Associate Professor Ciprian ȘIPOȘ, PhD
E-mail: ciprian.sipos@e-uvt.ro
Lecturer Ioana VIAȘU, PhD
E-mail: ioana.viasu@e-uvt.ro
Faculty of Economics and Business Administration
West University of Timisoara

LONG RUN MULTIPLE CAUSALITY MEASURE ON ECONOMIC GROWTH

Abstract. *In a recent published paper, the second author studied the problem of causality between economic growth and investment in education, by using the method developed by Dufour and Taamouti. In this paper we intend to extend this analysis by considering the case of three variables: gross domestic product, investment in education, and investment in physical capital, all variables being considered as per-capita quantities. We try to highlight the explicit form of a VAR model, to emphasize the evolutionary dynamics and to make a comparative study of different types of economies: Germany and France on the one hand and Romania on the other. The main aim of this paper is to determine the measure of causality effect of the two types of investments on economic growth. The results largely confirm the theoretical assumptions of the endogenous models.*

Keywords: *causality measures, economic growth, vector autoregressive model.*

JEL Classification:C01, C32, C51

1. Introduction

The theoretical notion of causality was first studied by Wiener (Wiener, 1956) and a few years later by Granger (Granger, 1969) and may be considered as a fundamental concept to analyze the dynamic relations between the time series. This concept was defined in terms of predictability at horizon one, of a variable X from its own past and the past of a variable Y. The following definition of causality was formulated by Granger and it has multiple advantages, among them its facility to be tested by econometric methods:"A variable y_t causes the variable x_t, if the variance of the predicted errors of the variable x_t, by using its own past and the past of variable y_t, is lower than the variance of the predicted errors of the variable x_t, obtained only by knowing its own past":

$$\sigma_\epsilon^2(x_t|x_{t-1},x_{t-2},\dots,y_{t-1},y_{t-2},\dots) \leq \sigma_\epsilon^2(x_t|x_{t-1},x_{t-2},\dots), \qquad (1)$$

This theory is well-known today as the Wiener-Granger causality theory and has risen to an appreciable number of papers. Among them, the paper of Geweke (Geweke, 1982), can be considered as the reference work in the field.

If there are only two variables x and y in a stationary bivariate VAR(1) model

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \tag{2}$$

then a necessary and sufficient condition for variable y not to Granger-cause variable x is that $a_{12} = 0$ and this condition is good for all forecast horizons h. Another problem that has needed to be clarified is the way in which the causality is transmitted. In order to understand this aspect we consider the case of the following stationary bivariate VAR(1) model:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0.50 & 0.70 \\ 0.40 & 0.35 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix} \tag{3}$$

so that x_t is given by the equation:

$$x_t = 0.50x_{t-1} + 0.70y_{t-1} + u_t \tag{4}$$

The coefficient of y_{t-1} in equation (4) is equal to 0.70 and thus we can claim that y causes x in the sense of Granger. However, this information does not clarify if a causality at horizons larger than one exists or not, nor on the degree of intensity of this one. We try now to analyse the existence of causality at horizon two, considering the above system (3) at time t + 1 to obtain:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.530 & 0.595 \\ 0.340 & 0.402 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0.50 & 0.70 \\ 0.40 & 0.35 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} \tag{5}$$

In particular, x_{t+1} is given by

$$x_{t+1} = 0.530x_{t-1} + 0.595y_{t-1} + 0.50u_t + 0.70v_t + u_{t+1}. \tag{6}$$

The coefficient of y_{t-1} in equation (6) is equal to 0.595 and consequently we can claim that y causes x at horizon two. The question is now if we can measure the degree of importance of this long-run causality. Examining the existing literature, we conclude that the classical measures do not answer to this question.

In economic systems there are usually more than two variables and consequently, it is highly desirable to extent this concept to higher dimensional systems. Suppose we have a three dimensional system with VAR representation

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 0.50 & 0 & 0.70 \\ 0 & 0.60 & 0 \\ 0 & 0.40 & 0.50 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} \tag{7}$$

so that $\mathrm{x_t}$ is given by the equation:

$$x_t = 0.50\, x_{t-1} + 0.70\, z_{t-1} + u_t \tag{8}$$

The coefficient of y_t in equation (8) is zero and based on the Granger causality, we can conclude that y does not cause x at horizon one. Let us now we consider the model (7) at time t + 1 and we get:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.28 & 0.70 \\ 0 & 0.36 & 0 \\ 0 & 0.44 & 0.25 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} 0.50 & 0 & 0.70 \\ 0 & 0.60 & 0 \\ 0 & 0.40 & 0.50 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ v_{t+1} \\ w_{t+1} \end{bmatrix} \tag{9}$$

so that $\mathrm{x_{t+1}}$ is given by the equation:

$$x_{t+1} = 0.25\, x_{t-1} + 0.28 y_{t-1} + 0.70\, z_{t-1} + 0.50 u_t + 0.70 w_t + u_{t+1} \tag{10}$$

Now we observe that the coefficient of $\mathrm{y_{t-1}}$ in equation (10) is equal to 0.28. Accordingly to the Granger definition of causality, y causes x at horizon two. Consequently, the absence of causality at horizon one does not exclude the possibility of a causality at horizon two. This effect is transmitted by the variable z, via the coefficients 0.40 (the coefficients of the one period effect of y on z) and 0.70 (the coefficients of the one period effect of z on x). At this point we have to answer to the same question as above that is to say, how to measure the importance of this indirect effect. Again, examining the existing literature, we conclude that the classical measures didn't find acceptable answers to our question. However, recent developments have arrived to clarify this question, in a relatively simple way. We mention here, especially the papers of Dufour and Renault (Dufour and Renault, 1998), respectively, Dufour and Taamouti (Dufour and Taamouti,2010), who propose a new definition of the causality measure at any time horizon h > 0.

This paper is organized in four sections, the first one being this introduction. In the second section we provide a VAR model, which will allow us to highlight the new measure of causality. The third section contains the main contribution of our paper and studies the causality between GDP per capita, investment allocated to physical capital and investments allocated to education, and the last section presents final comments and conclusions.

2. A measure of causality - a VAR approach

The starting point of the developments presented in this section is the model introduced by Dufour and Taamouti (Dufour and Taamouti, 2010). Without loss of generality we consider the case of a three-dimensional stationary vector process $W_t = [x_t, y_t, z_t]^T$, where the three variables are denoted by x, y and z, with zero mean, characterized by the following VAR(1) representation:

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \varphi_{xx} & \varphi_{xy} & \varphi_{xz} \\ \varphi_{yx} & \varphi_{yy} & \varphi_{yz} \\ \varphi_{zx} & \varphi_{zy} & \varphi_{zz} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} \tag{11}$$

or more compactly

$$\mathrm{W_t} = \phi \mathrm{W_{t-1}} + \mathrm{U_t}, \phi = \begin{bmatrix} \varphi_{xx} & \varphi_{xy} & \varphi_{xz} \\ \varphi_{yx} & \varphi_{yy} & \varphi_{yz} \\ \varphi_{zx} & \varphi_{zy} & \varphi_{zz} \end{bmatrix}, \ \mathrm{W_t} = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}, \ \mathrm{U_t} = \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix}, \tag{12}$$

where u_t, v_t and w_t are Wiener processes with zero means and constant variances and thus $\mathrm{U_t}$ is a three-dimensional Wiener process with nonsingular variance-covariance matrix Σ_u. Of course, Σ_u is a symmetric positive definite matrix. This model is also called the unconstrained model.

