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## PARAMETRIC LINEAR FRACTIONAL PROGRAMMING

***Abstract.*** *This paper considers a mathematical programming problem whose objective function is a linear fractional. The constraint set consists of linear inequalities with non-negative requirements on the variables. A parameter is introduced in the objective function of the problem. Optimum solutions are obtained for the various intervals of the parameter. A numerical example illustrates the steps of the proposed algorithm.*

***Key words:*** *Linear fractional programming; parameter; objective function.; intervals.*

## JEL Classification: 90 C30

### 1. Introduction

Development of parametric optimization tools are essential in the process design as they can offer significant analytical results to problems related to uncertainty objective optimization. In fact, the solution of the pertinent parametric optimization problems is the complete and exact solution from the mathematical point of view.

Although sensitivity analysis and parametric optimization problems have been addressed successfully in the linear programming case (Gal, 1979) they are still the subject of ongoing research for non-linear mathematical programming problems.(Gass, 1985) has very lucidly dealt with the parametric optimization in the case of linear programming problems;(Murty, 1980) has studied the computational complexity of parametric linear programming problems.( Singh et.al., 2011) have considered a multiparametric problem for a generalized transportation problem. (Mordukhorichet.al., 2009) have studied sub-gradients of marginal functions in parametric mathematical programming.(Aggarwal, 1968) has studied a linear fractional programming when the parameter appears in a very special structured objective function of the problem.

This paper addresses the behavior of solutions to a linear fractional programming problem when the coefficients of the objective function, in its most general form, are allowed to vary; i.e., for what ranges of coefficient values will the deterministic solution remain optimal? In general, a parametric linear fractional programming problem can be stated as:

let $\omega \le \mu \le \varphi$, where $\omega$ may be an arbitrary small, but finite number and $\varphi$ may be an arbitrary, algebraically large, but finite number. For each $\mu$ in this interval, find a vector **x** = ( $x_1, x_2, \ldots, x_n$) which maximizes

$$Z = \frac{\sum_{j=1}^{n} [c_j + \mu c'_j]\, x_j}{\sum_{j=1}^{n} [d_j + \mu d'_j] x_j}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,\ldots,m \qquad (1)$$

$$x_j \ge 0 \qquad j = 1,\ldots,n$$

where $c_j$, $c_j'$ ,$d_j$, $d_j'$ ,$a_{ij}$, and $b_i$ are given constants.

In matrix form above is same as:

$$Z = \frac{[\boldsymbol{c} + \mu \boldsymbol{c}']\, \boldsymbol{x}}{[\boldsymbol{d} + \mu \boldsymbol{d}']\, \boldsymbol{x}}$$

subject to

$$\boldsymbol{x} \in S.$$

Here $S$ = ( $\boldsymbol{Ax} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}$); $\boldsymbol{A} = (\boldsymbol{A_1}, \boldsymbol{A_2}., \ldots \boldsymbol{A_n})$ is m by n matrix;$\boldsymbol{c}, \boldsymbol{c}', \boldsymbol{d}, \boldsymbol{d}'$are n-component row vectors, $\boldsymbol{x}$ and $\boldsymbol{b}$ are n and m components column vectors respectively.

(Aggarwal, 1968) has studied three special cases of problem (1). In case one the objective function of the problem is:

$$\frac{\sum_{j=1}^{n} [c_j + \mu c'_j]\, x_j}{\sum_{j=1}^{n} [d_j + \mu c'_j]\, x_j};$$

in the second case objective function is:

$$\frac{\sum_{j=1}^{n} [c_j + \mu c'_j]\, x_j}{\sum_{j=1}^{n} d_j x_j};$$

the objective function in the last case is:

$$\frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} [d_j + \mu d'_j]\, x_j}$$

(Chadha, 1971) has studied the above three cases in a linear fractional programming problem when two parameters appear in the objective function. The intention here is to study the parametric linear fractional programming in its most general form as in (1). Preliminaries are given in section 2; the algorithm, in detail, is presented in section 3; the last section of the paper contains a numerical example. This example illustrates all the steps of the proposed algorithm..

## 2. Preliminaries

A linear fractional programming problem is given by

$$\text{Maximize } F(x) = \frac{\boldsymbol{Cx}}{\boldsymbol{Dx}} \qquad (2)$$

subject to

$\boldsymbol{x} \in S.$

Here $S = (\ \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0})$; $\boldsymbol{A} = (\boldsymbol{A_1}, \boldsymbol{A_2}, \dots \boldsymbol{A_n})$ is m by n matrix; $\boldsymbol{C}, \boldsymbol{D}$ are n-component row vectors, $\boldsymbol{x}$ and $\boldsymbol{b}$ are n and m components column vectors respectively.

