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A NEW RISK ASSESSMENT AND OPTIMIZATION MODEL WITH APPLICATIONS TO INSURANCE

Abstract. *In this paper we introduce the Limited Value-at-Risk and Limited Conditional Tail Expectation risk measures. Representation formulas are derived and their properties and advantages are investigated. Also, optimization problems using these mesures as objective functions are formulated and solved. Necessary and sufficient conditions for the existence of the optimal solution are obtained and analytical formulas for the solutions of the optimization problemes constructed are derived. The resultats extend previous results from the recent literature.*

Keywords: *risk management, Limited Value-at-Risk, Limited Conditional Tail Expectation, optimization problems, stop-loss reinsurance.*

JEL CLASSIFICATION: G1, G11, G12.

1. INTRODUCTION

Various quantitative techniques have been proposed for solving real problems which arise in economics, social sciences, engineering and many other domains. In this respect, there can be mentioned the contributions of Ausloos et al., 2015, Di Bernardino and Prieur, 2014, Rotundo et al., 2015, Preda et al., 2015, Georgescu, 2014, Filip, 2012, Costea et al., 2009 and Klugman, 2004. Modeling the trend of financial indices and portfolio selection topics have caught the interest of the researchers, see, for example, Şerban et al., 2015, Toma, 2014. Recently, Preda et al., 2014 and Toma and Leoni-Aubin, 2013 investigated a variety of methods for financial data modeling.

Risk management plays a very important role in decision problems under uncertainty. The approach based on risk measures provides realistic techniques and efficient computational tools for modeling and solving real problems from finance and insurance. Recently, many research papers are devoted to the topic of decision making using different risk measures, see for example, Cousin and Di Bernardino, 2014, Cai and Tan, 2007, Toma, 2013, Dedu and Ciumara, 2010, Georgescu, 2007, Dedu, 2012 and Şerban et al., 2013.

In this paper we develop some risk management techniques based on two new risk measures. Some fundamental concepts of risk theory and the most important risk measures used in financial data modeling are presented in Section 2. The Limited Value-at-Risk and Limited Conditional Tail Expectation risk measures are introduced in Section 3. Also, representation formulas for these measures are derived and their properties are studied. In Section 4 we formulate and solve optimization problems with applications to insurance, using the risk measures introduced as objective functions. Necessary and sufficient condtions for the existence of the optimal solution are obtained and analytical formulas for the solutions of the optimization problems are derived. The results obtained are used to solve decisional problems under uncertainty in Section 5. The performance of the methods proposed is analized in case of some financial applications The resultats extend previous results from the recent literature. The conclusions of the paper are presented in Section 6.

2. FUNDAMENTAL CONCEPTS OF RISK THEORY

Let (Ω, K, P) be a probability space and $\boldsymbol{X}$ the set of random variables defined on the elementary events space, $\boldsymbol{X}=\{X : \Omega \rightarrow \mathbf{R} \mid X \text{ random variable}\}$.

Definition 2.1. A risk measure is a functional ρ defined as follows:

$\rho : \boldsymbol{X} \rightarrow \mathbf{R} \cup \{\pm\infty\}$.

Risk measures play an important role in modeling economic phenomena under uncertainty and provide an important class of objective functions for the problems regarding decision optimization. The results obtained in statistics and optimization and also the development powerful computation and simulation techniques determined the growing interest for the research regarding risk measures used for solving a wide variety of real world problems.

Let $X : \Omega \rightarrow \mathbf{R}$ be a random variable with cumulative distribution function $F_X(x) = P(X \le x)$ and survival function $S_X(x) = P(X > x)$. Let $\alpha \in (0,1)$ fixed.