The model from relation (12) can also be written:

$$\Phi(L)W_t = U_t \tag{13}$$

The statement of stationarity can be replaced by the claim that the value of the roots of the lag polynomial

$$det[\Phi(L)] = 0 \tag{14}$$

are all superior, in absolute value to one, where $\Phi(L) = I_3 - \phi L$ and ϕL signifies the multiplication of matrix ϕ with variable L:

$$\Phi(L) = \begin{bmatrix} \varphi_{xx}(L) & \varphi_{xy}(L) & \varphi_{xz}(L) \\ \varphi_{yx}(L) & \varphi_{yy}(L) & \varphi_{yz}(L) \\ \varphi_{zx}(L) & \varphi_{zy}(L) & \varphi_{zz}(L) \end{bmatrix} = \begin{bmatrix} 1-\varphi_{xx}L & -\varphi_{xy}L & -\varphi_{xz}L \\ -\varphi_{yx}L & 1-\varphi_{yy}L & -\varphi_{yz}L \\ -\varphi_{zx}L & -\varphi_{zy}L & 1-\varphi_{zz}L \end{bmatrix} \quad (15)$$

The matrix $\Phi^*(\mathrm{L})$, also called the adjoint matrix of the matrix $\Phi(\mathrm{L})$ is given by:

$$\Phi^*(\mathrm{L}) = \begin{bmatrix} \varphi^*_{xx}(L) & \varphi^*_{xy}(L) & \varphi^*_{xz}(L) \\ \varphi^*_{yx}(L) & \varphi^*_{yy}(L) & \varphi^*_{yz}(L) \\ \varphi^*_{zx}(L) & \varphi^*_{zy}(L) & \varphi^*_{zz}(L) \end{bmatrix} \quad (16)$$

With

$$\begin{gathered} \varphi^*_{xx}(L) = 1-\left(\varphi_{yy}+\varphi_{zz}\right)L + \left(\varphi_{yy}\varphi_{zz}-\varphi_{yz}\varphi_{zy}\right)L^2, \\ \varphi^*_{xy}(L) = \varphi_{yx}L + \left(\varphi_{yz}\varphi_{zx}-\varphi_{yx}\varphi_{zz}\right)L^2, \\ \varphi^*_{xz}(L) = \varphi_{zx}L + \left(\varphi_{yx}\varphi_{zy}-\varphi_{zx}\varphi_{yy}\right)L^2, \\ \varphi^*_{yx}(L) = \varphi_{xy}L + \left(\varphi_{xz}\varphi_{zy}-\varphi_{xy}\varphi_{zz}\right)L^2, \\ \varphi^*_{yy}(L) = 1-(\varphi_{xx}+\varphi_{zz})L + (\varphi_{xx}\varphi_{zz}-\varphi_{zx}\varphi_{xz})L^2, \quad (17) \\ \varphi^*_{yz}(L) = \varphi_{zy}L + \left(\varphi_{zx}\varphi_{xy}-\varphi_{zy}\varphi_{xx}\right)L^2, \\ \varphi^*_{zx}(L) = \varphi_{xz}L + \left(\varphi_{xy}\varphi_{yz}-\varphi_{xz}\varphi_{yy}\right)L^2, \\ \varphi^*_{zy}(L) = \varphi_{yz}L + \left(\varphi_{xz}\varphi_{yx}-\varphi_{yz}\varphi_{xx}\right)L^2, \\ \varphi^*_{zz}(L) = 1-\left(\varphi_{xx}+\varphi_{yy}\right)L + \left(\varphi_{xx}\varphi_{yy}-\varphi_{xy}\varphi_{yx}\right)L^2 \end{gathered}$$

The following equation is obviously true

$$\Phi^*(L)\Phi(L) = det[\Phi(L)]I_3 \quad (18)$$

And finally obtain:

$$\mathrm{Det}[\Phi(\mathrm{L})] = 1-\varphi_1\mathrm{L}+\varphi_2\mathrm{L}^2 - \varphi_3\mathrm{L}^3 \quad (19)$$

Where:

$$\begin{gathered} \varphi_1 = \varphi_{xx}+\varphi_{yy}+\varphi_{zz}, \\ \varphi_2 = \varphi_{yy}+\varphi_{xx}\varphi_{zz}+\varphi_{yy}\varphi_{zz}-\varphi_{xy}\varphi_{yx}-\varphi_{xz}\varphi_{zx}-\varphi_{yz}\varphi_{zy}, \\ \varphi_3 = \varphi_{xx}\varphi_{yy}\varphi_{zz}-\varphi_{xx}\varphi_{yz}\varphi_{zy}-\varphi_{xy}\varphi_{yx}\varphi_{zz}+\varphi_{xy}\varphi_{yz}\varphi_{zx}+ \\ +\varphi_{xz}\varphi_{yx}\varphi_{zy}-\varphi_{xz}\varphi_{yy}\varphi_{zx} \end{gathered} \quad (20)$$

Under stationarity, W_t has the following $VAR(\infty)$ representation:

$$W_t = \Psi(L)U_t, \ \ \Psi(L) = \Phi^{-1}(L) = \sum_{j=0}^{\infty} \Psi_j L^j, \Psi_0 = I_3 \qquad (21)$$

$$W_t = \sum_{j=0}^{\infty} \psi_j \, U_{t-j}, \text{where} \ \ \psi_j = \phi^j \ \text{and} \ \psi_0 = \phi^0 = I_3 \qquad (22)$$

Accordingly to Dufour and Taamouti (Dufour and Taamouti, 2010), the two causality measures: from y to x, denoted by $CL_{yx}(h)$ and from z to x, denoted by $CL_{zx}(h)$, at any horizon h, are given by:

$$CL_{yx}(h) = \ln\left[\frac{Var[x_{t+h}|x_t, z_t]}{Var[x_{t+h}|x_t, y_t, z_t]}\right] \text{ and } CL_{zx}(h) = \ln\left[\frac{Var[x_{t+h}|x_t, y_t]}{Var[x_{t+h}|x_t, y_t, z_t]}\right] \ (23)$$

The values $Var[x_{t+h}|x_t, z_t]$ and $Var[x_{t+h}|x_t, y_t]$ represent the variances of the constrained models and $Var[x_{t+h}|x_t, y_t, z_t]$ represents the variance of the unconstrained model. Regarding the predictability, this can be judged as the amount of information produced, by the past of the variable y and respectively, by the past of the variable z, that can improve the forecast of x (t+h). As a result of the Geweke causality definition, this measure can be interpreted, as a proportional reduction of the variance of the forecast error of x (t + h) obtained by taking into account the past of y and respectively the past of z. From the two above relations in equation (23), we deduce that the measures of causality are defined in terms of variance-covariance matrices of the constrained and unconstrained forecast errors. To compute these measures, we need to determine the structure of the constrained model and this one can be deduced from the structure of the unconstrained model (12) using the following proposition (Lütkepohl, 1993).

