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# A SERIES SOLUTION OF BLACK-SCHOLES EQUATION UNDER JUMP DIFFUSION MODEL

**Abstract.** *We introduce a series solution for a partial integro-differential equation which arises in option pricing when the Black-Scholes partial differential equations are considered under jump diffusion models. We construct a polynomial chaos solution using the Taylor expansion with respect to Hermite polynomials, which simplifies the integral term and derives a system of deterministic ordinary differential equations. Numerical examples show that the proposed method efficiently gives the desired accuracy for pricing options*.

**Key words:** *Black-Scholes equation, jump-diffusion, polynomial chaos, partial integro-differential equation, option pricing.*

**JEL Classification: G13, C63, C02**

## 1. Introduction

Options in finance are contracts that give the owner the right, but not the obligation, to buy or sell underlying assets at pre-defined price called strike price on the expiration date of contract. Since an investor of options can set the strike price of underlying assets in advance, these options are powerful tools for hedging risk in financial market. Therefore the trading volumes of options are increasing all over the world. From the seminal papers of Black and Scholes (1973) and Merton (1976), the no-arbitrage price of an option can be computed by the solution of the Black-Sholes partial differential equation.

However there have been observations which show that continuous movement of Brownian motion is not enough to describe discontinuous nature of the price in the market. Indeed Bates (1996) and Bakshi *et. al.* (1997) pointed out that a jump term is important for pricing options and internal consistency. Merton (1973) introduced a jump-diffusion model in option pricing to take into account the discontinuities in the price movements. Kou (2002) proposed a double exponential jump-diffusion model. For a complete and detailed explanation on jump diffusion, see Cont and Tankov (2004).

Based on jump diffusion models, the price of options can be computed by solving a partial integro-differential equation (PIDE). Since the closed form solution of PIDE is very limited, there have been different numerical approaches to solve the proposed PIDE. Andersen and Andreasen (2000) proposed a finite difference method with fast Fourier transform. Briani *et. al.* (2004) applied explicit finite difference methods to solve the PIDE. Cont and Voltchkova (2005) proposed an explicit-implicit finite difference method using the notion of viscosity solution and see also Briani *et. al.* (2007). Matache *et. al.* (2004) discretized the PIDE by the $\theta$-scheme in time and a wavelet Galerkin method in space and Matache *et. al.* (2005) applied discontinuous Galerkin methods in time and wavelet discretization in space. However, all of these methods give only approximations of the solution of the PIDE.

In this paper, we apply polynomial chaos expansions for PIDEs in option pricing, which gives an efficient series solution of PIDE. Therefore we have a semi-analytic solution of PIDE. Polynomial chaos expansions by Xiu and Karniadakis (2002) and Xiu (2010) are getting popular for solving stochastic differential equations. Using the polynomial chaos, stochastic solutions are expressed as a series with respect to polynomials, which depends upon input random parameters. It is the novelty of the present work that we newly apply the polynomial chaos method for solving PIDEs in option pricing with Hermite basis polynomials. Numerical examples for European options in section 4 show that the proposed polynomial chaos method gives a good series solution.

The outline of the paper is as follows. In Section 2, we describe the option pricing problem under jump diffusion models. In Section 3, we describe the polynomial chaos method for solving PIDEs in option pricing. In Section 4, we present numerical results of the proposed method for European options. We summarize our results and present future research directions in Section 5. Detailed proofs and derivations of important equations are in the Appendix.

## 2. Option valuation under jump diffusion model

In order to reflect the discontinuous nature of stock prices, Merton (1976) introduces a jump process in option pricing. In the Merton model (1976), the price of underlying asset $S(\tau)$ allows compound Poisson jumps:

$$S(\tau) = S(0)\ \exp\left[\mu\tau + \sigma W(\tau) + \sum_{i=1}^{N(\tau)} Y_i\right], \tag{1}$$

where $\mu$ is an expected rate of return, $\sigma$ is a diffusion volatility and $W(\tau)$ is a Brownian motion. Here $N(\tau)$ is a Poisson process with intensity $\lambda$, which is independent from $W$ and jump sizes $Y_i$ are independent identically distributed with standard deviation $\delta$. See Cont and Tankov (2004) for detailed properties of jump processes.