Under the assumptions that the

(i) set S is regular, i.e. non-empty and bounded,

(ii) $\boldsymbol{Dx} > 0$for all $\boldsymbol{x}in \in S$,

and (iii) the problem is non-degenerate,

it has been proved by (Martos, 1964)and (Swarup, 1965) that a basic feasible solution, $\boldsymbol{x^0} = (\boldsymbol{x_B}, \boldsymbol{0})$ solves the problem (2) if

$$\Delta_j = Z_2 \left[\boldsymbol{C_B P_j} - c_j\right] - Z_1 [\boldsymbol{D_B P_j} - d_j] \geq 0 \ ; \quad j = 1, 2, \dots, n. \qquad (3)$$

Here $c_j$, and $d_j$are the jth-elements of the vectors $\boldsymbol{C}$ and $\boldsymbol{D}$ respectively; $\boldsymbol{C_B}$, and$\boldsymbol{D_B}$are the sub-vectors of **C**, and **D** respectively. Corresponding to the basis matrix$\boldsymbol{B}$of$\boldsymbol{A}$; $\boldsymbol{P_j} = \boldsymbol{B^{-1} A_j}, \boldsymbol{x_B} = \boldsymbol{B^{-1} b},\ Z_2 = \boldsymbol{D_B x_B}$, and $Z_1 = \boldsymbol{C_B x_B}$.

## 3. Description of the algorithm

We start with a basic feasible solution, $\boldsymbol{x^0} = (\ \boldsymbol{x_B}, \boldsymbol{0}\ )$ for problem (1),with$\boldsymbol{A} \equiv [\boldsymbol{B}, \boldsymbol{N}]$. Next we calculate $\Delta'_j s$ associated with this basic feasible solution for all$j in \boldsymbol{N}$.

$$\Delta_j = (\boldsymbol{d_B} + \mu \boldsymbol{d'_B})\boldsymbol{x_B}\ [(\boldsymbol{c_B} + \mu \boldsymbol{c'_B})\boldsymbol{B^{-1}A_j} - (c_j + \mu c'_j)] - (\boldsymbol{c_B} + \mu \boldsymbol{c'_B})\boldsymbol{x_B}[(\boldsymbol{d_B} + \mu \boldsymbol{d'_B})\boldsymbol{B^{-1}A_j} - (d_j + \mu d'_j)]$$

The above expression simplifies to be a quadratic expression in $\mu$ and hence can be expressed as

$$\Delta_j = \alpha_j + \mu\beta_j + \mu^2\gamma_j. \tag{4}$$

**Steps of the algorithm:**

(a) Solve the quadratic equations$\alpha_j + \mu\beta_j + \mu^2\gamma_j = 0$for all $j \in N$. Let all the roots be complex numbers and let all the quadratic expressions be positive i.e. $\alpha_j + \mu\beta_j + \mu^2\gamma_j > 0$for all $j \in N$.In this case the current solution is optimum over$\omega \leq \mu \leq \varphi$.Bu if$\alpha_j + \mu\beta_j + \mu^2\gamma_j$<0for any $j \in N$then move to an adjacent basic feasible solution.

(b) Solve the quadratic equations$\alpha_j + \mu\beta_j + \mu^2\gamma_j$= 0 for all $\in N$ . Mark the real values on a number line and find the interval (intervals) of $\mu$ when $\Delta_j = \alpha_j + \mu\beta_j + \mu^2\gamma_j \geq 0$.

(c) Find an intersection set of all the intervals found in step (b) and let that interval be $[\underline{\mu}\ ,\ \overline{\mu}\ ]$.

(d) In this case$\boldsymbol{x^0} = (\ \boldsymbol{x_B}, \boldsymbol{0}\ )$ solves the problem (1) for all $\mu$ in the interval $\underline{\mu} \leq \mu \leq \overline{\mu}$ if all the quadratic expressions with complex roots are positive i.e. $\alpha_j + \mu\beta_j + \mu^2\gamma_j > 0$ for all j's with complex roots. If $\alpha_j + \mu\beta_j + \mu^2\gamma_j < 0$ for at least one non-basic vector then the problem has no solution over the interval $\underline{\mu} \leq \mu \leq \overline{\mu}$.

**(e)** Next, let vector $\boldsymbol{A}_j$ to enter the basis that corresponds to $\underline{\mu}$ or$\overline{\mu}$ and follow steps (a) through (d) to find another optimum solution for problem (1) along with new range of the parameter$\mu$.