Proposition 1(Linear transformation of a VAR(p) process): Let U_t be a p-dimensional white noise process with nonsingular variance-covariance matrix, W_t be a p-dimensional VAR(p) process and let F be a (m, k) matrix of rank m. Then the process $V_t = FW_t$, has an invertible VARMA($\bar{p}, \bar{q}$) representation with $\bar{p} \leq kp$ and $\bar{q} \leq (k-1)p$.

Suppose now that we are interested in measuring the causality from y to x at a given horizon h, for the case of our VAR(1) process, that is for p = 1, m = 2 and k = 3. Consequently we have: $\bar{p} \leq 3$ *and* $\bar{q} \leq 2$.We need to apply Proposition 1 to obtain the structure of process $W_t = [x_t, z_t]^{T\prime}$. If we left-multiply equation (13) by the adjoint matrix of $\Phi(L)$, denoted $\Phi^*(L)$, we get

$$\Phi^*(L)\Phi(L)W_t = \Phi^*(L)U_t, \qquad (24)$$

Where:

$$\Phi^*(L)\Phi(L) = \det[\Phi(L)]I_3 \qquad (25)$$

Thus, equation (24) can be written as follows:

$$\det[\Phi(\mathrm{L})]\mathrm{W_t} = \Phi^*(\mathrm{L})\mathrm{U_t} \qquad (26)$$

The equation (26) is another representation of the stationary invertible VAR process W_t, also called the marginal representation form. The model of the process $\mathrm{V_t} = [\mathrm{x_t}, \mathrm{z_t}]^{\mathrm{T}}$ can be obtained by choosing $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

On pre-multiplying (26) by F, we get

$$\det[\Phi(\mathrm{L})]\mathrm{V_t} = \mathrm{F}\Phi^*(\mathrm{L})\mathrm{U_t} \qquad (27)$$

The right-hand side of (27) is a linearly transformed finite-order VMA process which, by Proposition 1, has a VMA($\bar{q}$) representation with $\bar{q} \leq p(k-1) = 2$. Thus, we get the model:

$$\theta(L)\mathcal{E}_t = \begin{bmatrix} \theta_{xx}(L) & \theta_{xz}(L) \\ \theta_{zx}(L) & \theta_{zz}(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{z,t} \end{bmatrix}, \mathcal{E}_t = \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{z,t} \end{bmatrix}, E(\mathcal{E}_t) = 0,$$

$$E(\mathcal{E}_t \mathcal{E}_s) = \begin{cases} \Sigma_\varepsilon \; if \; t = s \\ 0 \; if \; t \neq s \end{cases} \qquad (28)$$

The elements of the matrix $\theta(L)$ can be determined as follows: $\theta_{xx}(L) = 1 + \theta_{11}L + \bar{\theta}_{11}L^2$, $\theta_{zz}(L) = 1 + \theta_{22}L + \bar{\theta}_{22}L^2$, $\theta_{xz}(L) = \theta_{12}L + \bar{\theta}_{12}L^2$, $\theta_{zx}(L) = \theta_{21}L + \bar{\theta}_{21}L^2$ and thus we can write:

$$\theta(L)\mathcal{E}_t = \begin{bmatrix} \varepsilon_{x,t} + (\theta_{11}\varepsilon_{x,t} + \theta_{12}\varepsilon_{z,t})L + (\bar{\theta}_{11}\varepsilon_{x,t} + \bar{\theta}_{12}\varepsilon_{z,t})L^2 \\ \varepsilon_{z,t} + (\theta_{21}\varepsilon_{x,t} + \theta_{22}\varepsilon_{z,t})L + (\bar{\theta}_{21}\varepsilon_{x,t} + \bar{\theta}_{22}\varepsilon_{z,t})L^2 \end{bmatrix} \qquad (29)$$

The right-hand side of the system (27) can also be written:

$$\mathrm{F}\Phi^*(\mathrm{L})\mathrm{U_t} = \begin{bmatrix} \varphi^*_{xx}(L)u_t + \varphi^*_{xy}(L)v_t + \varphi^*_{xz}(L)w_t \\ \varphi^*_{zx}(L)u_t + \varphi^*_{zy}(L)v_t + \varphi^*_{zz}(L)w_t \end{bmatrix} \qquad (30)$$

Substituting now the corresponding relations from equation (16) we get:

$$\mathrm{F\Phi^*(L)U_t} = \begin{bmatrix} u_t + a_{11}(u_t, v_t, w_t)L + a_{12}(u_t, v_t, w_t)L^2 \\ w_t + a_{21}(u_t, v_t, w_t)L + a_{22}(u_t, v_t, w_t)L^2 \end{bmatrix} \tag{31}$$

With

$$\begin{gathered} a_{11}(u_t, v_t, w_t) = -(\varphi_{yy} + \varphi_{zz})u_t + \varphi_{yx}v_t + \varphi_{zx}w_t, \\ a_{12}(u_t, v_t, w_t) = (\varphi_{yy}\varphi_{zz} - \varphi_{yz}\varphi_{zy})u_t + (\varphi_{yz}\varphi_{zx} - \varphi_{yx}\varphi_{zz})v_t + \\ +(\varphi_{yx}\varphi_{zy} - \varphi_{zx}\varphi_{yy})w_t, \\ a_{21}(u_t, v_t, w_t) = \varphi_{xz}u_t + \varphi_{yz}v_t - (\varphi_{xx} + \varphi_{yy})w_t \\ a_{22}(u_t, v_t, w_t) = (\varphi_{xy}\varphi_{yz} - \varphi_{xz}\varphi_{yy})u_t + (\varphi_{xz}\varphi_{yx} - \varphi_{xx}\varphi_{yz})v_t + \\ +(\varphi_{xx}\varphi_{yy} - \varphi_{xy}\varphi_{yx})w_t \end{gathered} \tag{32}$$

The left-hand side of (27) will generate the following two-dimensional VAR(3) model:

$$V_t = \varphi_1 V_{t-1} + \varphi_2 V_{t-2} - \varphi_3 V_{t-3} \tag{33}$$

The right-hand side of (27) will generate the model:

$$\mathrm{M1}_t = \begin{bmatrix} \varepsilon_{x,t} + \theta_{11}\varepsilon_{x,t-1} + \theta_{12}\varepsilon_{z,t-1} + \bar{\theta}_{11}\varepsilon_{x,t-2} + \bar{\theta}_{12}\varepsilon_{z,t-2} \\ \varepsilon_{z,t} + \theta_{21}\varepsilon_{x,t-1} + \theta_{22}\varepsilon_{z,t-1} + \bar{\theta}_{21}\varepsilon_{x,t-2} + \bar{\theta}_{22}\varepsilon_{z,t-2} \end{bmatrix}, \tag{34}$$

and right-hand side of (27), via (28) will generate the following model:

$$M2_t = \begin{bmatrix} u_t + a_{11}u_{t-1} + a_{12}v_{t-1} + a_{13}w_{t-1} + b_{11}u_{t-2} + b_{12}v_{t-2} + b_{13}w_{t-2} \\ w_t + a_{21}u_{t-1} + a_{22}v_{t-1} + a_{23}w_{t-1} + b_{21}u_{t-2} + b_{22}v_{t-2} + b_{23}w_{t-2} \end{bmatrix} \tag{35}$$