Definition 2.2. The α-Value-at-Risk measure (α-VaR) of the random variable X corresponding to the probability level α is defined by:

$$\mathrm{VaR}_{\alpha}(X) = \inf\{x \in \mathbf{R} \mid \mathrm{P}(X \le x) \ge \alpha\} = \inf\{x \in \mathbf{R} \mid \mathrm{P}(X > x) \le 1-\alpha\}$$

or

$$\mathrm{VaR}_{\alpha}(X) = \inf\{x \in \mathbf{R} \mid F_X(\mathrm{x}) \ge \alpha\} = \inf\{x \in \mathbf{R} \mid S_X(\mathrm{x}) \le 1-\alpha\}.$$

Remark 2.3. If the cumulative distribution function of the random variable X is given by a one-to-one continuous function, then the $\mathrm{VaR}_{\alpha}(X)$ measure is the unique solution of one of the following equations:

$$\mathrm{P}(X \leq \mathrm{VaR}_{\alpha}(X)) = \alpha$$

or

$$\mathrm{P}(X > \mathrm{VaR}_{\alpha}(X)) = 1 - \alpha .$$

In this case, we have:

$$\mathrm{VaR}_{\alpha}(X) = F_X^{-1}(\alpha) = S_X^{-1}(1 - \alpha) .$$

If the random variable X models the loss associated to a set of decisions, then α-VaR evaluates the maximal loss which can occur with probability α. The financial institutions or insurance companies use the α-VaR measure to evaluate the minimal capital which must be allocated for the reserve fund in order to avoid ruin. Even though the α-VaR measure represents a key instrument for risk assessment, it lacks some important properties, such as subadditivity, convexity and continuity. In the same time, the α-VaR measure is not sensitive to the shape of the distribution beyond α-VaR threshold. Thus, the maximal loss beyond the α-VaR threshold may significantly increase and the α-VaR value does not change. The α-VaR measure does not provide any information about the long tail of a distribution. This issue is very important if we take into account that an institution for risk regulation is concerned not only about the frequency of the claims, but also about the severity of the claims. The insensitivity of the α-VaR measure to the extreme tails of the distributions is an undesirable property, which allows taking big and uncontrollable risks. Therefore, the big losses with a very small probability of occurence can not be quantified using α-VaR measure. The α-VaR measure has some shortcomings, as it is not coherent and it does not provide any information about the tail of the distribution, which evaluates the severity of risk. The lack of these important properties stimulated the research for finding new risk measures. The Conditional Tail Expectation measure was introduced for estimating the mean value of the losses beyond the α-VaR threshold.

Definition 2.4. The α-Conditional Tail Expectation measure (α-CTE) of the random variable X at the probability level α is defined by:

$$\mathrm{CTE}_{\alpha}(X) = \mathrm{E}[\mathrm{X} \mid \mathrm{X} \geq \mathrm{VaR}_{\alpha}(X)].$$

Remark 2.5. The α-CTE measure evaluates the mean intensity of the risk beyond the α-VaR threshold and estimates the severity of risk corresponding to the right tail of the distribution. The α-CTE measure evaluates the mean loss in the most unfavorable α% situations, therefore the α-CTE measure estimates the value of the losses which affect in the highest degree the investor, being sensitive with respect to the shape of the distribution of the risk corresponding to the tail of the distribution. This is a very important feature in the case the losses corresponding to the upper tail of the distribution.

Various representation formulas for α-CTE measure have been proposed in the literature, as it follows.

Theorem 2.6. The α-Conditional Tail Expectation of the random variable X corresponding to the probability level α can be expressed using the equivalent formula:

$$\mathrm{CTE}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma .$$

3. NEW RISK MEASURES USED FOR SOLVING OPTIMIZATION PROBLEMS IN INSURANCE

We will use a Value-at-Risk-based risk measure, introduced in Tudor and Dedu, 2012. Let (Ω,K,P) be a probability space, $X : \Omega \to \mathbf{R}$ a random variable, the probability level $\alpha \in (0,1)$ and $l_0 \in \mathbf{R}\cup\{\infty\}$ such that the condition $\mathrm{P}(X \le l_0) > 0$ holds.

