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A FINANCIAL WAVE MODEL FOR STOCK INDICES

Abstract: *The advanced analysis is to highlight application possibilities of some analogies between quantum mechanical and financial systems into modelling stock indices dynamics. Schrodinger`s equation and Hamilton-Jacobi mechanics are used as instruments to build the "financial wave". Endogenous and exogenous factors` influence on system dynamics is measured through the study of "financial energy" components—"financial kinetic energy" and "financial potential energy", respectively. The model can be used for policy impact assessment—a change in investor expectations with regards to price movements can be studied by programming the probability amplitude, while the impact of various events on the system can be assessed by analysing the "energy" released in that period.*

Keywords: *quantum finance, Schrodinger equation, Hamilton-Jacobi mechanics, kinetic energy, potential energy.*

JEL Classification: C22, C29, G12, G15

1.Introduction

A specific research direction in econophysics is the analysis of financial phenomena by applying methods developed for the study of quantum systems. Principles from quantum physics can be applied if financial markets are considered complex systems where investors interact with each other similar to the way particles interact in a quantum environment. Due to the uncertainty that characterises financial markets, many instruments developed to study their behaviour originate from statistical physics (e.g., stochastic processes and non-linear dynamics). Meyer [1999] and Eisert et al. [1999] proposed quantum physics principles to be used in the elaboration of transaction strategies, which lead to the development of quantum game theory. Segal and Segal [1998] argue that irregularities in the price of financial assets can be attributed to a certain extent to quantum effects. Haven [2003, 2005] uses principles from quantum mechanics to describe the underlying stochastic processes that generate movements in the prices of financial assets. In such an environment, the "micro" characteristics of a financial system are reflected at a "macro" level through its dynamics. Choustova [2007] proposes a quantum model for financial markets, nevertheless without testing it empirically.

A model that quantifies the influence of market characteristics on the dynamics of financial asset prices is developed. With this objective in mind, a series of analogies between quantum mechanical and financial systems are realised, and the Schrodinger equation (see Drabik [2011] and Nastasiuk [2014]) and the Hamilton-Jacobi mechanics (see Choustova [2007]) are used as instruments for "financial wave" modelling. The system dynamics is analysed by the study of two "energy" components determined by the action of market factors, namely the "financial kinetic energy" (a measure of the system's intrinsic forces) and "financial potential energy" (a measure of the system's inertial forces). The market energy is interpreted as a form of system volatility determined by the movement of financial time series (i.e., price level and trading volume). Furthermore, an analytical formula for calculating the "financial wave" is proposed.

The model facilitates the study of a financial system`s dynamics by analysing the magnitude of the "energy" components released by the system, differentiating between "normal" periods (with low "energy" releases) and more turbulent periods (with high "energy" releases). Also, the impact of various events on the market can be assessed by analysing the "energy" released in the event occurrence period[1].

The paper is organized as follows: Section two presents the methodology, including analogies between quantum and financial processes, as well as the notions and instruments used, while Section three offers a detailed description of the "financial wave" model. Section four presents the dataset used for empirical testing, and Section five showcases the results and subsequent analysis. The final conclusions are drawn in Section six.

1. Methodology

The general theoretical framework of the model is given by the quantum mechanics theory. The instruments and theories are borrowed from classical and quantum mechanics, combining elements from particle dynamics and mechanical systems with probability theory. The financial market is similar to a quantum system, whose component particles` interactions generate system dynamics. At a financial market level, interactions between investors generate a dynamics in the system and ultimately the price changes accordingly. The market factors are the catalysts of all system dynamics. The general hypothesis is that the behaviour of investors is not rational[2] at all times; moreover, they have different preferences and expectations about the evolution of financial asset prices. The information is

[1] The causality between the event and the market change still needs to be proven

[2] Investors have bounded rationality, constrained by their financial education and utility function; limited financial education and an utility function that does not depend solely on profit may induce a behaviour that can be considered irational

disseminated heterogeneously among investors, according to certain restraints such as financial education and access to information. The complex interactions between investors, coupled with the influence of market characteristics, generate non-random price movements. The financial system can comprise a single asset (e.g., stock, bond, currency pair, and financial derivative) or multiple assets (e.g., market index).