Based on the Merton model, the value $V(S,\tau)$ of European option satisfies the following second-order partial integro-differential equation (PIDE):

$$V_\tau + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \lambda\bar{k})SV_S - rV + \lambda\left(\int_0^\infty V(S\eta,\tau)\tilde{\Gamma}_\delta(\eta)\,d\eta - V\right) = 0, \tag{2}$$

where

$$\tilde{\Gamma}_\delta(\eta) = \frac{1}{\sqrt{2\pi}\delta\eta}\exp\left(-\frac{1}{2}\left(\frac{\log\eta}{\delta}\right)^2\right), \quad \bar{k} = \int_0^\infty (e^\eta - 1)\tilde{\Gamma}_\delta(\eta)\ d\eta.$$

Let us consider the change of variables

$$x = \frac{1}{\delta}\log S, \quad z = \frac{1}{\delta}\log\eta, \quad t = T - \tau$$

so that $S = e^{x\delta}$, $\eta = e^{z\delta}$ and $S\eta = e^{(x+z)\delta}$. When $v(x,t)$ is defined by

$$V(S,\tau) = V(e^{x\delta}, T-t) = v(x,t),$$

we can derive

$$V_\tau = -v_t, \quad V_S = \frac{1}{\delta S}v_x, \quad V_{SS} = \frac{-1}{\delta S^2}v_x + \frac{1}{(\delta S)^2}v_{xx}$$

and the equation (2) can be written as

$$v_t(x,t) = Dv(x,t) + \lambda\ Iv(x,t), \tag{3}$$

where the differential term $Dv(x,t)$ and the integral term $Iv(x,t)$ are

$$Dv(x,t) = \frac{1}{2}\sigma^2\left(\frac{-1}{\delta}v_x(x,t) + \frac{1}{\delta^2}v_{xx}(x,t)\right) + (r - \lambda\bar{k})\frac{1}{\delta}v_x(x,t) - (r+\lambda)v(x,t),$$

$$Iv(x,t) = \int_{-\infty}^{\infty} v(x+z,t)\Gamma(z)\ \mathrm{dz},$$

and

$$\Gamma(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right). \tag{4}$$

Note that the partial integro-differential equation (3) has two variables, namely time and normalized log price, which has stochastic nature. We first apply the polynomial chaos expansion to determine the evolution of uncertainty of price. Then we apply the Runge-Kutta method for the remaining deterministic systems of equations.

**3. Polynomial chaos**

The polynomial chaos expansion represents a stochastic process as an infinite series using orthogonal basis functions. The polynomial chaos expansion by Xiu and Karniadakis (2002) and Xiu (2010) has been successfully applied to many areas including fluid dynamics, solid mechanics and robust control etc. Here we newly apply the polynomial chaos expansion with Hermite polynomial basis to option pricing under jump diffusion models.

Hermite polynomial $H_n^*(x)$ of order $n$ is defined by

$$H_n^*(x) = (-1)^n e^{x^2/2}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2}\right)$$

and the normalized Hermite polynomial of order $n$ is defined by

$$H_n(x) = \frac{1}{\sqrt{n!}} H_n^*(x).$$

Since the random variable we consider in this study is Gaussian, we write $v(x,t)$ as the Hermite-Fourier series

$$v(x,t) = \sum_m v_m(t) H_m(x),$$

where $H_m(x)$ is the normalized Hermite polynomial of order $m$. See Xiu (2010) for details. Then, by (14) in the Appendix

$$\begin{aligned} &v(x+z,t) \\ &= \sum_m v_m(t) H_m(x+z) \\ &= \sum_m v_m(t) 2^{-m/2} \sum_{r=0}^{m} \binom{m}{r}^{1/2} H_{m-r}(\sqrt{2}x) H_r(\sqrt{2}z) \\ &= \sum_m v_m(t) 2^{-m} \sum_{r=0}^{m} \binom{m}{r}^{1/2} \sum_{p=0}^{m-r} \binom{m-r}{p}^{1/2} H_p(x) H_{m-r-p}(x) \sum_{q=0}^{r} \binom{r}{q}^{1/2} H_q(z) H_{r-q}(z). \end{aligned}$$