(f) In case the intersection set of all the intervals found in step (c) is empty then go to another basic feasible solution by letting vector $\boldsymbol{A_j}$ to enter the basis for which$\Delta_j = 0$. Follow steps (a) through (e).

(g) Repeat steps (a) - (f) until the entire range $\omega \leq \mu \leq \varphi$ of the parameter $\mu$has been examined.

**4.Numerical Example**

Maximize Z = $\frac{(0+\mu)\,x_1+(1+\mu)x_2}{(1+2\mu)x_1+(1+3\mu)x_2+2}$

Subject to

$x_1 + x_2 \leq 4$

$x_1 + 3x_2 \leq 6$(5)

$x_1, x_2 \geq 0,$

First basic feasible solution can be read from table 1.

**Table 1**

| $d_B$ | $c_B$ | Basic variables | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b |
|---|---|---|---|---|---|---|---|
| 0 | 0 | $x_3$ | 1 | 1 | 1 | 0 | 4 |
| 0 | 0 | $x_4$ | 1 | 3 | 0 | 1 | 6 |

$\boldsymbol{x^0} = (0, 0, 4, 6)$with, Z = $\frac{z_1}{z_2}$=$\frac{0}{2}$ .

$\Delta_j^{'}s$ associated with this basic feasible solution are:

$\Delta_1$ =[(2)(0$-\mu$)]- [(0)] = - 2$\mu$

$\Delta_2$ = [ (2)($0 - 1 - \mu$)]- [(0)] = 2($-1 - \mu$)].

The intersection interval for $\Delta_1 \geq 0$ and for $\Delta_2 \geq 0$ is$(-\infty, -1]$.

Over this interval the condition,$\boldsymbol{Dx} > 0$for all $x \in S$, gets violated. Thus problem has no solution over the interval$(-\infty, -1]$. We move to another basic feasible solution by letting $\boldsymbol{A_2}$ to enter and$\boldsymbol{A_4}$to depart from the basis.
Table 2 yields the new solution.

**Table 2**

| $d_B$ | $c_B$ | Basic variables | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b |
|---|---|---|---|---|---|---|---|
| 0 | 0 | $x_3$ | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 2 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1+3$\mu$ | 1+$\mu$ | $x_2$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |

At this solution, $\boldsymbol{x^0} = (0, 2, 2, 0)$, Z = $\frac{z_1}{z_2} = \frac{\mu+1}{3\mu+2}$.

$\Delta_j^{'}s$ at this basic feasible solution are:

$\Delta_1$= $[(3\mu + 2)(\frac{1}{3} + \frac{1}{3}\mu - \mu)] - [(1+\mu)(\frac{1}{3} + \mu - 1 - 2\mu)]$

$= -\mu^2 + \frac{4}{3}\mu + \frac{4}{3}$

$\Delta_4 = [(3\mu + 2)(\frac{1}{3} + \frac{1}{3}\mu)] - [(1+\mu)(\frac{1}{3} + \mu)] = \frac{1}{3} + \frac{1}{3}\mu$

$\Delta_1 = 0$ yields $\mu = 2, \; -\frac{2}{3}$; and the interval over which $\Delta_1 \geq 0$ is given by $[-\frac{2}{3}, 2]$. But for $\mu \leq -\frac{2}{3}$ the condition, $\boldsymbol{Dx} > 0$ for all $x \in S$, gets violated, therefore, he interval over which $\Delta_1 \geq 0$ is given by $(-\frac{2}{3}, 2]$. $\Delta_4 \geq 0$ is true for $\mu \geq -1$.

Their intersection interval is $(-\frac{2}{3}, 2]$. Thus $\boldsymbol{x^0} = (0, 2, 2, 0)$ solves the problem for $-\frac{2}{3} < \mu \leq 2$.

Next, $\mu = 2$ makes $\Delta_1 = 0$. A new basic feasible solution is obtained by letting $\boldsymbol{A_1}$ to enter and $\boldsymbol{A_3}$ to depart from the basis. This solution is given by table 3.

**Table 3**

| $D_B$ | $C_B$ | Basic variables | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b |
|---|---|---|---|---|---|---|---|
| $1 + 2\mu$ | $\mu$ | $x_1$ | 1 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 3 |
| 1+3 $\mu$ | 1+ $\mu$ | $x_2$ | 0 | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

At this solution, $\boldsymbol{x^0} = (3, 1, 0, 0)$, Z = $\frac{z_1}{z_2} = \frac{4\mu+1}{9\mu+6}$.

