\end{aligned}
$$

In the following asymptotic analysis, the so-called centering (or solvability) condition on a Poisson equation is crucial, where the Poisson equation is given by $\mathcal{L}_{00}H+g=0$for some function g. If the function g is smooth enough to satisfy $g\in C^2(R)$such that a solution H of the Poisson equation exists, then the condition$\langle g\rangle=0$, which is called the centering condition, has to be satisfied, where$\langle\cdot\rangle$denotes average with respect to the invariant distribution of the processY_t, i.e.,

$$\langle g\rangle=\frac{1}{\sqrt{2\pi u^2}}\int_{-\infty}^{\infty}g(y)e^{-\frac{(y-m)^2}{2u^2}}dy.$$

Refer to the Fredholm alternative theorem (cf. Ramm, 2001) or Fouque et al. (2011) for more details.

The following theorem states that the leading order term P_{00} and the correction terms $P_{10},P_{01},P_{20},P_{11},P_{02},P_{30}$ and P_{12} are independent of the variable y. This fact becomes an important tool to derive each of the terms.

Theorem 3.1. Assume that P_{ij}does not grow as much as $\frac{\partial P_{ij}}{\partial y}\sim e^{\frac{y^2}{2}}$as y goes to infinity for each$i,j=0,1,2,\cdots$. Then $P_{00},P_{10},P_{01},P_{20},P_{11},P_{02},P_{30}$and P_{12} are independent of the variable y.

Proof. From the order $\frac{1}{\varepsilon},\frac{\sqrt{\delta}}{\varepsilon}$and $\frac{\delta}{\varepsilon}$terms in (8), $P_{00},P_{01}$and $P_{02}$are solutions for the Poisson equation

$$\mathcal{L}_{00}P_{ij}(t,s,v,y)=0,\qquad ij=00,01,02.$$

Solving this equation forP_{ij}, we obtain

$$P_{ij}(t,s,v,y)=k_1(t,s,v)\int_{-\infty}^{y}e^{\frac{(z-m)^2}{2u^2}}dz+k_2(t,s,v)$$

for some y-independent functions $k_1$and k_2. From the assumed growth condition, $k_1=0$must hold. So, P_{ij} is independent of the y variable. This can be expressed as $P_{ij}=P_{ij}(t,s,v)$, where $ij=00,01,02$. From the order$\frac{1}{\sqrt{\varepsilon}},\frac{\sqrt{\delta}}{\sqrt{\varepsilon}}and\frac{\delta}{\sqrt{\varepsilon}}$ terms in (8) and the y-independence of P_{00},P_{01} and P_{02} which has been just derived, the y-

independence of P_{10}, P_{11} and P_{12} can be also derived similarly. Also, the y-independence of $P_{10}$and P_{00} leads to the y-independence of P_{20} from the order$O(1)$ term. Lastly, the y-independence of P_{20} yields that the order $\sqrt{\varepsilon}$ term becomes $\mathcal{L}_{00}P_{30} + \mathcal{L}_{20}P_{10} = 0$.By applying the centering condition for a Poisson equation and the y-independence of P_{10} to this equation, we have

$$\mathcal{L}_{20}P_{10} = \langle \mathcal{L}_{20}P_{10} \rangle = 0 \tag{9}$$

Thus the order$\sqrt{\varepsilon}$ termisreducedto$\mathcal{L}_{00}P_{30} = 0$,andso P_{30}isindependentonthevariable y. ■

Next,we derive PDEs for the leading order and correction termsP_{ij}, $ij = 00, 01, 10, 20, 11, 02$.

Theorem 3.2.P_{ij}'s ($ij = 00, 10, 01, 20, 11, 02$) satisfy the following PDEs

$$\mathcal{L}_{20}P_{ij}(t,s,v) = \begin{cases} 0 & : ij = 00, 10, 20 \\ -\bar{\mathcal{L}}_{21}P_{00} & : ij = 01 \\ \mathcal{H}P_{00} & : ij = 11 \\ -(\bar{\mathcal{L}}_{21}P_{01} + \bar{\mathcal{L}}_{22}P_{00}) & : ij = 02, \end{cases} \tag{10}$$