Since $H_q(z)H_{r-q}(z)$ above is the function of $z$ only, the integral term $Iv(x,t)$ in (3) becomes

$$Iv(x,t) = \sum_m v_m(t) 2^{-m} \sum_{r=0}^{m} \binom{m}{r}^{1/2} \left( \sum_{p=0}^{m-r} \binom{m-r}{p}^{1/2} H_p(x) H_{m-r-p}(x) \right)$$

$$\times\left(\sum_{q=0}^{r}\binom{r}{q}^{1/2}\int_{-\infty}^{\infty}H_q(z)H_{r-q}(z)\Gamma(z)\,\mathrm{d}z\right)$$

$$=\sum_m v_m(t)2^{-m}\sum_{r=0}^{m}\binom{m}{r}^{1/2}\sum_{p=0}^{m-r}\binom{m-r}{p}^{1/2}H_p(x)H_{m-r-p}(x)\sum_{q=0}^{r}\binom{r}{q}^{1/2}\boldsymbol{\delta}_{q,r\text{-}q}$$

$$=\sum_m v_m(t)2^{-m}\sum_{q=0}^{m}\sum_{r=q}^{m}\binom{m}{r}^{1/2}\sum_{p=0}^{m-r}\binom{m-r}{p}^{1/2}H_p(x)H_{m-r-p}(x)\binom{r}{q}^{1/2}\boldsymbol{\delta}_{q,r\text{-}q}\ ,$$

where $\boldsymbol{\delta}_{q,r\text{-}q}=1$ when $q=r-q$ (i.e. when $r=2q$ ), and 0 otherwise. Since $2q>m\geq r$ for $q>m/2$, $\boldsymbol{\delta}_{r,r\text{-}q}=0$ if $q>m/2$. Thus,

$$Iv(x,t)=\sum_m v_m(t)2^{-m}\sum_{q=0}^{\lfloor m/2\rfloor}\binom{m}{2q}^{1/2}\sum_{p=0}^{m-2q}\binom{m-2q}{p}^{1/2}H_p(x)H_{m-2q-p}(x)\binom{2q}{q}^{1/2}$$

$$=\sum_m v_m(t)2^{-m}\sqrt{m!}\sum_{q=0}^{\lfloor m/2\rfloor}\frac{1}{q!}\sum_{p=0}^{m-2q}\frac{1}{\sqrt{p!(m-2q-p)!}}H_p(x)H_{m-2q-p}(x),$$

where $\lfloor x\rfloor$ is the largest integer not greater than $x$ . It is also known that $\{H_m(x)\}$ satisfy

$$H_m(x)H_n(x)=\sum_{k=0}^{m\wedge n}\binom{m}{k}^{1/2}\binom{n}{k}^{1/2}\binom{m+n-2k}{m-k}^{1/2}H_{m+n-2k}(x).$$

Then

$$H_p(x)H_{m-2q-p}(x)=\sum_{k=0}^{p\wedge(m-2q-p)}\binom{p}{k}^{1/2}\binom{m-2q-p}{k}^{1/2}\binom{m-2q-2k}{p-k}^{1/2}H_{p+(m-2q-p)-2k}(x)$$

and the integral term becomes

$$Iv(x,t)=\sum_m\frac{v_m(t)\sqrt{m!}}{2^m}\sum_{q=0}^{\lfloor m/2\rfloor}\frac{1}{q!}\sum_{p=0}^{m-2q}\sum_{k=0}^{p\wedge(m-2q-p)}\frac{\sqrt{(m-2q-2k)!}}{(p-k)!k!(m-2q-p-k)!}H_{m-2q-2k}(x).\quad(5)$$