$\Delta_j^{'}s$ at this basic feasible solution are:

$\Delta_3$= $[(9\mu + 6)(\frac{3}{2}\mu - \frac{1}{2} - \frac{\mu}{2})] - [(4\mu + 1)(\frac{3}{2} + 3\mu - \frac{1}{2} - \frac{3}{2}\mu)]$

$= 3\,\mu^2 - 4\,\mu - 4$

$\Delta_4$= $[(9\mu + 6)(-\frac{\mu}{2} + \frac{1}{2} + \frac{1}{2}\mu)] - [(4\mu + 1)(-\frac{1}{2} - \mu + \frac{1}{2} + \frac{3}{2}\mu)]$

$= -2\,\mu^2 + 4\,\mu + 3$

$\Delta_3$= 0 yields $\mu$ = 2, $-\frac{2}{3}$; and the interval over which $\Delta_3 \geq 0$ is given by

$(-\infty, \,, -\frac{2}{3}] \cup [2, \infty)$.

$\Delta_4 = 0$ gives $\mu = 2.5, -0.58$ and the interval over which $\Delta_4 \geq 0$ is given by

[-0.58, 2.5].Their intersection interval is $[2, 2.5]$. Thus $\boldsymbol{x^0} = (3, 1, 0, 0)$ solves the problem for $2 \leq \mu \leq 2.5$.

Next, $\mu = 2.5$ makes $\Delta_4 = 0$. A new basic feasible solution is obtained by letting $\boldsymbol{A_4}$to enter and $\boldsymbol{A_2}$ to depart from the basis. Table 4 yields the new solution

**Table 4**

| $D_B$ | $C_B$ | Basic variables | $A_1$ | $A_2$ | $A_3$ | $A_4$ | b |
|---|---|---|---|---|---|---|---|
| $1 + 2\mu$ | $\mu$ | $x_1$ | 1 | 1 | 1 | 0 | 4 |
| 0 | 0 | $x_4$ | 0 | 2 | $-1$ | 1 | 2 |

At this solution, $\boldsymbol{x^0} = (4, 0, 0, 2)$, Z = $\frac{z_1}{z_2}$ = = $\frac{2\mu}{4\mu+3}$ .

$\Delta_j^{'}s$ at this basic feasible solution are:

$\Delta_2 = [(4\mu + 3)(\mu - 1 - \mu)] - [(2\mu)(1 + 2\mu - 1 - 3\mu)]$

$= 2\,\mu^2 - 4\,\mu - 3$

$\Delta_3$= $[(4\mu + 3)(\mu)] - [(2\mu)(1 + 2\mu) = \mu$

$\Delta_2 = 0$ gives $\mu = 2.5, -0.58$ and the interval over which $\Delta_2 \geq 0$ is given by

$(-\infty, -0.58] \cup [\, 2.5, \infty)$. The intersection interval of $\mu$ for which $\Delta_2$ and $\Delta_3$ are $\geq 0$ is given by $[\, 2.5, \infty)$.

Thus $\boldsymbol{x^0} = (4, 0, 0, 2)$ solves the problem for $2.5 \leq \mu < \infty$.

## 5. Conclusion

This work completes an exhaustive study of a linear fractional programming problem when the parameter appears in the objective function of the problem.

## REFERENCES

[1]**Aggarwal, S.P.(1968),*Parametric Linear Fractional Functions Programming*;**Metrika, vol. 12;
[2]**Chadha, S.S.(1971),*A Linear Fractional Functional Program with a two Parameter ObjectiveFunction*;**ZAAM, vol. 51, no. 6;
[3]**Gass, S.I. (1985),*Linear Programming: methods and applications,* 5th ed.** *Mcgraw-Hill*, New York;
[4]**Gal, T. (1979), *PostoptimalAnalysis, Parametric Programming and Related Topics.*** *McGraw- Hill* book company , New York;
[5]**Martos,B.(1964), *Hyperbolic Programming***; Naval Research Logistics Quarterly II, (translated by Whinston, A);
[6]**Mordukhorich,B.S., Nam.,N,M., Yen, N.D. (2009),*Subgradients of Marginal Functions in Parametric Mathematical Programming***. Mathematical Programming Series B;
[7]**Murty, K.G. (1980),*Computational Complexity of Parametric Linear Programming***. Mathematical Programming, vo. 19, issue 1;
[8]**Singh,S.,Gupta.,P.,Milan, V. (2011),*On MultiparametricAnalysis in Generalized TransportationProblems.***Computational Science and its Applications, III;
**[9]Swarup, K. (1965),*Linear Fractional Functions Programming.*** Operations Research 13/6.