Where:

$$\begin{gathered} a_{11} = -(\varphi_{yy} + \varphi_{zz}), a_{12} = \varphi_{yx}, a_{13} = \varphi_{zx}, \\ a_{21} = \varphi_{xz}, a_{22} = \varphi_{yz}, a_{23} = -(\varphi_{xx} + \varphi_{yy}), \\ b_{11} = \varphi_{yy}\varphi_{zz} - \varphi_{yz}\varphi_{zy},\ b_{12} = \varphi_{yz}\varphi_{zx} - \varphi_{yx}\varphi_{zz}, b_{13} = \varphi_{yx}\varphi_{zy} - \varphi_{zx}\varphi_{yy}, \\ b_{21} = \varphi_{xy}\varphi_{yz} - \varphi_{xz}\varphi_{yy},\ b_{22} = \varphi_{xz}\varphi_{yx} - \varphi_{xx}\varphi_{yz},\ b_{23} = \varphi_{xx}\varphi_{yy} - \varphi_{xy} \end{gathered}$$

We obviously have $\mathrm{M1}_t = \mathrm{M2}_t$ and we denote this by Ω_t.Thus we have, in the same time:

$$V_t = \varphi_1 V_{t-1} - \varphi_2 V_{t-2} + \varphi_3 V_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} + \bar{\theta}\varepsilon_{t-2} \tag{36}$$

$$V_t = \varphi_1 V_{t-1} - \varphi_2 V_{t-2} + \varphi_3 V_{t-3} + \overline{U}_t + AU_{t-1} + BU_{t-2} \tag{37}$$

In which,

$$V_t = \begin{bmatrix} x_t \\ z_t \end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{z,t} \end{bmatrix}, \theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}, \bar{\theta} = \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} \\ \bar{\theta}_{21} & \bar{\theta}_{22} \end{bmatrix},$$

$$\overline{U}_t = \begin{bmatrix} u_t \\ w_t \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \tag{38}$$

To determine the parameters of constrained model in terms of parameters of unconstrained model, using first the relation (36), we obtain:

$$C = E(\Omega_t \Omega_t^T) = \begin{bmatrix} Var(\omega_1) & Cov(\omega_1, \omega_2) \\ Cov(\omega_2, \omega_1) & Var(\omega_2) \end{bmatrix},$$

$$C = E[(\varepsilon_t + \theta\varepsilon_{t-1} + \bar{\theta}\varepsilon_{t-2})(\varepsilon_t^T + \varepsilon_{t-1}^T\theta^T + \varepsilon_{t-2}^T\bar{\theta}^T)],$$

$$C = \Sigma_\varepsilon + \theta\Sigma_\varepsilon\theta^T + \bar{\theta}\Sigma_\varepsilon\bar{\theta}^T = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}, \Sigma_\varepsilon = \begin{bmatrix} \sigma_{\varepsilon x}^2 & \sigma_{\varepsilon xz} \\ \sigma_{\varepsilon xz} & \sigma_{\varepsilon z}^2 \end{bmatrix} \tag{39}$$

More explicitly we can write:

$$\begin{aligned} Var(\omega_1) = (1 + \theta_{11}^2 + \bar{\theta}_{11}^2)\sigma_{\varepsilon x}^2 + (\theta_{12}^2 + \bar{\theta}_{12}^2)\sigma_{\varepsilon z}^2 + \\ +2(\theta_{11}\theta_{12} + \bar{\theta}_{11}\bar{\theta}_{12})\sigma_{\varepsilon xz} \end{aligned} \tag{40}$$

$$\begin{aligned} Cov(\omega_1, \omega_2) = (\theta_{11}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2 + (\theta_{12}\theta_{22} + \bar{\theta}_{12}\bar{\theta}_{22})\sigma_{\varepsilon z}^2 + \\ +(1 + \theta_{11}\theta_{22} + \theta_{12}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{22} + \bar{\theta}_{12}\bar{\theta}_{21})\sigma_{\varepsilon xz} \end{aligned} \tag{41}$$

$$\begin{aligned} Var(\omega_2) = (\theta_{21}^2 + \bar{\theta}_{21}^2)\sigma_{\varepsilon x}^2 + (1 + \theta_{22}^2 + \bar{\theta}_{22}^2)\sigma_{\varepsilon z}^2 + \\ +2(\theta_{21}\theta_{22} + \bar{\theta}_{21}\bar{\theta}_{22})\sigma_{\varepsilon xz} \end{aligned} \tag{42}$$

Then, using the relation (37), we obtain:

$$C = E[(\overline{U}_t + AU_{t-1} + BU_{t-2})(\overline{U}_t^T + U_{t-1}^T A^T + U_{t-2}^T B^T)] =$$

$$= \Sigma_{\overline{U}} + A\Sigma_U A^T + B\Sigma_U B^T = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}, \Sigma_{\overline{U}} = \begin{bmatrix} \sigma_u^2 & \sigma_{uw} \\ \sigma_{uw} & \sigma_w^2 \end{bmatrix},$$

$$\Sigma_U = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} & \sigma_{uw} \\ \sigma_{uv} & \sigma_v^2 & \sigma_{vw} \\ \sigma_{uw} & \sigma_{vw} & \sigma_w^2 \end{bmatrix} \tag{43}$$

The elements of the matrix C given by (43) are all known. Equating now (40), (41) and (42) with the corresponding elements of (43) we obtain the first system of three equations.

What we need now is to determine the covariance function of the two white noise processes, $\omega_{1,t}$ and $\omega_{2,t}$.Thus, from equation (36) we obtain:

$$C1 = E(\Omega_{t+1}, \Omega_t) = E[(\varepsilon_{t+1} + \theta\varepsilon_t + \bar{\theta}\varepsilon_{t-1})(\varepsilon_t^T + \varepsilon_{t-1}^T\theta^T + \varepsilon_{t-2}^T\bar{\theta}^T)],$$

$$C1 = \theta\Sigma_\varepsilon + \bar{\theta}\Sigma_\varepsilon\theta^T = \begin{bmatrix} C1_{11} & C1_{12} \\ C1_{21} & C1_{22} \end{bmatrix} \tag{44}$$

More explicitly we can write:

$$Cov(\omega_{1,t+1}, \omega_{1,t}) = \theta_{11}(1 + \bar{\theta}_{11})\sigma_{\varepsilon x}^2 + [\theta_{12}(1 + \bar{\theta}_{11}) + \theta_{11}\bar{\theta}_{12}]\sigma_{\varepsilon xz} + \\ + \theta_{12}\bar{\theta}_{12}\sigma_{\varepsilon z}^2 \tag{45}$$

$$Cov(\omega_{1,t+1}, \omega_{2,t}) = \theta_{21}\bar{\theta}_{11}\sigma_{\varepsilon x}^2 + (\theta_{11} + \theta_{21}\bar{\theta}_{12} + \theta_{22}\bar{\theta}_{11})\sigma_{\varepsilon xz} + \\ + (\theta_{12} + \theta_{22}\bar{\theta}_{12})\sigma_{\varepsilon z}^2 \tag{46}$$

$$Cov(\omega_{2,t+1}, \omega_{1,t}) = (\theta_{21} + \theta_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2 + \\ + (\theta_{22} + \theta_{11}\bar{\theta}_{22} + \theta_{12}\bar{\theta}_{21})\sigma_{\varepsilon xz} + \theta_{12}\bar{\theta}_{22}\sigma_{\varepsilon z}^2 \tag{47}$$

$$Cov(\omega_{2,t+1}, \omega_{2,t}) = \theta_{21}\bar{\theta}_{21}\sigma_{\varepsilon x}^2 + [\theta_{21}(1 + \bar{\theta}_{22}) + \theta_{22}\bar{\theta}_{21}]\sigma_{\varepsilon xz} + \\ + \theta_{22}(1 + \bar{\theta}_{22})\sigma_{\varepsilon z}^2 \tag{48}$$