The following notations are used:

q – The space coordinate of the quantum particle and the price level of the financial asset;

p – The momentum of the quantum particle`s position and of the price level, respectively;

m – The mass of the quantum particle and the trading volume of the financial asset;

Ψ – The wave function of quantum and financial system, respectively;

H – The system`s Hamiltonian, or the total energy of the quantum and financial system, respectively;

T – The kinetic energy of quantum and financial systems, respectively;

V – The potential energy of quantum and financial systems, respectively;

S – The phase space function of quantum and financial waves, respectively;

R – The probability amplitude, or the square root of the probability density of the particle`s position and price level, respectively;

h – Planck`s reduced constant;

i – The imaginary unit.

Similar to the definition from classical physics[3], the price momentum is calculated as the product of trading volume with price velocity (a measure for the rate of change of price).

1.1. Schrodinger Equation

Introduced by Ervin Schrodinger in 1926 to describe the movement of quantum microstructures (e.g., atoms, molecules and other subatomic particles), the equation is also used in the study of macroscopical system dynamics (e.g., superfluids), specifically due to the contributions of authors such as Leggett [1980] and Ghirardi et al. [1986]. Ultimately, it found applications outside the study of physical systems, in social sciences such as economics, where the cyclical nature of phenomena (e.g., financial asset prices) makes it plausible to associate a wave to its dynamics. Haven [2002] showed that the Black-Scholes-Merton equation, which gives the price evolution of a call/put option, is a particular case of the Schrodinger equation.

[3] $p = mv = m\frac{\partial q}{\partial t}$

The non-relativistic, time dependent form of the equation if given by relation (1):

$$ih\frac{\partial\Psi}{\partial t}(q,t) = H(q,t)\Psi(q,t) \quad (1)$$

Interpreting it from quantum mechanics perspective, the wave function offers complete description of a physical system, with the system Hamiltonian incorporating the effects of market factors whose influence on the system dynamics are under study.

As with the position of a quantum particle, the price of a financial asset is influenced by stochastic processes and cannot be determined with certainty; however, one can assign to each possible position a certain probability[4], so that at each moment, a probability space can be estimated for the position of the price. This interpretation can be transposed to financial markets as the evolution of asset prices is characterised by uncertainty and investors can only associate a certain probability (R^2) to future price levels. Moreover, the trajectory of financial asset prices is explained by the wave function, through the action of phase space function (S). In the probabilistic interpretation of quantum mechanics, the solution to the Schrodinger Equation can be written in polar coordinates as given by relation (2).

$$\Psi(q,t) = R(q,t)e^{iS(q,t)/h} \quad (2)$$

Here, the interpretation of Planck`s constant is that it represents a core structural characteristic of the financial system, which is constant over longer periods. To simplify the subsequent modelling process, it is considered to be the unity, $h = 1$.

1.2. Hamilton-Jacobi Mechanics

The Hamilton-Jacobi mechanics completes the framework required to model the movement of a particle in a mechanical system (according to classical mechanics principles) as a wave, which can be then determined by Schrodinger`s equation (according to quantum mechanics principles). This is the particle-wave duality characteristic of a physical object. More importantly, it gives the relationship between a system`s total energy (the Hamiltonian) and its oscillation path (the phase space function), as per relation (3).

$$H + \frac{\partial S}{\partial t} = 0 \quad (3)$$

The “energy” released by a financial system is directly tied to the change rate of the oscillation path, which is a measurement of its volatility.

An expression for the variation of the phase space function is obtained[5]:

$$dS = pdq - Hdt \quad (4)$$

[4] More precisely, a probability is assigned so that the particle is in a certain space area; e.g., an interval for one dimension space as is the case of asset prices

[5] For a detailed demonstration, see Demonstration 1 from Appendix

The trajectory path of the system is a function of its energy and the dynamics of the particle`s position. In the financial interpretation, the financial system`s trajectory depends on the price change multiplied by its momentum and the "energy" released on the market.