respectively, where

$$\begin{aligned}\mathcal{H} = {} & \sqrt{2}\rho_{sy}\, u\sigma_s\, s\left[\sigma_s^2\langle\psi'\rangle\left(2s\ \ln s\frac{\partial^2}{\partial s^2} + s\frac{\partial^2}{\partial s^2} + s^2\ln s\frac{\partial^3}{\partial s^3}\right)\right. \\ & \left. + \rho_{sv}\sigma_s\,\sigma_v\langle\psi'\rangle v\left(\ln s\frac{\partial^2}{\partial s\,\partial v} + \frac{\partial^2}{\partial s\,\partial v} + s\ln s\frac{\partial^3}{\partial s^2\,\partial v}\right)\right] \\ & +\sqrt{2}\rho_{vy}\, u\sigma_v v\left[\sigma_s^2\langle\psi'\rangle s^2\ln s\frac{\partial^3}{\partial s^2\,\partial v}\right. \\ & \left. + \rho_{sv}\sigma_s\sigma_v\langle\psi'\rangle s\ln s\left(\frac{\partial^2}{\partial s\,\partial v} + v\ \frac{\partial^3}{\partial s\,\partial v^2}\right)\right] \\ & -\sqrt{2}u\left[\sigma_s^2\langle\Lambda\psi'\rangle s^2\ln s\frac{\partial^2}{\partial s^2} + \rho_{sv}\sigma_s\sigma_v\langle\Lambda\psi'\rangle vs\ln s\frac{\partial^2}{\partial s\,\partial v}\right],\end{aligned}$$

$$\bar{\mathcal{L}}_{2i} = \frac{1}{2}\sigma_s^2 s^2\frac{\langle f^i\rangle(2\ln s)^i}{i!}\frac{\partial^2}{\partial s^2} + \rho_{sv}\sigma_s\sigma_v sv\frac{\langle f^i\rangle(\ln s)^i}{i!}\frac{\partial^2}{\partial s\,\partial v}, i = 1, 2.$$

Here, ψ denotes the solution of$\mathcal{L}_{00}\psi(y) = f(y) - \langle f\rangle$.

Proof. The proof of Theorem 3.1 has already produced a PDE forP_{00} which is

$$\mathcal{L}_{20}P_{00}(t,s,v) = 0. \tag{11}$$

From (9), we also have$\mathcal{L}_{20}P_{10}(t,s,v) = 0$.

Next, from the y-independence of$P_{11}$and P_{10}, the order$\sqrt{\delta}$terms in (8) lead to

$$\mathcal{L}_{00}P_{21} + \mathcal{L}_{20}P_{01} + \mathcal{L}_{21}P_{00} = 0. \tag{12}$$

Then, applying the centering condition to this Poisson equation, we obtain the PDE

$$P_{01}(t,s,v) = -\bar{\mathcal{L}}_{21}P_{00.} \tag{13}$$

From the y-independenceof P_{12}, P_{11} and P_{10}, theorder δ termsin(8)gives $\mathcal{L}_{00}P_{22} + \mathcal{L}_{20}P_{02} + \mathcal{L}_{21}P_{01} + \mathcal{L}_{22}P_{00} = 0$.Then, again by applying the centering condition to this Poisson equation, we obtain a PDE for P_{02} given by

$$\mathcal{L}_{20}P_{02}(t,s,v) = -(\bar{\mathcal{L}}_{21}P_{01.} + \bar{\mathcal{L}}_{22}P_{00.}). \tag{14}$$

The y-independence of P_{30} and the centering condition lead to

$$\mathcal{L}_{20}P_{20} = 0 \tag{15}$$

from the order εterm.

Note that the PDEs (9) and (15) have vanishing solutions once the terminal conditions $P_{10}(T,s,v) = 0$ and$P_{20}(T,s,v)$are imposed. In fact, this result could be expected from the scale structure of the underlying model (1)-(2).

It remains to derive a PDE for $P_{11}(t,s,v)$. First, we define a function ψ by the solution of

$$\mathcal{L}_{00}\psi(y) = f(y) - \langle f \rangle.$$

Subtracting (13) from (12), we obtain$\mathcal{L}_{00}P_{21} = -(\mathcal{L}_{21} - \bar{\mathcal{L}}_{21})P_{00}$, that is,

$$\mathcal{L}_{00}P_{21} = -\sigma_s^2(f(y) - \langle f \rangle)s^2 \ln s \frac{\partial^2 P_{00}}{\partial s^2} - \rho_{sv}\sigma_s\sigma_v(f(y) - \langle f \rangle)vs \ln s \frac{\partial^2 P_{00}}{\partial sv}.$$