From equation (37) we obtain:

$$C1 = E[(\bar{U}_{t+1} + AU_t + BU_{t-1})(\bar{U}_t^T + U_{t-1}^T A^T + U_{t-2}^T B^T)],$$

$$C1 = A\Sigma_{U1} + B\Sigma_U A^T = \begin{bmatrix} C1_{11} & C1_{12} \\ C1_{21} & C1_{22} \end{bmatrix}, \Sigma_{U1} = \begin{bmatrix} \sigma_u^2 & \sigma_{uw} \\ \sigma_{uv} & \sigma_{vw} \\ \sigma_{uw} & \sigma_w^2 \end{bmatrix} \tag{49}$$

The elements of the matrix C1 given by (49) are all known. Equating now (45) – (48) with the corresponding elements of (49) we obtain the second system of four equations. Thus, from equation (36) we obtain:

$$C2 = E(\Omega_{t+2}, \Omega_t) = E[(\varepsilon_{t+2} + \theta\varepsilon_{t+1} + \bar{\theta}\varepsilon_t)(\varepsilon_t^T + \varepsilon_{t-1}^T\theta^T + \varepsilon_{t-2}^T\bar{\theta}^T)]\,,$$

$$C2 = \bar{\theta}\Sigma_\varepsilon = \begin{bmatrix} C2_{11} & C2_{12} \\ C2_{21} & C2_{22} \end{bmatrix} \quad (50)$$

More explicitly we can write:

$$Cov(\omega_{1,t+2}, \omega_{1,t}) = \bar{\theta}_{11}\sigma_{\varepsilon x}^2 + \bar{\theta}_{12}\sigma_{\varepsilon xz}, \quad (51)$$

$$Cov(\omega_{1,t+2}, \omega_{2,t}) = \bar{\theta}_{11}\sigma_{\varepsilon xz} + \bar{\theta}_{12}\sigma_{\varepsilon z}^2, \quad (52)$$

$$Cov(\omega_{2,t+2}, \omega_{1,t}) = \bar{\theta}_{21}\sigma_{\varepsilon x}^2 + \bar{\theta}_{22}\sigma_{\varepsilon xz}, \quad (53)$$

$$Cov(\omega_{2,t+2}, \omega_{2,t}) = \bar{\theta}_{21}\sigma_{\varepsilon xz} + \bar{\theta}_{22}\sigma_{\varepsilon z}^2 \quad (54)$$

From equation (37) we get:

$$C2 = E[(\bar{U}_{t+2} + AU_{t+1} + BU_t)(\bar{U}_t^T + U_{t-1}^T A^T + U_{t-2}^T B^T)] =$$

$$= B\Sigma_{U1} = \begin{bmatrix} C2_{11} & C2_{12} \\ C2_{21} & C2_{22} \end{bmatrix} \quad (55)$$

The elements of the matrix C2 given by (55) are all known. Equating now (51) – (54) with the corresponding elements of (55) we obtain the third system of four equations. Combining the three systems determined above we finally obtain a non-linear system of eleven equations with eleven unknowns. We denote this system by following equations:

$$(1 + \theta_{11}^2 + \bar{\theta}_{11}^2)\sigma_{\varepsilon x}^2 + 2(\theta_{11}\theta_{12} + \bar{\theta}_{11}\bar{\theta}_{12})\sigma_{\varepsilon xz} + (\theta_{12}^2 + \bar{\theta}_{12}^2)\sigma_{\varepsilon z}^2 = C_{11}$$

$$(\theta_{11}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2 + (1 + \theta_{11}\theta_{22} + \theta_{12}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{22} + \bar{\theta}_{12}\bar{\theta}_{21})\sigma_{\varepsilon xz} + (\theta_{12}\theta_{22} + \bar{\theta}_{12}\bar{\theta}_{22})\sigma_{\varepsilon z}^2 = C_{12}$$

$$(\theta_{21}^2 + \bar{\theta}_{21}^2)\sigma_{\varepsilon x}^2 + 2(\theta_{21}\theta_{22} + \bar{\theta}_{21}\bar{\theta}_{22})\sigma_{\varepsilon xz} + (1 + \theta_{22}^2 + \bar{\theta}_{22}^2)\sigma_{\varepsilon z}^2 = C_{22}$$

$$\theta_{11}(1 + \bar{\theta}_{11})\sigma_{\varepsilon x}^2 + [\theta_{12}(1 + \bar{\theta}_{11}) + \theta_{11}\bar{\theta}_{12}]\sigma_{\varepsilon xz} + \theta_{12}\bar{\theta}_{12}\sigma_{\varepsilon z}^2 = C1_{11}$$

$$\theta_{21}\bar{\theta}_{11}\sigma_{\varepsilon x}^2 + (\theta_{11} + \theta_{21}\bar{\theta}_{12} + \theta_{22}\bar{\theta}_{11})\sigma_{\varepsilon xz} + (\theta_{12} + \theta_{22}\bar{\theta}_{12})\sigma_{\varepsilon z}^2 = C1_{12}$$

$$(\theta_{21}+\theta_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2+(\theta_{22}+\theta_{11}\bar{\theta}_{22}+\theta_{12}\bar{\theta}_{21})\sigma_{\varepsilon xz}+\theta_{12}\bar{\theta}_{22}\sigma_{\varepsilon z}^2=C1_{21}$$
$$\theta_{21}\bar{\theta}_{21}\sigma_{\varepsilon x}^2+[\theta_{21}(1+\bar{\theta}_{22})+\theta_{22}\bar{\theta}_{21}]\sigma_{\varepsilon xz}+\theta_{22}(1+\bar{\theta}_{22})\sigma_{\varepsilon z}^2=C1_{22}$$
$$\bar{\theta}_{11}\sigma_{\varepsilon x}^2+\bar{\theta}_{12}\sigma_{\varepsilon xz}=C2_{11}$$
$$\bar{\theta}_{11}\sigma_{\varepsilon xz}+\bar{\theta}_{12}\sigma_{\varepsilon z}^2=C2_{12}$$
$$\bar{\theta}_{21}\sigma_{\varepsilon x}^2+\bar{\theta}_{22}\sigma_{\varepsilon xz}=C2_{21}$$
$$\bar{\theta}_{21}\sigma_{\varepsilon xz}+\bar{\theta}_{22}\sigma_{\varepsilon z}^2=C2_{22} \tag{56}$$