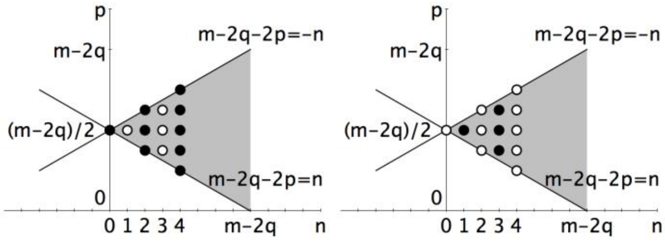
Set $n=m-2q-2k$ so that $k=(m-2q-n)/2$. Then when $k$ changes from 0 to $p\wedge(m-2q-p)$ , $n$ increases from $|m-2q-2p|$ to $m-2q$ with the increment of 2. For the notational simplicity, let us define a symbol $\sum^*$ in this study to represent the summation whose index increases with the increment of 2. For example,

$$\sum_{i=1}^{5}{}^{*}a_i=a_1+a_3+a_5.$$

When the summation in $k$ in (5) is replaced by that in $n$ ,

$$Iv(x,t)=\sum_m \frac{v_m(t)m!}{2^m}\sum_{q=0}^{\lfloor m/2\rfloor}\frac{1}{q!}\sum_{p=0}^{m-2q}\sum_{n=|m-2q-2p|}^{m-2q}{}^{*}\frac{\sqrt{n!}H_n(x)}{\left(\frac{m-n}{2}-q\right)!\left(p+q-\frac{m-n}{2}\right)!\left(\frac{m+n}{2}-p-q\right)!}$$

**Figure 1 : $np$ domain (Left) when $m$ is even and (Right) when $m$ is odd.**




When $m$ is even, $m-2q$ is also even so that $(m-2q)/2$ is an integer. $|m-2q-2p|\le n\le m-2q$ implies that $n$ varies from the maximum between $n$ values of $m-2q-2p=\pm n$ to $n=m-2q$ with the increment of 2. See Figure 1 (Left). Thus, when $m$ is even,

$$\sum_{p=0}^{m-2q}\sum_{n=|m-2q-2p|}^{m-2q}{}^{*}=\sum_{n=0}^{m-2q}{}^{*}\sum_{p=(m-2q-n)/2}^{(m-2q+n)/2}$$

When $m$ is odd, $(m-2q)/2$ is not integer-valued and we obtain

$$\sum_{p=0}^{m-2q}\sum_{n=|m-2q-2p|}^{m-2q}{}^{*}=\sum_{n=1}^{m-2q}{}^{*}\sum_{p=(m-2q-n)/2}^{(m-2q+n)/2}$$

See Figure 1 (Right). If we define $sgn(m)$ by

$$sgn(m)=\begin{cases}0\ , & \text{if } m \text{ is even}\\ 1\ , & \text{if } m \text{ is odd}\end{cases},$$

we can combine these two cases into one:

$$\sum_{p=0}^{m-2q}\sum_{n=|m-2q-2p|}^{m-2q}{}^{*}=\sum_{n=sgn(m)}^{m-2q}{}^{*}\sum_{p=(m-2q-n)/2}^{(m-2q+n)/2}.$$

and the integral term becomes

$$Iv(x,t)$$

$$=\sum_m v_m(t)2^{-m}\sqrt{m!}\sum_{q=0}^{\lfloor m/2\rfloor}\frac{1}{q!}\sum_{n=sgn(m)}^{m-2q}{}^{*}\frac{1}{\left(\frac{m-n}{2}-q\right)!}\sum_{p=(m-2q-n)/2}^{(m-2q+n)/2}\frac{\sqrt{n!}H_n(x)}{\left(p+q-\frac{m-n}{2}\right)!\left(\frac{m+n}{2}-p-q\right)!}$$

Note that for each fixed $n$ and $q$, the summation in $p$ can be simplified as

$$\sum_{p=(m-2q-n)/2}^{(m-2q+n)/2} \frac{\sqrt{n!}}{\left(p+q-\frac{m-n}{2}\right)!\left(\frac{m+n}{2}-p-q\right)!} = \frac{1}{\sqrt{n!}}\sum_{p=0}^{n}\frac{n!}{p!(n-p)!} = \frac{2^n}{\sqrt{n!}}$$

and

$$Iv(x,t) = \sum_m v_m(t)2^{-m}\sqrt{m!}\sum_{q=0}^{\lfloor m/2\rfloor}\frac{1}{q!}\sum_{n=sgn(m)}^{m-2q}{}^{*}\frac{1}{\left(\frac{m-n}{2}-q\right)!}\frac{2^n H_n(x)}{\sqrt{n!}}$$