Then, using the function ψ, the solution P_{21} can be expressed as

$$P_{21}(t,s,v,y) = -\sigma_s^2\big(\psi(y) + c(t,s,v)\big)s^2 \ln s \frac{\partial^2 P_{00}}{\partial s^2} - \rho_{sv}\sigma_s\sigma_v\big(\psi(y) + c(t,s,v)\big)vs \ln s \frac{\partial^2 P_{00}}{\partial sv}. \tag{16}$$

for some function$c(t,s,v)$independentof y. On the other hand, applying the y-independence of P_{20} and the fact$P_{10} = 0$to the order $\sqrt{\varepsilon\delta}$ terms in (8) leads to $\mathcal{L}_{00}P_{31} + \mathcal{L}_{10}P_{21} + \mathcal{L}_{20}P_{11} = 0$.Then the centering condition for this Poisson equation yields$\mathcal{L}_{20}P_{11} = -\langle \mathcal{L}_{10}P_{21} \rangle$. Substituting (16) into this PDE, we obtain a PDE for P_{11} given by

$$\mathcal{L}_{20}P_{11}(t,s,v) = \mathcal{H}P_{00}(t,s,v), \tag{17}$$

where$\mathcal{H}$ is given in the theorem. The results (11), (9), (13), (15), (17) and (14) obtained above are put together and lead to (10).■

Next, we solve the PDEs obtained above inTheorem3.2 explicitly. First, nothing that the leading order term P_{00} is exactly the same as the price corresponding to the Black-Scholes model which was driven by Klein (1996), we quote the formula in terms of our notation as follows.

Theorem 3.3. The solution $P_{00}(t,s,v)$of the PDE problem$\mathcal{L}_{20}P_{00}(t,s,v) = 0$with the terminal condition$P_{00}(\mathrm{T,s,v}) = (s-K)^+\left(1_{\{v \geq D^*\}} + 1_{\{v < D^*\}}\frac{(1-\alpha)v}{D}\right)$is given by

$$P_{00}(t,s,v) = sN_2(a_1,a_2,\rho_{sv}) - e^{-r(T-t)}KN_2(b_1,b_2,\rho_{sv}) + \frac{(1-\alpha)v}{D}\Big(se^{(r+\rho_{sv}\sigma_s\sigma_v)(T-t)}N_2(c_1,c_2,-\rho_{sv}) - KN_2(d_1,d_2,-\rho_{sv})\Big),$$

where N_2 is the bivariate normal cumulative distribution function and

$$a_1 = \frac{\ln\left(\frac{s}{K}\right) + \left(r + \frac{\sigma_s^2}{2}\right)(T-t)}{\sigma_s\sqrt{T-t}}, \qquad a_2 = \frac{\ln\left(\frac{v}{D^*}\right) + \left(r - \frac{\sigma_v^2}{2} + \rho_{sv}\sigma_s\sigma_v\right)(T-t)}{\sigma_v\sqrt{T-t}},$$

$$b_1 = \frac{\ln\left(\frac{s}{K}\right) + \left(r - \frac{\sigma_s^2}{2}\right)(T-t)}{\sigma_s\sqrt{T-t}}, \; b_2 = \frac{\ln\left(\frac{v}{D^*}\right) + \left(r - \frac{\sigma_v^2}{2}\right)(T-t)}{\sigma_v\sqrt{T-t}},$$

$$c_1 = \frac{\ln\left(\frac{s}{K}\right) + \left(r + \frac{\sigma_s^2}{2} + \rho_{sv}\sigma_s\sigma_v\right)(T-t)}{\sigma_s\sqrt{T-t}}, c_2 = -\frac{\ln\left(\frac{v}{D^*}\right) + \left(r + \frac{\sigma_v^2}{2} + \rho_{sv}\sigma_s\sigma_v\right)(T-t)}{\sigma_v\sqrt{T-t}},$$

$$d_1 = \frac{\ln\left(\frac{s}{K}\right) + \left(r - \frac{\sigma_s^2}{2} + \rho_{sv}\sigma_s\sigma_v\right)(T-t)}{\sigma_s\sqrt{T-t}}, \qquad d_2 = -\frac{\ln\left(\frac{v}{D^*}\right) + \left(r + \frac{\sigma_v^2}{2}\right)(T-t)}{\sigma_v\sqrt{T-t}}.$$

Proof. Refer to Klein (1996). ■

The correction terms P_{ij} of the PDE problem(10)with $P_{ij}(\mathrm{T},\mathrm{s},\mathrm{v}) = 0$ for $ij = 01, 10, 20, 11, 02$ are given by the following theorem, respectively.