One can notice that relation (3) does not constrain the total system energy to a particular form. The Hamiltonian depends on the characteristics of the market, which subsequently determine the system`s dynamics. Hence, the Hamiltonian of a closed market (i.e., no exogenous factors) will not include a component to take into consideration the influence of external factors. However, in this study, the case of an open market is considered, hence the components of the Hamiltonian will need to include the effects of both endogenous and exogenous factors—the "financial kinetic energy" specific for endogenous market factors and the "financial potential energy" specific for exogenous market factors.

$$H = T + V \tag{5}$$

2. Financial Wave Model

It is assumed that the financial system is completely described by the "financial wave" as per relation (1). The "financial wave" must be interpreted as an information field that displays a deterministic nature (described by Schrodinger`s equation), generating fluctuations in the price of financial assets. Price changes are determined exclusively by new information about market characteristics (e.g., resource availability, existing investment barriers, market liquidity and industrial/sectorial performance) entering the market. These changes act on the financial system in a mechanical way similar to the way kinetic and potential energy from classical mechanics influence the dynamics of a mechanical system. The "kinetic financial energy" is the result of the intrinsic movement of the financial system, and a measure of instantaneous force of the system determined by the influence of endogenous factors. The "financial potential energy" is the result of exogenous factors` action on the system and a measure of the inertial force. The "financial kinetic energy" has short memory (it measures the instantaneous force of the system), while the "financial potential energy" has long memory (it measure the inertial force of the system accumulated in time), hence the amplitude of the "financial kinetic energy" is generally lower than that of "financial potential energy". This observation might prove useful when constructing an instrument that differentiates between an exogenous shock recorded by the market and a structural change in the functioning of the system.

2.1. Financial Energy

Unlike Choustova [2007], who proposed the form for the "potential energy", an analytical formula is derived from the definitions of the energy and force in classical mechanics. The "financial energy" is the total force generated within the financial system by the dynamics of price and trading volume. Similar to classical mechanics, the energy is calculated as the integral of "financial force" over the price variation (relation [6]).

$$E = \int f(q,t)dq \tag{6}$$

E is the "financial energy" and f the "financial force".

A method for calculating the "financial force" of the system is derived from the definition of force from classical mechanics: the force is the time derivative of the momentum[6] and the momentum is the product between mass and velocity. In the classical framework, the mass of a particle is constant; hence the force is the product between the mass and the acceleration $f = ma = m\frac{\partial v}{\partial t}$. However, the transaction volume (the "mass" of the financial system) changes over time, therefore the formula for calculating the financial needs to take into consideration both price and volume changes. Relation (7) gives the revised analytical expression for calculating the "financial energy" generated by market factors[7].

$$E = \frac{2}{3}\frac{\partial m}{\partial t}q\frac{\partial q}{\partial t} + m\frac{\left(\frac{\partial q}{\partial t}\right)^2}{2} \tag{7}$$

This expression comprises both "kinetic" and "potential" energies. As the second term is the expression of the kinetic energy from classical mechanics.

$$T = m\frac{\left(\frac{\partial q}{\partial t}\right)^2}{2} \tag{8}$$

The first term of from relation (7) is the expression for the "financial potential energy".

$$V = \frac{2}{3}\frac{\partial m}{\partial t}q\frac{\partial q}{\partial t} \tag{9}$$

Further, to calculate the "financial energy", a method to estimate the velocity of price and trading volume, respectively, is required. To calculate the "energy" released by the S&P500 and DJIA indices, Negrea [2014] used as an estimator for the velocity of the annualised volatility of the indices, $\hat{\sigma}$, multiplied with a correction factor, τ. The annualised volatility is the standard deviation of the relative price change. This method is used to estimate the velocity of both the price and trading volume. Considering that the standard deviation of relative price change is not constant over time, a rolling window of fixed length N is used to calculate the velocity for each period t. The standard deviation is given by relation (10) and the annualised volatility by relation (11).

$$\hat{\sigma}_t = \left(\frac{1}{N-1}\sum_{i=2}^{N}\left(\frac{x_i - x_{i-1}}{x_{i-1}} - \hat{\mu}_t\right)^2\right)^{\frac{1}{2}} \tag{10}$$

[6] $f = \frac{\partial p}{\partial t}$

[7] For a detailed demonstration see Demonstration 2 from Appendix

X is the variable for which the velocity is calculated (i.e., price and trading volume) and $\hat{\mu}_t = \frac{1}{N}\sum_{i=2}^{N}\frac{X_t - X_{i-1}}{X_{t-1}}$ is the mean of relative changes of X over a window of length N.

$$\hat{a}_t = \frac{\hat{\sigma}_t}{\tau} \tag{11}$$

The length of rolling window is determined based on the frequency of time series. It should be large enough to capture the volatility of turbulent periods, but not too large to smoothen out their effects. The use of daily data justifies a rolling window of length 30 days, N=30, and a correction coefficient τ=252 representing the number of trading days in a year (see Negrea [2014]).