Definition 3.1. The (α,l_0)-Limited Value-at-Risk of the random variable X corresponding to the probability level α and threshold l_0 is defined by:

$$\mathrm{LVaR}_{\alpha,l_0}(X) = \inf\{x \in \mathbf{R} \mid \mathrm{P}(X > x \mid X \le l_0) \le 1-\alpha\}.$$

We will denote by (α,l_0)-LVaR the (α,l_0)-Limited Value-at-Risk measure.

Remark 3.2. Since for any $x \ge l_0$ we have $\mathrm{P}(X > x \mid X \le l_0) = 0 \le 1 - \alpha$, it follows that

$$\mathrm{LVaR}_{\alpha,l_0}(X) < l_0 . \tag{1}$$

Proposition 3.3. If $X : \Omega \to \mathbf{R}$ is a random variable, $\alpha \in (0,1)$ and $l_0 \in \mathbf{R}\cup\{\infty\}$ such that $\mathrm{P}(X \le l_0) > 0$, then we have:

$$\mathrm{LVaR}_{\alpha,l_0}(X) = \mathrm{VaR}_{\beta(\alpha,l_0)}(X) \tag{2}$$

where

$$\beta(\alpha, l_0) = \alpha\, \mathrm{F}_X(l_0).$$

Proof. Using (1) it results that we will study the inequality $\mathrm{P}(X > x \mid X \le l_0) \le 1 - \alpha$ only if $x < l_0$. In this case we have:

$$\mathrm{P}(X > x \mid X \le l_0) \le 1 - \alpha$$

if and only if

$$\mathrm{P}(X > x) \le \alpha F_X(l_0).$$

We denote by $\beta(\alpha, l_0) = \alpha\, \mathrm{F}_X(l_0)$ and the conclusion follows.

Using the properties stated in (2) we can prove that the (α,l_0)-LVaR measure has the properties stated in the following proposition.

Proposition 3.4. For any random variable X, for any $\alpha \in (0,1)$ and $l_0 \in \mathbf{R} \cup \{\infty\}$ such that the condition $\mathrm{P}(X \le l_0) > 0$ holds, we have:

a) $\beta(\alpha, l_0) \in [0, \alpha]$;

b) $\lim_{l_0 \to \infty} \beta(\alpha, l_0) = \alpha$;

c) $\lim_{l_0 \to \infty} \mathrm{LVaR}_{\alpha, l_0}(X) = \mathrm{VaR}_{\alpha}(X)$.

Remark 3.5. The (α,l_0)-LVaR measure of a random variable that models risk evaluates the maximal loss which can occur with the probability $1-\alpha$, if the values of the risk random variable does not exceed a certain threshold l_0. The risk level is modeled by the probability l_0 of the threshold and also by the value of the probability level. Since the α-VaR corresponding to the same probability level α overestimates risk, it results that the approach based on using (α,l_0)-LVaR is more realistically for modeling risk. We can obtain further information regarding the behavior of the loss random variable corresponding to the right tail of the distribution by increasing the value l_0 of the threshold corresponding to the (α,l_0)-LVaR measure.

Definition 3.6. The (α,l_0)-Limited Conditional Tail Expectation of the random variable X corresponding to the probability level α and the threshold l_0 is defined by:

$$\mathrm{LCTE}_{\alpha, l_0}(X) = \mathrm{E}\left[\mathrm{X} \mid \mathrm{X} \ge \mathrm{LVaR}_{\alpha, l_0}(X)\right].$$

We denote by (α,l_0)-LCTE the (α,l_0)-Limited Conditional Tail Expectation measure.

Remark 3.7. We have:

$$\mathrm{LCTE}_{\alpha, \infty}(X) = \mathrm{CTE}_{\alpha}(X).$$

Using Proposition 3.4 we get the following representation formulas.

Proposition 3.8. We have:

$$\mathrm{LCTE}_{\alpha,l_0}(X) = \mathrm{CTE}_{\alpha}(X),$$

where $\beta(\alpha, l_0) = \alpha \, \mathrm{F}_X(l_0)$.