We can now try to measure the causality from z to x at a given horizon h, for the case of the same VAR(1) process, that is for p = 1, m = 2 and k = 3. Using the same procedure as above, the model of the process $V_t=[x_t,y_t]^T$ can be obtained by choosing $F=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ and thus we get:

$$\theta(L)\mathcal{E}_t=\begin{bmatrix}\varepsilon_{x,t}+(\theta_{11}\varepsilon_{x,t}+\theta_{12}\varepsilon_{y,t})L+(\bar{\theta}_{11}\varepsilon_{x,t}+\bar{\theta}_{12}\varepsilon_{y,t})L^2\\ \varepsilon_{y,t}+(\theta_{21}\varepsilon_{x,t}+\theta_{22}\varepsilon_{y,t})L+(\bar{\theta}_{21}\varepsilon_{x,t}+\bar{\theta}_{22}\varepsilon_{y,t})L^2\end{bmatrix} \tag{57}$$

$$F\Phi^*(L)U_t=\begin{bmatrix}u_t+a_{11}(u_t,v_t,w_t)L+a_{12}(u_t,v_t,w_t)L^2\\ v_t+a_{21}(u_t,v_t,w_t)L+a_{22}(u_t,v_t,w_t)L^2\end{bmatrix} \tag{58}$$

Where,

$$a_{11}(u_t,v_t,w_t)=-(\varphi_{yy}+\varphi_{zz})u_t+\varphi_{yx}v_t+\varphi_{zx}w_t,$$
$$a_{12}(u_t,v_t,w_t)=(\varphi_{yy}\varphi_{zz}-\varphi_{yz}\varphi_{zy})u_t+(\varphi_{yz}\varphi_{zx}-\varphi_{yx}\varphi_{zz})v_t + (\varphi_{yx}\varphi_{zy}-\varphi_{zx}\varphi_{yy})w_t,$$
$$a_{21}(u_t,v_t,w_t)=\varphi_{xy}u_t-(\varphi_{xx}+\varphi_{zz})v_t+\varphi_{zy}w_t$$
$$a_{22}(u_t,v_t,w_t)=(\varphi_{xz}\varphi_{zy}-\varphi_{xy}\varphi_{zz})u_t+(\varphi_{xx}\varphi_{zz}-\varphi_{zx}\varphi_{xz})v_t+ (\varphi_{zx}\varphi_{xy}-\varphi_{zy}\varphi_{xx})w_t \tag{59}$$

This time the left-hand side of (27) will generate the two-dimensional VAR(3) model:

$$V_t-\varphi_1V_{t-1}+\varphi_2V_{t-2}-\varphi_3V_{t-3} \tag{60}$$

The right-hand side of equation (27) will generate directly the model:

$$\mathrm{M1}_t = \begin{bmatrix} \varepsilon_{x,t} + \theta_{11}\varepsilon_{x,t-1} + \theta_{12}\varepsilon_{y,t-1} + \bar{\theta}_{11}\varepsilon_{x,t-2} + \bar{\theta}_{12}\varepsilon_{y,t-2} \\ \varepsilon_{y,t} + \theta_{21}\varepsilon_{x,t-1} + \theta_{22}\varepsilon_{y,t-1} + \bar{\theta}_{21}\varepsilon_{x,t-2} + \bar{\theta}_{22}\varepsilon_{y,t-2} \end{bmatrix} \tag{61}$$

and via (28) will generate the following model:

$$M2_t = \begin{bmatrix} u_t - a_{11}u_{t-1} + a_{12}v_{t-1} + a_{13}w_{t-1} + b_{11}u_{t-2} + b_{12}v_{t-2} + b_{13}w_{t-2} \\ v_t + a_{21}u_{t-1} - a_{22}v_{t-1} + a_{23}w_{t-1} + b_{21}u_{t-2} + b_{22}v_{t-2} + b_{23}w_{t-2} \end{bmatrix} \tag{62}$$

Where,

$$a_{11} = -\left(\varphi_{yy} + \varphi_{zz}\right), a_{12} = \varphi_{yx}, a_{13} = \varphi_{zx},$$
$$a_{21} = \varphi_{xy}, a_{22} = -(\varphi_{xx} + \varphi_{zz}), a_{23} = \varphi_{zy},$$
$$b_{11} = \varphi_{yy}\varphi_{zz} - \varphi_{yz}\varphi_{zy},\ b_{12} = \varphi_{yz}\varphi_{zx} - \varphi_{yx}\varphi_{zz}, b_{13} = \varphi_{yx}\varphi_{zy} - \varphi_{zx}\varphi_{yy},$$
$$b_{21} = \varphi_{xz}\varphi_{zy} - \varphi_{xy}\varphi_{zz},\ b_{22} = \varphi_{xx}\varphi_{zz} - \varphi_{zx}\varphi_{xz},\ b_{23} = \varphi_{zx}\varphi_{xy} - \varphi_{zy}\varphi_{xx}. \tag{63}$$

Proceeding as above and denoting $\Omega_t = \mathrm{M1}_t = \mathrm{M2}_t$ we can write:

$$V_t = \varphi_1 V_{t-1} - \varphi_2 V_{t-2} + \varphi_3 V_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} + \bar{\theta}\varepsilon_{t-2} \tag{64}$$

$$V_t = \varphi_1 V_{t-1} - \varphi_2 V_{t-2} + \varphi_3 V_{t-3} + \bar{U}_t + AU_{t-1} + BU_{t-2} \tag{65}$$

Where this time we have:

$$V_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \Sigma_\varepsilon = \begin{bmatrix} \sigma_{\varepsilon x}^2 & \sigma_{\varepsilon xy} \\ \sigma_{\varepsilon xy} & \sigma_{\varepsilon z}^2 \end{bmatrix}, \Sigma_{U1} = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \\ \sigma_{uw} & \sigma_{vw} \end{bmatrix} \tag{66}$$

And thus we obtain an alternative system of equations of (56) where, we simply replace z by y. We denote this system by following equations:

$$(1 + \theta_{11}^2 + \bar{\theta}_{11}^2)\sigma_{\varepsilon x}^2 + 2(\theta_{11}\theta_{12} + \bar{\theta}_{11}\bar{\theta}_{12})\sigma_{\varepsilon xy} + (\theta_{12}^2 + \bar{\theta}_{12}^2)\sigma_{\varepsilon y}^2 = C_{11}$$
$$(\theta_{11}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2 + (1 + \theta_{11}\theta_{22} + \theta_{12}\theta_{21} + \bar{\theta}_{11}\bar{\theta}_{22} + \bar{\theta}_{12}\bar{\theta}_{21})\sigma_{\varepsilon xy}$$
$$+ (\theta_{12}\theta_{22} + \bar{\theta}_{12}\bar{\theta}_{22})\sigma_{\varepsilon y}^2 = C_{12}$$
$$(\theta_{21}^2 + \bar{\theta}_{21}^2)\sigma_{\varepsilon x}^2 + 2(\theta_{21}\theta_{22} + \bar{\theta}_{21}\bar{\theta}_{22})\sigma_{\varepsilon xy} + (1 + \theta_{22}^2 + \bar{\theta}_{22}^2)\sigma_{\varepsilon y}^2 = C_{22}$$