When $m$ is even,

$$\sum_{q=0}^{\lfloor m/2\rfloor}\sum_{n=sgn(m)}^{m-2q}{}^{*} = \sum_{q=0}^{m/2}\sum_{n=0}^{m-2q}{}^{*} = \sum_{n=0}^{m}{}^{*}\sum_{q=0}^{(m-n)/2}$$

since both $m$ and $n$ are even. When $m$ is odd,

$$\sum_{q=0}^{\lfloor m/2\rfloor}\sum_{n=sgn(m)}^{m-2q}{}^{*} = \sum_{q=0}^{(m-1)/2}\sum_{n=1}^{m-2q}{}^{*} = \sum_{n=1}^{m}{}^{*}\sum_{q=0}^{(m-n)/2}$$

since both $m$ and $n$ are odd. Then, we can combine these two cases into one:

$$\sum_{q=0}^{\lfloor m/2\rfloor}\sum_{n=sgn(m)}^{m-2q}{}^{*} = \sum_{n=sgn(m)}^{m}{}^{*}\sum_{q=0}^{(m-n)/2}$$

and the integral becomes

$$Iv(x,t) = \sum_m v_m(t)2^{-m}\sqrt{m!}\sum_{n=sgn(m)}^{m}{}^{*}\frac{2^n H_n(x)}{\sqrt{n!}}\sum_{q=0}^{(m-n)/2}\frac{1}{q!\left(\frac{m-n}{2}-q\right)!}$$

$$= \sum_n \frac{2^n H_n(x)}{\sqrt{n!}}\sum_{m=n}^{\infty}{}^{*} v_m(t)2^{-m}\sqrt{m!}\sum_{q=0}^{(m-n)/2}\frac{1}{q!\left(\frac{m-n}{2}-q\right)!}$$

Note also that for each fixed $m$ and $n$, the summation in $q$ can be simplified as

$$\sum_{q=0}^{(m-n)/2}\frac{1}{q!\left(\frac{m-n}{2}-q\right)!} = \frac{1}{\left(\frac{m-n}{2}\right)!}\sum_{q=0}^{(m-n)/2}\frac{\left(\frac{m-n}{2}\right)!}{q!\left(\frac{m-n}{2}-q\right)!} = \frac{2^{(m-n)/2}}{\left(\frac{m-n}{2}\right)!}$$

Thus the partial integro-differential equation (3) can be written as

$$\left(\sum_n v_n(t)H_n(x)\right)_t = a\left(\sum_n v_n(t)H_n(x)\right)_x + b\left(\sum_n v_n(t)H_n(x)\right)_{xx}$$
$$+c\left(\sum_n v_n(t)H_n(x)\right) + \lambda\sum_n \frac{H_n(x)}{\sqrt{n!}}\sum_{m=n}^{\infty}{}^{*}\frac{v_m(t)\sqrt{m!}}{2^{(m-n)/2}\left(\dfrac{m-n}{2}\right)!}, \qquad (6)$$

where

$$a=\frac{1}{\delta}\left(r-\lambda\bar{k}-\frac{1}{2}\sigma^2\right),\quad b=\frac{\sigma^2}{2\delta^2},\quad c=-(r+\lambda).$$

From (15) and (16) in the Appendix,

$$\left(\sum_n v_n(t)H_n(x)\right)_x = \sum_{n=1}^{\infty} v_n(t)\sqrt{n}H_{n-1}(x) = \sum_{n=0}^{\infty}\sqrt{n+1}v_{n+1}(t)H_n(x)$$

and

$$\left(\sum_n v_n(t)H_n(x)\right)_{xx} = \sum_{n=2}^{\infty} v_n(t)\sqrt{n(n-1)}H_{n-2}(x) = \sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}v_{n+2}(t)H_n(x).$$

In addition, since $\{H_n(x)\}$ are orthonormal (6) implies a following *deterministic* system of ordinary differential equations,

$$v_n{}'(t) = D_d v_n(t) + \lambda\, I_d v_n(t),\quad n = 0,1,2,\dots\ , \qquad (7)$$

where

$$D_d v_n(t) = a\sqrt{n+1}v_{n+1}(t) + b\sqrt{(n+1)(n+2)}v_{n+2}(t) + cv_n(t)$$

and

$$I_d v_n(t) = \frac{1}{\sqrt{n!}}\sum_{m=n}^{\infty}{}^{*}\frac{v_m(t)\sqrt{m!}}{2^{(m-n)/2}\left(\dfrac{m-n}{2}\right)!}\ .$$