4. OPTIMIZATION IN STOP-LOSS REINSURANCE, USING LIMITED RISK MEASURES

4.1. THE STOP-LOSS REINSURANCE MODEL

Reinsurance represents a risk management technique used by insurance companies to protect themselves against the risk of losses, transferring the risk to a second insurance institution. The former part is called the cedent or the insurer and the latter is the reinsurer. Reinsurance can be: proportional (quota share, surplus) or non-proportional (excess of loss, stop-loss).

Stop-loss is a special type of reinsurance. The stop-loss agreement states that the liability of the insurer is limited to a specified amount, called retention and the reinsurer will pay the cedent's losses which exceed the retention level.

We will model the aggregated loss corresponding to a portfolio composed by insurance contracts using a nonnegative random variable *X*, with cumulative distribution function $F_X(x) = P(X \le x)$ and survival function $S_X(x) = P(X > x)$ and expectation $\mathrm{E}(X) > 0$. We assume that X has a one-to-one continuous distribution function on $[0, \infty)$. Let X_I and X_R the random variables which model the loss of the insurer and respectively the loss of the reinsurer for a stop-loss reinsurance contract, hence $X = X_I + X_R$. We denote by d>0 the retention value. Under the reinsurance agreement, the reinsurer undertakes the risk which exceeds the retention level, while the insurer is protected from the risk of a potential large loss by limiting its liability to the retention level. The relationship between the random variables X_I, X_R and X can be expressed by $X_I = X$, if $X \le d$ and $X_I = d$, if $X > d$. Therefore, we have $X_R = 0$, if $X \le d$ and $X_R = X - d$, if $X > d$.

The reinsurance premium can be evaluated using one of the premium principles. Using the mean value principle, it becomes:

$$\pi(d) = \mathrm{E}(X_R)$$

The next result establishes a representation formula for the stop-loss premium under the mean value principle.

Proposition 4.1. (Klugman et al., 2004) The reinsurance premium corresponding to a stop-loss reinsurance contract is given by:

$$\pi(d) = E(X_R) = E\left[(X-d)^+\right]$$

It follows that the gross stop-loss reinsurance premium $\delta(d) = (1+\rho)\pi(d)$ can be expressed as follows:

$$\pi(d) = E(X_R) = \int_d^\infty S_X(x)\mathrm{d}x,$$

where $\rho > 0$ represents the safety loading coefficient.

Remark 4.2. The function δ is decreasing with respect to the argument d.

We denote by T the random variable which models the total cost of the insurer in a stop-loss reinsurance model. T has two components: the loss of the reinsurer and the insurance premium:

$$T = X_I + \delta(d).$$

We will study the optimization problem for a stop-loss reinsurance model, which means deriving the retention level d which assures an optimal trade-off between the risk of the insurer and the risk of the reinsurer. Different optimization criteria can be used for solving this problem, for example minimizing the ruin probability, maximizing the utility of the insurer resulted from the risk transfer or minimizing certain risk measures. Recent research approached the optimization of Value-at-Risk, Conditional Tail Expectation or Conditional Value-at-Risk measures.

We propose to formulate some optimization problems with applications in stop-loss reinsurance, using the risk measures introduced in this paper as objective functions. An effective risk management from the point of view of the insurer aims minimizing the risk measures corresponding to the total cost of the insurer T. We will study the problem of deriving the optimal retention using as optimization criterion the minimization of the risk measures introduced in the previous section. Let $\alpha \in (0,1)$ and $l_0 \in \mathbf{R} \cup \{\infty\}$, such that $\mathrm{P}(X \le l_0) > 0$. The analytical formulas of the (α, l_0)-LVaR and (α, l_0)-LCTE measures of the random variables X_I and T, corresponding to the retention d and to the probability level 1-α are defined as follows:

$$\mathrm{LVaR}_{\alpha, l_0}(X_I, d) = \inf\left\{x \in \mathbf{R} \mid \mathrm{P}(X_I > x \mid X_I \le l_0) \le 1-\alpha\right\}$$