$$\theta_{11}(1+\bar{\theta}_{11})\sigma_{\varepsilon x}^2+[\theta_{12}(1+\bar{\theta}_{11})+\theta_{11}\bar{\theta}_{12}]\sigma_{\varepsilon xy}+\theta_{12}\bar{\theta}_{12}\sigma_{\varepsilon y}^2=C1_{11}$$
$$\theta_{21}\bar{\theta}_{11}\sigma_{\varepsilon x}^2+(\theta_{11}+\theta_{21}\bar{\theta}_{12}+\theta_{22}\bar{\theta}_{11})\sigma_{\varepsilon xy}+(\theta_{12}+\theta_{22}\bar{\theta}_{12})\sigma_{\varepsilon y}^2=C1_{12}$$
$$(\theta_{21}+\theta_{11}\bar{\theta}_{21})\sigma_{\varepsilon x}^2+(\theta_{22}+\theta_{11}\bar{\theta}_{22}+\theta_{12}\bar{\theta}_{21})\sigma_{\varepsilon xy}+\theta_{12}\bar{\theta}_{22}\sigma_{\varepsilon y}^2=C1_{21}$$
$$\theta_{21}\bar{\theta}_{21}\sigma_{\varepsilon x}^2+[\theta_{21}(1+\bar{\theta}_{22})+\theta_{22}\bar{\theta}_{21}]\sigma_{\varepsilon xy}+\theta_{22}(1+\bar{\theta}_{22})\sigma_{\varepsilon y}^2=C1_{22}$$
$$\bar{\theta}_{11}\sigma_{\varepsilon x}^2+\bar{\theta}_{12}\sigma_{\varepsilon xy}=C2_{11}$$
$$\bar{\theta}_{11}\sigma_{\varepsilon xy}+\bar{\theta}_{12}\sigma_{\varepsilon y}^2=C2_{12}$$
$$\bar{\theta}_{21}\sigma_{\varepsilon x}^2+\bar{\theta}_{22}\sigma_{\varepsilon xy}=C2_{21}$$
$$\bar{\theta}_{21}\sigma_{\varepsilon xy}+\bar{\theta}_{22}\sigma_{\varepsilon y}^2=C2_{22} \qquad (67)$$

To quantify the degree of causality from y to x and from z to x, at horizon h, we have to examine the two models: unconstrained and constrained. The two equations describing the constraint and unconstrained model are defined as follows:

$$\mathrm{W_t}=\phi\mathrm{W_{t-1}}+\mathrm{U_t} V_t=\varphi_1 V_{t-1}-\varphi_2 V_{t-2}+\varphi_3 V_{t-3}+\varepsilon_t+\theta\varepsilon_{t-1}+\bar{\theta}\varepsilon_{t-2} (68)$$

The corresponding variances will be computed as follows (see Dufour and Taamouti, 2010):

$$\mathrm{V_1(h)}=\mathrm{Var}[\mathrm{W_t}]=\sum_{i=0}^{h-1} J_1^T\psi_i\Sigma_u\,\psi_i^T J_1, \qquad h=1, \qquad h=2,\ldots \qquad (69)$$
$$\mathrm{V_2(h)}=\mathrm{Var}[V_t]=\sum_{i=0}^{h-1} J_0^T\bar{\psi}_i\Sigma_\varepsilon\,\bar{\psi}_i^T J_0, \qquad h=1, \qquad h=2,\ldots \qquad (70)$$

Where $J_1^T=[1\quad 0\quad 0]^T$, $J_0^T=[1\quad 0]^T$ and $\bar{\psi}_i$ will be computed similarly as ψ_i from the equation (12). Of course, the degree of causality could be determined for any value of h but here we limit our analysis only for h=1, h=2 and h=3. From equation (68) we obtain:

$$\mathrm{W_{t+1}}=\phi\mathrm{W_t}+\mathrm{U_{t+1}},$$
$$\mathrm{W_{t+2}}=\phi^2\mathrm{W_t}+\phi\mathrm{U_{t+1}}+\mathrm{U_{t+2}},$$
$$\mathrm{W_{t+3}}=\phi^3\mathrm{W_t}+\phi^2\mathrm{U_{t+1}}+\phi\mathrm{U_{t+2}}+\mathrm{U_{t+3}},$$
$$V_{t+1}=\varphi_1 V_t-\varphi_2 V_{t-1}+\varphi_3 V_{t-2}+\varepsilon_{t+1}+\theta\varepsilon_t+\bar{\theta}\varepsilon_{t-1},$$
$$V_{t+2}=(\varphi_1^2-\varphi_2)V_t-(\varphi_1\varphi_2-\varphi_3)V_{t-1}+\varphi_1\varphi_3 V_{t-2}+$$

$$+\varphi_1\bar{\theta}\varepsilon_{t-1} + (\varphi_1\theta + \bar{\theta})\varepsilon_t + (\varphi_1 I_2 + \theta)\varepsilon_{t+1} + \varepsilon_{t+2},$$

$$\begin{aligned} V_{t+3} &= [\varphi_1(\varphi_1^2 - \varphi_2) - \varphi_1\varphi_2 + \varphi_3]V_t - [\varphi_1(\varphi_1\varphi_2 - \varphi_3) - \varphi_2^2]V_{t-1} + \\ &+ \varphi_3(\varphi_1^2 - \varphi_2)V_{t-2} + \bar{\theta}(\varphi_1^2 - \varphi_2)\varepsilon_{t-1} + [\varphi_1(\varphi_1\theta + \bar{\theta}) - \varphi_2\theta]\varepsilon_t + \\ &+ [(\varphi_1^2 - \varphi_2)I_2 + \varphi_1\theta + \bar{\theta}]\varepsilon_{t+1} + (\varphi_1 I_2 + \theta)\varepsilon_{t+2} + \varepsilon_{t+3,} \end{aligned} \quad (71)$$

From where we deduce that:

$$COV_1(1) = Var(\mathrm{W_{t+1}}) = \Sigma_\mathrm{u}\,,$$

$$COV_1(2) = Var(\mathrm{W_{t+2}}) = \Sigma_\mathrm{u} + \phi\Sigma_\mathrm{u}\phi^T,\ \psi_0 = I_3,\ \psi_1 = \phi,$$

$$COV_1(3) = Var(\mathrm{W_{t+3}}) = \Sigma_\mathrm{u} + \phi\Sigma_\mathrm{u}\phi^T + \phi^2\Sigma_\mathrm{u}[\phi^T]^2,$$

$$COV_2(1) = Var(\mathrm{V_{t+1}}) = \Sigma_\varepsilon,$$

$$COV_2(2) = Var(\mathrm{V_{t+2}}) = \Sigma_\varepsilon + \overline{\psi}_1\Sigma_\varepsilon\overline{\psi}_1^{\,T}, \qquad \overline{\psi}_0 = I_2, \qquad \overline{\psi}_1 = \varphi_1 I_2 + \theta,$$

$$COV_2(3) = Var(\mathrm{V_{t+3}}) = \Sigma_\varepsilon + \overline{\psi}_1\Sigma_\varepsilon\overline{\psi}_1^{\,T} + \overline{\psi}_2\Sigma_\varepsilon\overline{\psi}_2^{\,T},$$

$$\overline{\psi}_2 = (\varphi_1^2 - \varphi_2)I_2 + \varphi_1\theta + \bar{\theta} \quad (72)$$

Now we can apply this procedure to determine the causality measure from y to x into the model described by equation (36), and to determine the causality measure from z to x into the model (64). Of course, this causality measure is positive. It can be equal to zero, only in the case where the causal relationship between the considered variables is not present. Consequently, we can claim that a higher causality effect generates a higher causality measure.