2.2. Phase Space Function

In relation (4), the differential operator d is replaced with the diference operator Δ to obtain analytical expression for the phase space function, which depends on variables that are either known or can be determined:

$$\Delta S = p\Delta q - H\Delta t \tag{12}$$

Relation (12) is equivalent with $\mathrm{S}_t - \mathrm{S}_{t-1} = \mathrm{p}_t(\mathrm{q}_t - \mathrm{q}_{t-1}) - H_t(t-(t-1))$, from where a recurrent formula for calculating the phase space function is derived:

$$\mathrm{S}_t = \mathrm{S}_{t-1} + \mathrm{m}_t\hat{a}_t(\mathrm{q}_t - \mathrm{q}_{t-1}) - H_t \tag{13}$$

The phase space function is completely determined if the initial value S_0 is known. The price and trading volume levels are known, while the annualised volatility and the Hamiltonian (“financial energy”) can be calculated. The change in the financial system`s trajectory path depends on the system`s “energy” and the price change multiplied by the price momentum.

2.3. Probability Amplitude

Lastly, a method to estimate the probability density of price level is required, from where the probability amplitude R can then be calculated. The analysis of histograms in Figure (1) suggests that the probability densities are multimodal, regardless of the length of the sample window. The multimodality of the time series is getting more obvious as the number of breaks in the histogram increases from 32 to 64 and then to 256 (see Figure [1]). Although over the medium and long term, trend patterns emerge in the price series, over shorter periods, the trajectory of the price is less predictable with more frequent trend-reverting episodes. The presence of smaller patterns within long-term trends characterised by relatively frequent trend changes determine the multimodality of time series` probability distribution.

To estimate the empirical probability distribution of price, a weighted mix of Gaussian distributions is used. A rolling window of length w=252 (the number of trading days in a year) is used to estimate the empirical probability distribution for the price at each moment t. The length of the window is long enough for the sample to generate statistically significant results, and sufficiently short for the empirical distribution to characterise the preferences and expectations of investors at that specific moment. At each period t a series (q_i) with $i \in \overline{(t-w+1):t}$ is considered, for which the empirical probability distribution is estimated using a weighted mix of Gaussian distributions[8]. The selection of Gaussian distribution is based on the Bayesian Information Criterion[9]. The series $(w_i)_s$, $(\mu_i)_s$ and $(\sigma_i)_s$ representing the weights, means and standard deviations of the Gaussians are then used to compute the estimated probability density (s is the number of Gaussians in the mix). The estimated probability distribution is given by relation (14) and the probability amplitude (the square root of the probability distribution) by relation (15).

$$P(X=q) = \left(\sum_{i=1}^{s} w_i \frac{1}{\sigma_i \sqrt{2\pi}} e^{\frac{-(q-u_i)^2}{2\sigma_i^2}} \right) \quad (14)$$

$$R = \left(\sum_{i=1}^{s} w_i \frac{1}{\sigma_i \sqrt{2\pi}} e^{\frac{-(q-u_i)^2}{2\sigma_i^2}} \right)^{\frac{1}{2}} \quad (15)$$

2.4. Financial Wave

Knowing the expression for the probability amplitude and the phase space function, the "financial wave" can be computed by replacing relations (13) and (15) in relation (2). It can be interpreted as the visible effect of market factors on the financial system`s dynamics.

$$\Psi(q,t) = \left(\sum_{i=1}^{s} w_i \frac{1}{\sigma_i \sqrt{2\pi}} e^{\frac{-(q-u_i)^2}{2\sigma_i^2}} \right)^{\frac{1}{2}} e^{i(S_{t-1}+m_t \hat{a}_t (q_t - q_{t-1}) - H_t)} \quad (16)$$

[8] The estimation is done by using the densityMclust function from the Mclust package developed by Fraley et al. [2012] and implemented in R-Software.
[9] As per an algorithm developed by Fraley and Raftery [2012]

3. Data Set

The data set used in the analysis comprises the financial series (price and trading volume) of the American index S&P500[10]. The logarithmic series are used in order to "normalise" the "energy", which might by distorted if high values were to be introduced in the model (e. g., transactional volumes of the order of billions of dollars). Also, due to the fact that it allows a better comparison of results over longer periods, evading at least partially the effects of inflation and the exponential growth of trading volume. The time series was downloaded from the web portal Yahoo Finance on 19 October 2013 and refers to the period 15 October 2003–18 October 2013. The return is calculated as the difference between two consecutive values of the log-price level.
Figure (2) shows that the weighted mix of Gaussian distributions describes quite well the empirical probability distributions.