and

$$\mathrm{LVaR}_{\alpha, l_0}(T, d) = \inf\left\{x \in \mathbf{R} \mid \mathrm{P}(T > x \mid T \le l_0) \le 1-\alpha\right\},$$

respectively

$$\mathrm{LCTE}_{\alpha, l_0}(X_I, d) = \mathrm{E}\left[X_I \mid X_I \ge \mathrm{LVaR}_{\alpha, l_0}(X_I)\right]$$

and

$$\mathrm{LCTE}_{\alpha,l_0}(T,d) = \mathrm{E}\lfloor T \mid T \geq \mathrm{LVaR}_{\alpha,l_0}(T)\rfloor.$$

Now we will formulate and solve two optimization problems using the risk measures proposed in the previous section.

4.2. OPTIMIZATION PROBLEM USING (α, l_0)-LVaR MEASURE

The objective consists in deriving the level d^* of retention for which the measure $\mathrm{LVaR}_{\alpha,l_0}(T,d)$ attains its minimal value, that reduces to solving the following optimization problem:

$$\mathrm{LVaR}_{\alpha,l_0}(T,d^*) = \min_{d>0} \mathrm{LVaR}_{\alpha,l_0}(T,d) \tag{3}$$

First we will establish a result regarding an analytical representation of $\mathrm{LVaR}_{\alpha,l_0}(T,d)$ measure.

Proposition 4.3. For any $d > 0$ and $0 < \alpha + (1\text{-}\alpha)\mathrm{S}_X(l_0) < \mathrm{S}_X(0)$ we have:

a) if $0 < d \leq S_X^{-1}\left(\alpha + (1\text{-}\alpha)\mathrm{S}_X(l_0)\right) < \mathrm{S}_X(0)$, then

$$\mathrm{LVaR}_{\alpha,l_0}(T,d) = d + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x;$$

b) if $d > S_X^{-1}\left(\alpha + (1\text{-}\alpha)\mathrm{S}_X(l_0)\right)$, then

$$\mathrm{LVaR}_{\alpha,l_0}(T,d) = S_X^{-1}\left(\alpha + (1\text{-}\alpha)\mathrm{S}_X(l_0)\right) + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x.$$

In the next theorem we obtain results regarding the solution of the optimization problem (3).

Theorem 4.4. *a*) The optimal solution $d^* > 0$ of the problem (3) exists if and only if the following conditions are fulfilled:

$$\alpha + (1\text{-}\alpha)\mathrm{S}_X(l_0) < \frac{1}{1+\rho} < \mathrm{S}_X(0)$$

and

$$S_X^{-1}(\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0)) \geq S_X^{-1}\left(\frac{1}{1+\rho}\right)+\delta\left(S_X^{-1}\left(\frac{1}{1+\rho}\right)\right).$$

b) If the optimal solution $d^* > 0$ of the problem (3) exists, then:

$$d^* = S_X^{-1}\left(\frac{1}{1+\rho}\right)$$

and the minimal value of the measure $\mathrm{LVaR}_{\alpha,l_0}(T,d)$ corresponding to the random variable T is given by:

$$\mathrm{LVaR}_{\alpha,l_0}(T,d^*) = d^* + \delta(d^*).$$

4.3. OPTIMIZATION PROBLEM USING (α,l$_0$)-LCTE MEASURE

The second approach consists in searching for the optimal level d^* of retention which minimizes the measure (α,l_0)-LCTE of the total cost corresponding to a given probability level α:

$$\mathrm{LCTE}_{\alpha,l_0}(T,d^*) = \min_{d>0} \mathrm{LCTE}_{\alpha,l_0}(T,d) \tag{4}$$