Theorem 3.4. The solutions P_{ij} of the PDE problem (10) with$P_{ij}(T,s,v) = 0$ for $ij = 10, 01, 20, 11, 02$ are given by

$$P_{ij} = \begin{cases} 0 & : ij = 10, 20 \\ e^{\alpha y + \beta z + \gamma\tau} Q_{ij}(\tau, y/\, m_2, z/\, n_2) & : ij = 01, 11, 02, \end{cases}$$

respectively, where

$$\tau = T - t, \qquad y = \rho_{sv}\sigma_v \ln v\,, \quad z = \rho_{sv}\sigma_s \ln v - \sigma_v \ln s,$$

$$\alpha = -\frac{m_1}{2{m_2}^2}, \qquad \beta = -\frac{n_1}{2{n_2}^2}\,, \qquad \gamma = -\frac{1}{4}\left(\frac{{m_1}^2}{{m_2}^2} + \frac{{n_1}^2}{{n_2}^2} + 4r\right),$$

$$Q_{ij}(\tau,\tilde{y},\tilde{z}) = \int_{s=0}^{\infty}\int_{\xi=-\infty}^{\infty}\int_{\eta=-\infty}^{\infty} \frac{1}{4\pi(\tau-s)} e^{-\frac{(\tilde{y}-\xi)^2 + (\tilde{z}-\eta)^2}{4(\tau-s)}}\, \tilde{g}_{ij}(\tau,\xi,\eta)\, d\eta\, d\xi\, ds,$$

$$\tilde{g}_{ij}(\tau,\xi,\eta) = -e^{\alpha m_2\xi + \beta n_2\eta + \gamma\tau} g_{ij}\left(T-\tau, e^{\frac{\sigma_s}{\sigma_v^2}m_2\xi - \frac{1}{\sigma_v}n_2\eta}, e^{\frac{1}{\rho_{sv}\sigma_v}m_2\xi}\right),$$

$$g_{ij}(t,s,v) = \begin{cases} -\bar{\mathcal{L}}_{21}P_{00}(t,s,v) & : ij = 01 \\ \mathcal{H}P_{00}(t,s,v) & : ij = 11, \\ -(\bar{\mathcal{L}}_{21}P_{01} + \bar{\mathcal{L}}_{22}P_{00})(t,s,v) & : ij = 02 \end{cases}$$

$$m_1 = r\rho_{sv}\sigma_v - \frac{1}{2}\rho_{sv}\sigma_v^3,\ m_2 = \frac{1}{\sqrt{2}}\rho_{sv}\sigma_v^2,$$
$$n_1 = \frac{1}{2}\sigma_s^2\sigma_v - r\sigma_v - \frac{1}{2}\sigma_v^2\rho_{sv}\sigma_s + r\rho_{sv}\sigma_s,$$
$$n_2 = \frac{1}{\sqrt{2}}\sigma_s\ \sigma_v\ \sqrt{1-\rho_{sv}{}^2}.$$

Proof. The result for the case $ij = 10$ has been already mentioned in the middle of the proof of Theorem 3.2. Similarly, one can obtain $P_{20} = 0$ which also could be expected from the scale structure of the underlying model (1)-(2).

Since each $P_{ij}$for $ij = 01, 02, 11$satisfies an inhomogeneous Black-Scholes PDE with the zero terminal condition, one can use the well-known Duhamel's principle (cf. John, 1982) to transform the problem into a homogenous Black-Scholes PDE with a non-zero terminal condition. Then one can use the same standard technique (cf. Kevorkian, 2000) asin the derivation of the classical Black-Scholes formula to derive a solution formula for each P_{ij}. We omit the detailed derivation here.■

4. Elasticity Correction Effects

In the previous section, we have obtained formally an approximate solution denoted by $\tilde{P}$ as follows.

$$P^{\varepsilon,\delta} \approx \tilde{P} := P_{00} + \tilde{P}_{01} + \tilde{P}_{11}, \qquad \tilde{P}_{01} := \sqrt{\delta}P_{01}, \qquad \tilde{P}_{11} := \sqrt{\varepsilon\delta}P_{11}.$$