4. Results and Analysis

Figure 3 presents the dynamics of S&P 500, through the evolution of the (log-) price level, return and (log-) trading volume. The (log-) price level witnessed a sharp decline between fall 2008 and mid-2009, post which it re-entered an ascending path. The volatility of return increases as expected during that period of maximum acuity of the financial crisis. The (log-) trading volume was on an upward trend until the end of 2008, post which it entered a general descending path throughout the end of analysed period.

By comparing Figure 4 and Figure 3, the close relationship between energy dynamics and time series dynamics (i.e., (log-) price, return and (log-) trading volume) is revealed. The "financial kinetic energy", the component with short memory, as expected, is smaller than the "financial potential energy", which has a long memory. An energy burst takes place at the end of 2008, which reaches its maximum on November 5th 2008 (6.919)—almost half of the "financial potential energy" level during that same period. For the entire analysed sample, the average value of the "financial kinetic energy" is 19% of the average value of the "financial potential energy". This can be explained by the structure of the S&P 500, which includes stocks of companies that have been seriously impacted by the financial crisis during that time (especially companies from the financial sector), including those that led to the crisis. The "financial potential energy" records its maximum on December 1st 2008 (14.094), which is also the date of maximum value for the total market "energy" (20. 182). The system`s dynamics is mainly determined by the potential energy, which is understandable for a stock index from a mature, globally interconnected financial market.

[10] The index comprises top 500 companies in terms of market capitalization listed on NYSE and NASDAQ

In Figure 5, there is a pronounced “compression” of the financial wave between 2008 and 2009, which applies to the period of high acuity of the financial crisis. The price level hits high values for that period as per investors expectations (associated low probability on reaching that value), which leads to a “financial wave” “compression” that persists until investors adapt to the new market conditions and corresponding price levels. All this time, the “financial energy” released by the market is high and growing, intensifying the informational flow. Investors have to process high quantities of information and often do not succeed to do this efficiently. The information dissemination and investment decision-making process are distorted, and do no longer function rationally and efficiently. Generally, investors adopt a mimetic behaviour to minimise utility loss during the financial crisis and maximise utility during financial bubbles, assuming that the other investors take the right decisions. When the market conditions stabilise and the released energy declines, the financial wave starts to expand and the price realigns with investors’ expectations. Starting with 2010, the price level entered a steady growth path, often above investors’ expectations, which explains a slight “compression” of the “financial wave” to the end of analysed period (the existence of a cloud of light blue dots).

5. Conclusions

The conceptual framework applied here is that of quantum mechanics, where Schrodinger’s equation and Hamilton-Jacobi mechanics are used as instruments of “financial wave” modelling. After the analogies have been done and the “financial wave” model established, empirical testing possibilities were assessed, considering the case of a mature financial market, namely the American stock index S&P 500.

The analysis highlighted the dynamics of information dissemination and shock transmission process through the energy components released by the system. Therefore, the “financial kinetic energy” captures the influence of market factors that are endogenous to the system and the “financial potential energy” captures the influence of market factors that are exogenous to the system. By analysing the dynamics of the “financial wave”, one can identify turbulent periods, characterised by price levels with low occurrence probability in regions where the wave is contracting.

Furthermore, the model could potentially be used to measure the impact of various policy scenarios on the dynamics of the financial system, by programming the model components. For example, a change in investor expectations with regards to price movements can be studied by programming the probability amplitude. Also, the impact of various events on the market can be assessed by analysing the “energy” released in that period[11].

However, as the “financial wave” can be quantified numerically only ex-post, its utility in the study of financial system dynamics is limited to historical and

[11] The causality between the event and the market change still needs to be proven

scenario-based analysis. Although it cannot be used directly to forecast price movements, if the fundamental characteristics of the financial system are stable over long periods, some conclusions can be drawn.