2. Causality between GDP and investment in physical capital, respectively, investment in education

As it is well-known, physical capital and human capital are the two essentials factors of production. One of the arguments in supporting the conclusion that investments in physical capital and in education do contribute to economic growth, is that almost all developed countries have a high rate of investment in physical capital and a labor force with high level of education. On the other hand, it is also obviously true that investments in physical capital and in education are essentially conditioned by its economic degree of development. The question is if we have enough arguments for all results obtained from mathematical models and using statistical observations.

The results presented in this paper confirm that the answer is positive. More than this by using various statistical tests to the results obtained from these models, we can conclude that the theoretical features are widely consistent with the reality, as expected Sims (Sims, 1980). Proceeding in this manner, we are able to validate some of the theoretical features taken into account in the construction of the endogenous growth models.

The data used in this study are: the gross domestic product, the investment in physical capital and the investment for education, for the period 1991 – 2014, all of them as per-capita quantities, at constant 2005 prices. The source of these data is the database of the World Bank Open Data and concerns the following three countries: France, Germany and Romania.

The test of unit roots was first utilized to decide if all these series are or not stationary. Because the result was negative, we decided to use the alternative series, calculated by the difference. In accordance with the test of unit roots, the transformed series are stationaries. We then estimated the parameters of the unconstrained VAR model via the standard method of least squares and the results for the three countries are given below:

France:

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 0.712490 & -0.731680 & 3.847909 \\ 0.160881 & 0.033472 & 1.093596 \\ 0.106788 & -0.231727 & 0.017128 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix}$$

Germany:

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 0.208608 & -0.222308 & 2.459385 \\ -0.083987 & 0.269590 & 0.814841 \\ 0.033888 & -0.030177 & 0.417882 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix}$$

Romania:

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 0.480092 & 0.098697 & -0.861671 \\ 0.020238 & 0.079549 & 1.685668 \\ 0.041933 & 0.050574 & -0.598364 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix}$$

The unit roots test gives the following values: (1.65; 2.48; -4.08), (1.70; 3.16; -120.42) and respectively (2.16; 5.81; -1.48) and we can observe that all these roots confirm the hypothesis of stationarity.

By using the procedure described above we can quantify the degree of causality from investment in physical capital to gross domestic product and from investment in education to gross domestic product, at horizon 1, 2 and 3. To evaluate the degree of causality we need to write the corresponding equations given by relations (36) and (64). Thus, for each country we have:

France:

$$x_t = 0.763x_{t-1} + 0.003x_{t-2} - 0.060x_{t-3} + \varepsilon_{x,t} - 0.046\varepsilon_{x,t-1}$$
$$+0.053\varepsilon_{z,t-1} + 0.261\varepsilon_{x,t-2} - 0.059\varepsilon_{z,t-2}$$
$$x_t = 0.763x_{t-1} + 0.003x_{t-2} - 0.060x_{t-3} + \varepsilon_{x,t} - 0.051\varepsilon_{x,t-1}$$
$$+0.163\varepsilon_{y,t-1} + 0.254\varepsilon_{x,t-2} + 0.120\varepsilon_{y,t-2}$$

Germany:

$$x_t = 0.896x_{t-1} - 0.179x_{t-2} - 0.002x_{t-3} + \varepsilon_{x,t} - 0.722\varepsilon_{x,t-1}$$
$$+0.285\varepsilon_{z,t-1} + 0.148\varepsilon_{x,t-2} - 0.652\varepsilon_{z,t-2}$$
$$x_t = 0.896x_{t-1} - 0.179x_{t-2} - 0.002x_{t-3} + \varepsilon_{x,t} - 0.687\varepsilon_{x,t-1}$$
$$-0.089\varepsilon_{y,t-1} + 0.137\varepsilon_{x,t-2} + 0.065\varepsilon_{y,t-2}$$

Romania:

$$x_t = -0.039x_{t-1} + 0.348x_{t-2} - 0.054x_{t-3} + \varepsilon_{x,t} + 0.579\varepsilon_{x,t-1}$$
$$+0.981\varepsilon_{z,t-1} - 0.143\varepsilon_{x,t-2} - 0.452\varepsilon_{z,t-2}$$
$$x_t = -0.039x_{t-1} + 0.348x_{t-2} - 0.054x_{t-3} + \varepsilon_{x,t} + 0.208\varepsilon_{x,t-1}$$
$$+0.824\varepsilon_{z,t-1} - 0.283\varepsilon_{x,t-2} + 0.509\varepsilon_{z,t-2}$$

For each country, the first equation enable us to determine the degree of causality from investment in physical capital (IPK) to gross domestic product (GDP) and the second equation enable us to determine the degree of causality from investment in education (IED) to gross domestic product, of course, at horizon 1, 2 and 3. The results are presented in the table below:

Table 1. The degree of causality from investment in physical capital, respectively, investment in education to GDP

Country	Causality of horizon 1		Causality of horizon 2		Causality of horizon 3	
	IPK	IED	IPK	IED	IPK	IED
France	0.362	0.406	0.319	0.358	0.293	0.341
Germany	0.331	0.405	0.312	0.361	0.309	0.347
Romania	0.393	0.274	0.388	0.253	0.386	0.231

4. Conclusion and consequences

The results we have obtained enable us to claim that for all the countries we considered, the measure of causality is not only positive but also it is persistent. This conclusion is true for the both effects: investment in physical capital and investment in education, on gross domestic product. If we examine the size of values obtained for causality measures, we can also confirm that each of these countries give a considerable importance to investments in education.

As we can observe, for the two developed countries, France and Germany, the size of causality is higher than for the Romanian economy for the investment in education and less for the investment in physical capital, but these results reflect undoubtedly a trend – those two countries are the European countries that allocate considerable resources for the education process. Furthermore, is very important to see that in all studied countries the causality effect is persistent.

REFERENCES

[1]**Dufour, J M., Taamouti, A. (2010),*Short and Long Run Causality Measures: Theory and Inference*;***Journal of Econometrics*, Elsevier, 154(1), pp 42-58;
[2]**Dufour, J.M., Renault, E. (1998), *Short-run and Long-run Causality in Time Series: Theory*;***Econometrica*, 66(5), pp 1099–1125;
[3]**Geweke, J. (1982), *Measurement of Linear Dependence and Feedback between Multiple Time Series;*** *J. Am. Stat. Assoc*., 77, pp 304–313;
[4]**Granger, C. (1969), *Investigating Causal Relations by Econometric Models and Cross Spectral Methods*;** *Econometrica*, 37, pp 424–438;
[5]**Lütkepohl, H. (1993), *Introduction to Multiple Time Series Analysis***; Second Edition, *Springer-Verlag, Berlin;*
[6]**Sims, C (1980), *Macroeconomics and Reality*;***Econometrica*, 48,1, pp 1-48;
[7]**Stern, D., Enflo, K. (2013),*Causality between Energy and Output in the Long-run*;** *Energy Economics*, Elsevier, 39(C), pp 135-146;
[8]**Viasu, I., (2015),*The Long-term Causality. A Comparative Study for some EU Countries*;** *Computational Methods in Social Sciences*, Vol. III, Issue 2/2015, pp. 28-35;
[9]**Wiener, N. (1956), *The Theory of Prediction.*** *In Modern Mathematics for Engineers*; *McGraw-Hill, New York*;
[10] **World Bank Open Data:**http://data.worldbank.org/,last accessed 10.12.2015.