Here, the term P_{02} has been omitted because of its marginal quantitative contribution to P_{01}. Since P_{00} is the Black-Scholes price of Klein (1996), it is also denoted by notation P_{BS}. Since $\tilde{P}_{01}$ is a correction term of the first order coming from the regular perturbation of the elasticity of variance, $P_{00} + \tilde{P}_{01}$ corresponds to a constant correction to the zero elasticity of variance of the Black-Scholes model and so it is a CEV option price which is denoted by P_{CEV}. The correction term $\tilde{P}_{11}$involves the scale parameter ε of the mean-reverting OU process driving the elasticity of variance and so it represents a correction effect driven by stochastic elasticity of variance. Note that accuracy with respect to δ and ε can be checked out by regular andsingular perturbation arguments as in Fouque*et al.* (2003) or Fouque*et al.* (2014).

The vulnerable options are derivatives traded personally in over-the-counter markets so that the data of bid and ask price are not recorded officially. Thus this paper generates the price data of a vulnerable option by using the Monte-Carlo simulation, which is frequently used for forecasting the price of a variety of options in financial industry (cf. Boyle, 1977). The CEV model is popularly used for modeling derivatives in stock markets. Typically, when the elasticity parameter γ_t is less than 1, the CEV price is similar to the real market data as studied by Chen *et al.* (2009). So, we use alternative market data generated by Monte-Carlo simulation with the CEV model as a benchmark of comparison.

Throughout this section, we use the following values of parameters whenever they are required to be fixed; $s = 50$, $v = 30$, $K = 50$, $r = 0$, $T = 3$, $t = 0$, $\sigma_v = 0.125 \sigma_s = 0.2, \rho_{sy} = 0.2$, $\rho_{vy} = 0.1$, $\rho_{sv} = 0$, $\alpha = 0.5$, D^*=50, $D = 60$, $\varepsilon = 0.01$, $\delta = 0.02$, $\langle \psi' \rangle = 8$, $\langle f \rangle = 0.3, \langle f^2 \rangle = 0.01$, and u=0.7.

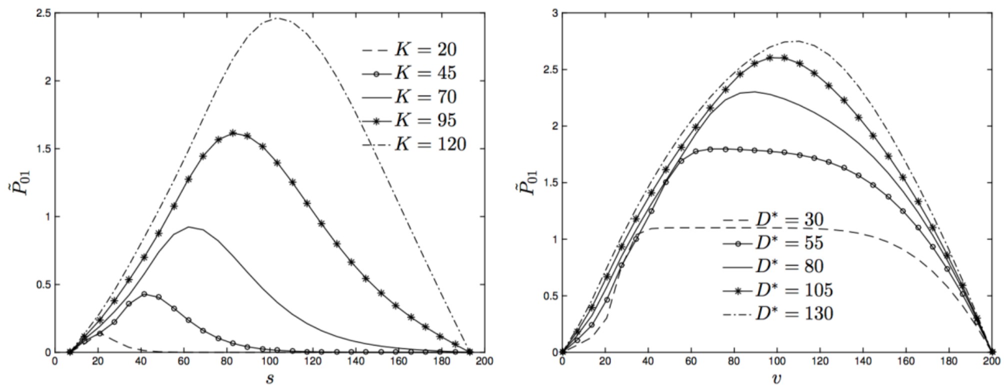
Table 1 shows a general comparison of the prices of a vulnerable European call option under the Black-Scholes model, the CEV model and the SEV model against the underlying asset price. It says that the CEV model raises the Black-Scholes prices but the SEV model lowers the CEV option prices.

Table 1. The prices of a vulnerable option under the Black-Scholes model, the CEV model and the SEV model

s	P_{BS}	P_{CEV}	$\tilde{P}$
35	0.2856	0.5548	0.5377
40	0.8768	1.2564	1.1781
45	1.8184	2.2617	2.1167
50	3.0799	3.5341	3.3379
55	4.6001	5.0239	4.8019
60	6.3128	6.6822	6.4584
65	8.1604	8.4665	8.2587

Now, we investigate the correction effects generated by the constant or stochastic elasticity of variance factor for different strike prices and default levels in detail. Note that the Black-Scholes model has a constant volatility and so it has zero elasticity of variance whereas the elasticity of variance of the SEV model changes randomly around zero in time. Recall that the option price can be decomposed into the intrinsic value and time value, where the time value partly arises from the uncertainty of movements of the underlying asset's future price. If one fixes conditions other than the volatility and the underlying asset price, one can observe the time value against the underlying asset price based on market data traded in financial market. In general, the time value appears to be relatively lager when the option is at-the-money than in-the-money or out-of-money. We want to observe how constant ($\tilde{P}_{01}$) as well as random $(\tilde{P}_{11})$change of elasticity of variance can affect the option price now.