From $v(x,0)=\sum_n v_n(0)H_n(x)$,

$$\int_{-\infty}^{\infty} v(x,0)H_m(x)\Gamma(x)\,\mathrm{d}x = \sum_n v_n(0)\int_{-\infty}^{\infty} H_m(x)H_n(x)\Gamma(x)\,\mathrm{d}x = \sum_n v_n(0)\boldsymbol{\delta}_{mn}$$

so that

$$v_n(0) = \int_{-\infty}^{\infty} v(x,0)H_n(x)\Gamma(x)\,\mathrm{d}x\ . \qquad (8)$$

Now (7) is an ODE in $t$ and its solution $v_n(T)$ at $t=T$ can be obtained using, say the Runge-Kutta method with $v_n(0)$ as the initial condition. Then $v(x,T)$ can be constructed by

$$v(x,T)=\sum_{n=0}^{\infty}v_n(T)H_n(x) \tag{9}$$

and the option price $V(S,0)=v(x,T)$ can be obtained.

*Remark.* The change of variable $X=x-x_0$ for a constant $x_0$ does not change the ODE (7). It has been observed that the computation with some $x_0\neq 0$ may generate faster convergence than with $x_0=0$. This seems to originate from the property of the Hermite functions, which gives better approximation near the center than away from the center.

**4. Numerical Experiments**

We now illustrate the performance of the proposed polynomial chaos method for pricing European vanilla options under jump diffusion models. We solve the systems of equation (7) with the initial condition based on Runge-Kutta method. The experiments have been performed on an Intel Core 1.7 GHz i5 computer. We use built-in Matlab function '*ode45*' for the numerical solution of ordinary differential equations, which integrates a system of the equation using fourth and fifth order Runge-Kutta formulas. We also use '*integral*' function in Matlab that approximates the integral of function using global adaptive quadrature.

A vanilla European call (put) option gives the holder the right to buy (sell, respectively) a predefined asset for a prescribed price, namely strike price $E$, at the expiration date, $T$. The following parameters are used for the European options: interest rate $r=5\%$, volatility $\sigma=20\%$, intensity of jump $\lambda=0.1$, standard deviation of jump sizes $\delta=0.8$, expiration date $T=1$, strike price $E=100$ and the present asset price $S(0)=100$.

We use a finite-dimensional approximation

$$v(x,t)\approx\sum_{n=0}^{N_{\max}}v_n(t)H_n(x) \tag{10}$$

and the computational results for the European call and put options for several $N_{\max}$ values are shown in Figure 2 and Table 1. Figure 2 shows that finite-dimensional approximations converge to the analytical price from the Merton model as the maximum order $N_{\max}$ increases. Table 1 presents the convergence for European call and put options under jump diffusion models and corresponding computational costs in seconds. It is shown that the errors for $N_{\max}=22$ are both less than a dime, which are computed within 0.2 seconds.

**Figure 2: Approximate solutions (10) from the proposed scheme (circle) for a European call option with jump diffusion process whose exact value (solid) is 13.2185.**

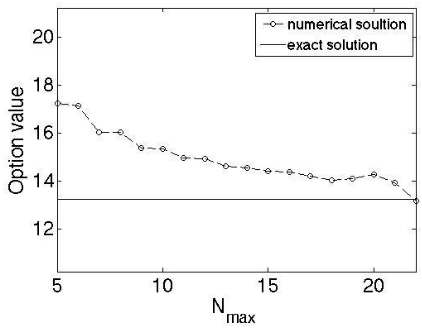



**Table 1: Approximate solutions (10) from the proposed scheme for European call and put options with jump diffusion process and corresponding computational CPU times in seconds.**

| $N_{\max}$ | Call | | Put | |
|---|---|---|---|---|
| | Value | CPU (sec) | Value | CPU (sec) |
| 6 | 17.1200 | 0.0423 | 12.1476 | 0.0441 |
| 10 | 15.3463 | 0.0712 | 10.4689 | 0.0820 |
| 14 | 14.5431 | 0.1129 | 9.6661 | 0.1199 |
| 18 | 14.0192 | 0.1401 | 9.1421 | 0.1588 |
| 22 | 13.1616 | 0.1962 | 8.2845 | 0.1971 |
| Exact | 13.2185 | | 8.3414 | |