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APPENDIX

Demonstration 1

The dynamics of price in Hamilton-Jacobi mechanics is described by the equations in relation (17).

$$\begin{cases} \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \\ \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q} \end{cases} \quad (17)$$

By replacing the second equation from relation (17) in relation (3), a direct relationship between the phase space function and the price momentum is obtained following the below-mentioned steps.

i. Relation (3) is derived by the spatial coordinate to get:

$$\frac{\partial^2 S}{\partial t \partial q} = -\frac{\partial H}{\partial q} \quad (18)$$

ii. Relation (18) is then introduced in the second equation of relation (17) to get relation (19):

$$\frac{\partial p}{\partial t} = \frac{\partial^2 S}{\partial t \partial q} \quad (19)$$

iii. Relation (19) is integrated over the time coordinate to get relation (20):

$$p = \frac{\partial S}{\partial q} \quad (20)$$

The differential of the phase space function is expressed as follows:

$$dS = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt \quad (21)$$

Finally, relations (3) and (21) are introduced in relation (21) to get a formula for the phase space function differential that is expressed on determinable variables:

$$dS = p dq - H dt \quad (22)$$

Demonstration 2

In the case of financial markets, where both the position (price) and mass (trading volume) are time dependent, the formula for the force is expressed as follows:

$$f = \frac{\partial}{\partial t}\left(m \frac{\partial q}{\partial t}\right) = \frac{\partial m}{\partial t}\frac{\partial q}{\partial t} + m \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right) \quad (23)$$

Relation (23) is introduced in relation (6) and to get expression (24) for the market energy:

$$E = \int \left(\frac{\partial m}{\partial t}\frac{\partial q}{\partial t} + m \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right)\right) dq = \int \frac{\partial m}{\partial t}\frac{\partial q}{\partial t} \partial q + \int m \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right) \partial q \quad (24)$$

Next, the trading volume is not supposed to be dependent on the price level (at least not directly). This supposition is plausible if changes in the trading volume are considered to be determined by the arrival of new information regarding market factors and not by the actual price level. Thus relation (24) becomes relation (25).

$$E = \frac{\partial m}{\partial t}\int \frac{\partial q}{\partial t}\partial q + m\int \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right)\partial q \qquad (25)$$

Further, the right side in relation (25) is transformed into an integral over time, $E = \frac{\partial m}{\partial t}\int \frac{\partial q}{\partial t}\frac{\partial q}{\partial t}\partial t + m\int \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right)\frac{\partial q}{\partial t}\partial t = \frac{\partial m}{\partial t}\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t + m\int \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right)\frac{\partial q}{\partial t}\partial t$, which is equivalent to relation (26):

$$E = \frac{\partial m}{\partial t}\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t + m\int \frac{\partial^2 q}{\partial t^2}\frac{\partial q}{\partial t}\partial t \qquad (26)$$

To arrive at an explicit form for the financial energy, the two integrals from relation (26) need to be solved.

The integral $\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t = \frac{2}{3}q\frac{\partial q}{\partial t}$ is obtained by following the next steps:

i. Let $\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t = I$; integrating by parts one gets:

$\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t = \int \frac{\partial q}{\partial t}\frac{\partial q}{\partial t}\partial t = q\frac{\partial q}{\partial t} - \int q\frac{\partial^2 q}{\partial t^2}\partial t = q\frac{\partial q}{\partial t} - \int q\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right)\partial t = q\frac{\partial q}{\partial t} - \int q\partial\frac{\partial q}{\partial t}$

This relation is equivalent with $I = q\frac{\partial q}{\partial t} - \int q\partial\frac{\partial q}{\partial t}$.

ii. The integral $\int q\partial\frac{\partial q}{\partial t}$ is obtained by expressing $q = \int \frac{\partial q}{\partial t}\partial t$:

$$\int q\partial\frac{\partial q}{\partial t} = \int \left(\int \frac{\partial q}{\partial t}\partial t\right)\partial\frac{\partial q}{\partial t} = \int \left(\int \frac{\partial q}{\partial t}\partial\frac{\partial q}{\partial t}\right)\partial t = \int \frac{\left(\frac{\partial q}{\partial t}\right)^2}{2}\partial t = \frac{1}{2}\int \left(\frac{\partial q}{\partial t}\right)^2 \partial t$$