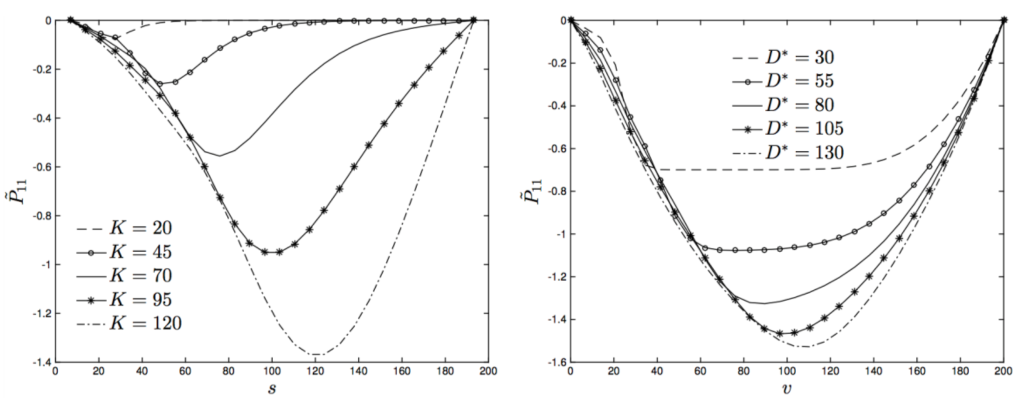
Figure 1 (a) and (b) are the graphs of $\tilde{P}_{01}$against the underlying price and the value of option writer's assets, respectively. Figure 1 (a) shows that the non-zero constant elasticity of variance raises Klein's option price and the effect is remarkable



(a) $\tilde{P}_{01}$ against S for different values of K

(b) $\tilde{P}_{01}$ against V for different values of D^*

Figure 1: The correction effect given by the term $\tilde{P}_{01}$



(a) $\tilde{P}_{11}$ against S for different values of K

(b) $\tilde{P}_{11}$ against V for different values of D^*

Figure 2: The correction effect given by the term $\tilde{P}_{11}$

near the money. Also, the raising effect increases as the strike price □ goes up. Figure 1 (b) shows that in general the raising effect increases as the default level □* rises. The effect reaches a maximum level as the value of option writer's assets approaches the default level and it tends to maintain the maximum level longer as the default level is lower.

On the other hand, Figure 2 shows the correction effect of stochastic elasticity of variance on the CEV option price. Contrary to the case of constant elasticity of variance, in which the non-zero elasticity of variance heightens the Klein price, stochastic elasticity of variance lowers the CEV option price (which includes the Klein price of course) and the effect is relatively large also near the money. The lowering effect increases as the strike price □ goes up while at each strike price it reaches a maximum level as the value of option writer's assets approaches the default level□*. The maximum level of the lowering effect stays longer as the default level is lower.

5. Conclusions

We have derived an analytic formula of the approximate price of a European-type vulnerable option when the underlying asset price follows a stochastic elasticity of variance diffusion process and compared with the corresponding price driven by the constant elasticity of variance model. The resultant option prices are extensions of the result of Klein (1996) which was built based upon the Black-Scholes model. From the formula, we have obtained the following findings. First, the non-zero constant elasticity of variance heightens Klein's option prices whereas the stochastic elasticity of variance lowers the option prices. Second, the heightening or lowering effect highly depends on the underlying price, the strike price, the value of option writer's assets and the default level. Third, the heightening or lowering effect comes close to the maximum level near the money at each strike price or when the value of option writer's assets approaches the default level. Fourth, the maximum level of the heightening of lowering effect tends to stay longer against the value of option writer's assets as the default level gets lower. Based upon this sensitivity analysis on the CEV and SEV models, we believe that one can design better a dynamic investment strategy reducing option writer's credit risk.

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