## 5. Conclusions

A numerical scheme has been proposed to derive a series solution of the partial integro-differential equations which arise in pricing options under jump diffusion models. The stochastic solution is represented as a Taylor expansion with respect to the Hermite polynomials. The convergence has been validated numerically. Numerical experiments also show that the method is efficient in the sense that the error tolerance less than a dime is obtained within 0.2 second.

The current study applies the Hermite polynomial as a basis because the Gaussian random variable is considered. The optimality of Hermite polynomials will be studied analytically in a future work. Finding optimal basis polynomials for distinct distributions will be another future research direction.

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## Appendix

$e^{wx-w^2/2}$ can be written as

$$e^{wx-w^2/2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^*(x) w^n, \tag{11}$$

i.e., $e^{wx-w^2/2}$ is a generating function of $H_n^*(x)$. Note that (11) gives

$$\left(e^{wx-w^2/2}\right)\left(e^{wy-w^2/2}\right) = e^{(\sqrt{2}w)(x+y)/\sqrt{2}-(\sqrt{2}w)^2/2} = \sum_{n=0}^{\infty} \frac{2^{n/2}}{n!} H_n^*\left(\frac{x+y}{\sqrt{2}}\right) w^n.$$

When $e^{wx-w^2/2}$ and $e^{wy-w^2/2}$ are expressed using (11), above expression implies

$$\sum_{n=0}^{\infty} \frac{2^{n/2}}{n!} H_n^*\left(\frac{x+y}{\sqrt{2}}\right) w^n = \left(\sum_{r=0}^{\infty} \frac{1}{r!} H_r^*(x) w^r\right)\left(\sum_{s=0}^{\infty} \frac{1}{s!} H_s^*(y) w^s\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \frac{1}{r!(n-r)!} H_r^*(x) H_{n-r}^*(y)\right) w^n$$

so that

$$\frac{2^{n/2}}{n!} H_n^*\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{r=0}^{n} \frac{1}{r!(n-r)!} H_r^*(x) H_{n-r}^*(y).$$

Thus, we can derive

$$2^{n/2} H_n^*\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{r=0}^{n} \binom{n}{r} H_r^*(x) H_{n-r}^*(y) \tag{12}$$

In particular, $x = y$ gives

$$2^{n/2} H_n^*\left(\sqrt{2}x\right) = \sum_{r=0}^{n} \binom{n}{r} H_r^*(x) H_{n-r}^*(x)$$

or

$$2^{n/2} H_n\left(\sqrt{2}x\right) = \sum_{r=0}^{n} \binom{n}{r}^{1/2} H_r(x) H_{n-r}(x). \tag{13}$$

When $x$ and $y$ are replaced by $\sqrt{2}x$ and $\sqrt{2}y$, respectively, we also obtain

$$2^{n/2} H_n^* \left( x + y \right) = \sum_{r=0}^{n} \binom{n}{r} H_r^* (\sqrt{2}x) H_{n-r}^* (\sqrt{2}y)$$

or

$$2^{n/2} H_n \left( x + y \right) = \sum_{r=0}^{n} \binom{n}{r}^{1/2} H_r (\sqrt{2}x) H_{n-r} (\sqrt{2}y) \tag{14}$$

We can easily show that $H_n(x)$ satisfies

$$H_n{}'(x) = \sqrt{n} H_{n-1}(x) \tag{15}$$

so that

$$H_n{}''(x) = \sqrt{n} \left( H_{n-1}(x) \right)' = \sqrt{n(n-1)} H_{n-2}(x). \tag{16}$$

It is also known that $\{H_n(x)\}$ are orthonormal in the sense that .

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2/2} / \sqrt{2\pi} \mathrm{d}x = \boldsymbol{\delta}_{mn} \tag{17}$$

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