This relation is equivalent with $\int q\partial\frac{\partial q}{\partial t} = \frac{1}{2}I$

iii. By replacing the relation obtained at step two in the relation from step one $I = q\frac{\partial q}{\partial t} - \frac{1}{2}I$ is obtained, or

$$I = \frac{2}{3}q\frac{\partial q}{\partial t} \qquad (27)$$

$\int \frac{\partial^2 q}{\partial t^2}\frac{\partial q}{\partial t}\partial t = \frac{\left(\frac{\partial q}{\partial t}\right)^2}{2}$ is derived similarly to the previous integral; let $\int \frac{\partial^2 q}{\partial t^2}\frac{\partial q}{\partial t}\partial t = J$ and integrating by parts one get:

$$\int \frac{\partial^2 q}{\partial t^2}\frac{\partial q}{\partial t}\partial t = \frac{\partial q}{\partial t}\frac{\partial q}{\partial t} - \int \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial t^2}\partial t$$

Which is equivalent with $J = \left(\frac{\partial q}{\partial t}\right)^2 - J$ or

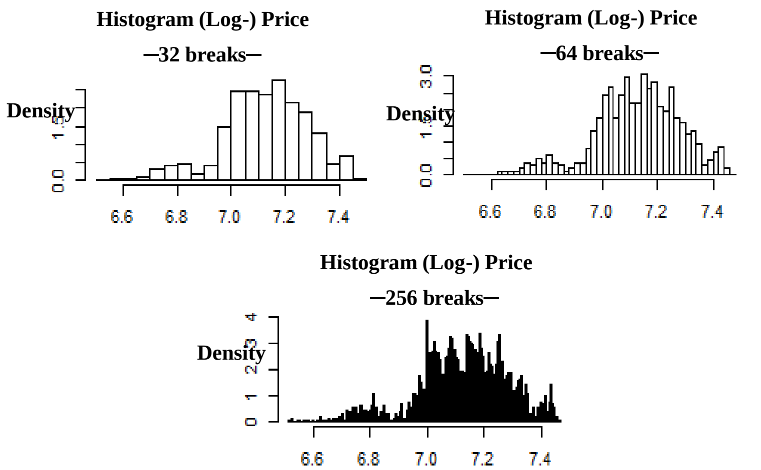
$$J = \frac{1}{2}\left(\frac{\partial q}{\partial t}\right)^2 \qquad (28)$$

Finally, by integrating relations (27) and (28) in relation (26), an analytical expression for the financial energy generated by the action of market factors is obtained:

$$E = \frac{2}{3}\frac{\partial m}{\partial t} q \frac{\partial q}{\partial t} + m \frac{\left(\frac{\partial q}{\partial t}\right)^2}{2} \qquad (29)$$