Proposition 4.5. For any $d > 0$ and $0 < \alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0) < \mathrm{S}_X(0)$ we have:

a) if $0 < d \leq S_X^{-1}(\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0)) < \mathrm{S}_X(0)$, then

$$\mathrm{LCTE}_{\alpha,l_0}(T,d) = d + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x;$$

b) if $d > S_X^{-1}(\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0))$, then

$$\mathrm{LCTE}_{\alpha,l_0}(T,d) = S_X^{-1}(\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0)) + (1+\rho)\int_d^{\infty} S_X(x)\mathrm{d}x + \frac{1}{\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0)}\int_{S_X^{-1}(\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0))}^{\infty} S_X(x)\mathrm{d}x$$

Theorem 4.6. *a*) The optimal solution $d^* > 0$ of the problem (4) exists if and only if the following conditions are fulfilled:

$$\alpha+(1\text{-}\alpha)\mathrm{S}_X(l_0) < \frac{1}{1+\rho} < \mathrm{S}_X(0)$$

and

$$S_X^{-1}\left(\alpha+(1-\alpha)\mathrm{S}_X(l_0)\right)\geq S_X^{-1}\left(\frac{1}{1+\rho}\right)+\delta\left(S_X^{-1}\left(\frac{1}{1+\rho}\right)\right).$$

b) If the optimal solution $d^* > 0$ of the problem (4) exists, then $d^*=S_X^{-1}\left(\frac{1}{1+\rho}\right)$ for $\alpha+(1-\alpha)\mathrm{S}_X(l_0)<\frac{1}{1+\rho}$ and $d^*=S_X^{-1}\left(\frac{1}{1+\rho}\right)$ for

$\alpha+(1-\alpha)\mathrm{S}_X(l_0)\geq\frac{1}{1+\rho}$

and the minimal value of the measure $\mathrm{LCTE}_{\alpha,l_0}(T,d)$ corresponding to the random variable T is given by:

$$\mathrm{LCTE}_{\alpha,l_0}(T,d^*)=d^*+(1+\rho)\int_{d^*}^{\infty}S_X(x)\mathrm{d}x.$$

5. CASE STUDY: RISK ESTIMATION AND OPTIMIZATION IN INSURANCE

In this section the previous results obtained are used in order to solve an application from insurance. We consider a database which contains the values of 32000 claims corresponding to a portfolio composed by motor insurance policies, which were grouped into 50 intervals. The descriptive statistics for these intervals are presented in Table 1.

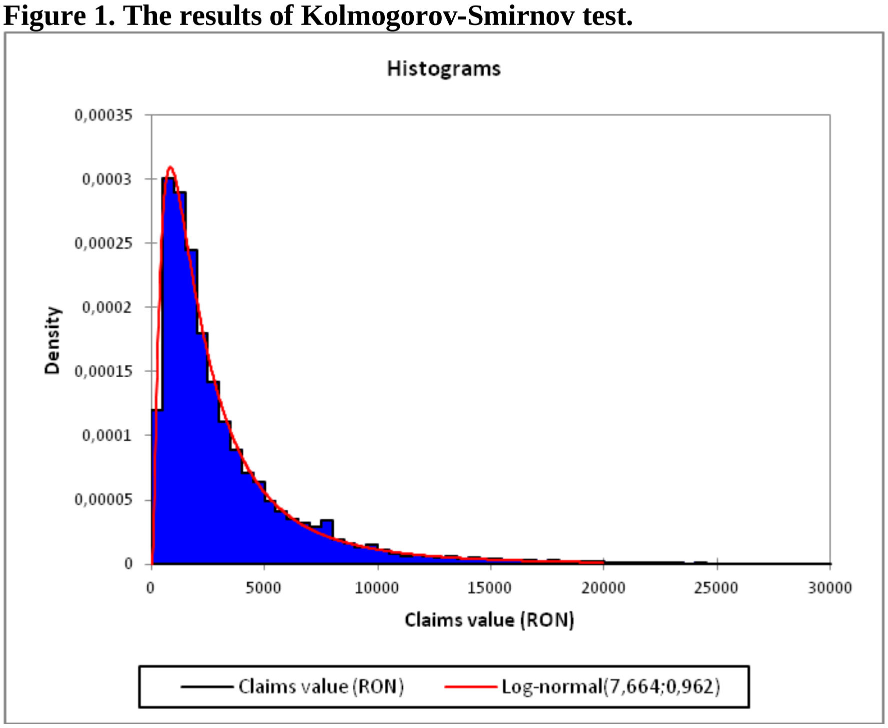
Table 1. The descriptive statistics of the data set