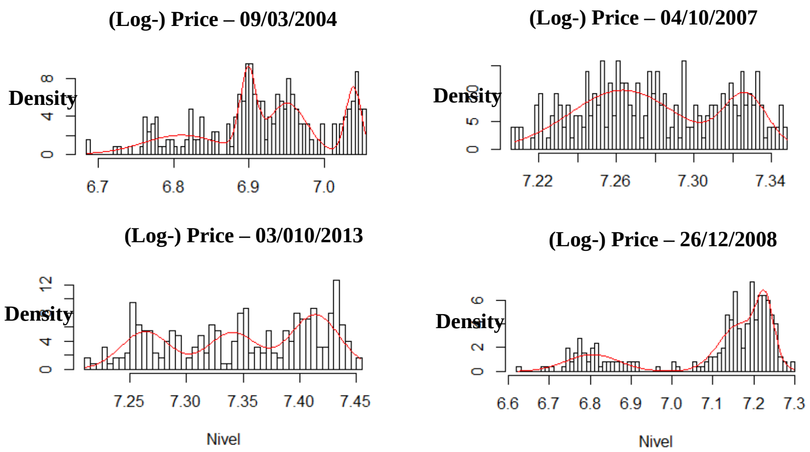
FIGURES

Figure 1: S&P 500 (Log-) Price – Distribution Multimodality



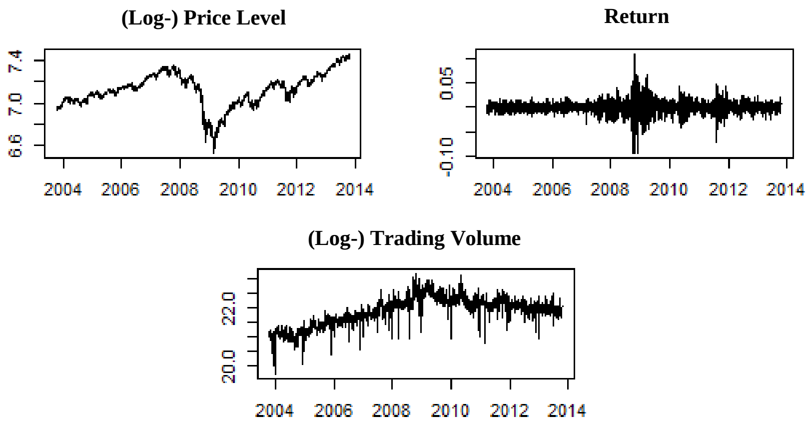
The histograms of S&P 500 index are presented for 32, 64 and 256 number of breaks, respectively, as returned by R-Software. The multimodal characteristic of the series gets stronger as the number of breaks increases.

Figure 2: S&P 500 (Log-) Price – Probability Distribution Estimate



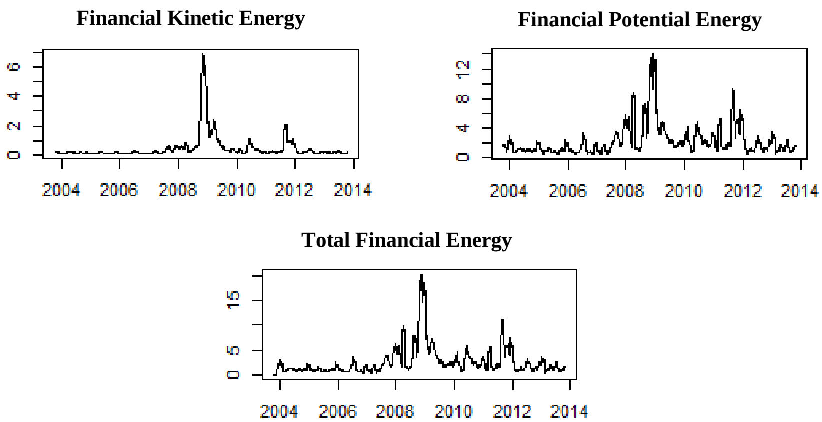
The histograms are for samples of 252 observations of the S&P 500 and the red line represents the weighted mix of Gaussian distributions. Four dates were chosen as examples, namely 9 March 2004, 4 October 2007, 26 December 2008 and 3 October 2013.

Figure 3: S&P 500 – (Log-) Price, Return and (Log-) Trading Volume



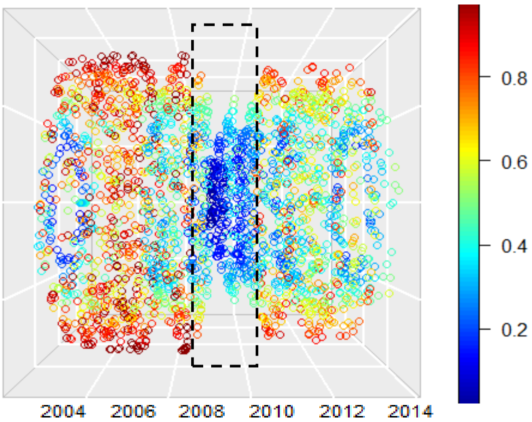
The price and trading volume series are logarithmic, and the return series is calculated as the difference between two successive values of the log-price series.

Figure 4: Energy Components of S&P 500 Index



The "financial kinetic energy" expresses the influence of market factors that are endogenous to the system, while the "financial potential energy" expresses the influence of market factors that are exogenous to the system. The total market "energy" is the arithmetic sum between the kinetic and potential components.

Figure 5: S&P 500 Financial Wave (Oct 2003–Oct 2013)



The colour coding is given by the scale on the right; red representing price values that investors assign for high probability of occurrence, and blue relates to low probability of occurrence. More precisely, as the expected probability of price to take any given discrete value is zero (similarly to the probability that a particle is at any given precise spatial coordinate), the probability of price was measured to be in the interval centered in the observed value, with upper and lower limits of ±10%.

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