Lower	Upper	Absolute frequency	Relative frequency	Lower	Upper	Absolute frequency	Relative frequency
0	500	1860	0.060	12500	13000	86	0.003
500	1000	4682	0.151	13000	13500	93	0.003
1000	1500	4510	0.145	13500	14000	71	0.002
1500	2000	3806	0.122	14000	14500	77	0.002
2000	2500	2793	0.090	14500	15000	57	0.002
2500	3000	2201	0.071	15000	15500	67	0.002
3000	3500	1731	0.056	15500	16000	54	0.002
3500	4000	1385	0.045	16000	16500	44	0.001
4000	4500	1112	0.036	16500	17000	51	0.002
4500	5000	998	0.032	17000	17500	31	0.001
5000	5500	763	0.025	17500	18000	43	0.001

5500	6000	635	0.020	18000	18500	39	0.001
6000	6500	550	0.018	18500	19000	37	0.001
6500	7000	500	0.016	19000	19500	32	0.001
7000	7500	460	0.015	19500	20000	41	0.001
7500	8000	523	0.017	20000	20500	22	0.001
8000	8500	293	0.009	20500	21000	25	0.001
8500	9000	256	0.008	21000	21500	24	0.001
9000	9500	212	0.007	21500	22000	16	0.001
9500	10000	233	0.007	22000	22500	17	0.001
10000	10500	179	0.006	22500	23000	19	0.001
10500	11000	129	0.004	23000	23500	15	0.000
11000	11500	97	0.003	23500	24000	7	0.000
11500	12000	106	0.003	24000	24500	17	0.001
12000	12500	100	0.003	24500	25000	0	0.000

After performing several concordance distribution tests it results the Log-Normal distribution with parameters 7.664 and 0.962 is accepted for modeling the data set. In Figure 1 we present the results of applying the Kolmogorov-Smirnov distribution fitting test.

Figure 1. The results of Kolmogorov-Smirnov test.



We check if the necessary and sufficient conditions from the theorems are fulfilled and, if they are, we calculate the optimal retention d^* and the minimal value of the Limited Value-at-Risk and Limited Conditional Tail expectation measures. In the following tables, the results obtained for different values of the probability level α and threshold l_0 and a 0.2 safety loading coefficient are presented.

Table 2. The optimal retention and the minimal value of the LVaR and CTE risk measures

α	l_0	d^*	LVaR	LCTE
0.95	15000	5832	5832.8	5833.2
0.95	20000	7143	7143.4	7144.5
0.95	22500	8621	8621.7	8622.6
0.99	15000	6213	6213.6	6214.4
0.99	20000	7849	7849.4	7850.3
0.99	22500	9241	9241.3	9242.7

5. CONCLUSIONS

In this paper we propose and study the properties of two new risk measures, useful for solving real problems from insurance. Optimization problems which use as objective functions the risk measures introduced are formulated and solved. We obtain necessary and sufficient conditions for the existence of the optimal solution and we derive analytical formulas for the solutions of the optimization problems constructed. In this way we extend the results obtained by Cai and Tan (2007) and Dedu and Ciumara (2010). The results presented are new, some of them representing generalizations of recent results from the literature. Computational results are provided, in order to illustrate the behavior of the proposed methods. The approach based on limited risk measures is important since it makes possible the assessment of the value of the reserve fund which must be allocated if some additional information regarding the severity of risk is available. In the case when the insurer limits its liability, the necessary capital may be overestimated if it does not take into account the value of the upper threshold.

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