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INTUITIONISTIC FUZZY LINGUISTIC NUMBERS GEOMETRIC AGGREGATION OPERATORS AND THEIR APPLICATION TO GROUP DECISION MAKING

Abstract. *Intuitionistic fuzzy linguistic numbers (IFLNs) are an extension of the linguistic variables and the intuitionistic fuzzy sets, which are proposed by Wang and Li (Wang J.Q., Li J.J. (2009) The multi-criteria group decision making method based on multi-granularity intuitionistic two semantics. Science & Technology Information (33), 8-9.). In this paper, we introduced some operational laws of IFLNs, and proposed the comparison method for IFLNs, firstly. Then, we developed an intuitionistic fuzzy linguistic numbers weighted geometric average (ILNWGA) operator, an intuitionistic fuzzy linguistic numbers ordered weighted geometric (ILNOWG) operator, and an intuitionistic fuzzy linguistic numbers hybrid geometric (ILNHG) operator which generalizes both the ILNWGA operator and the ILNOWG operator, and explored some desirable properties of these operators, such as commutativity, monotonicity, idempotency, and etc. Furthermore, we proposed a new method for the multiple attribute group decision making with intuitionistic fuzzy linguistic information based on these operators .Finally, we gave an illustrative example to verify the proposed method.*

Keywords*: multiple attribute group decision making; the intuitionistic fuzzy linguistic number; an intuitionistic fuzzy linguistic numbers weighted geometric average (ILNWGA) operator; an intuitionistic fuzzy linguistic numbers ordered weighted geometric (ILNOWG) operator; an intuitionistic fuzzy linguistic numbers hybrid geometric (ILNHG) operator.*

JEL classification: D81, M12, M51

1. Introduction

Since intuitionistic fuzzy set, which is used to character the fuzziness by membership degree and no-membership degree, is proposed by Atanassov (see Atanassov, 1986, 1989), it has received more and more attention, and many research results were achieved. Atanassov and Gargov (1989), Atanassov (1994) introduced the interval-valued intuitionistic fuzzy set (IVIFS) in which the membership function and non-membership function are expressed by intervals rather than crisp numbers. Obviously, IVIFS is a generalization of the IFS. Xu (2007a) proposed the multiple attribute decision making (MADM) method based on the ideal solution for the MADM problems in which attribute values take the form of the interval-valued intuitionistic fuzzy information and attribute weights are incompletely known or completely unknown. Wang (2006) proposed a nonlinear programming model to solve the attribute weight for the interval-valued intuitionistic fuzzy MADM with incompletely known weight information. Shu et al. (2006) gave the definition and operational rules of intuitionistic triangular fuzzy number, and proposed an algorithm of the intuitionistic fuzzy fault-tree analysis. Zhang and Liu (2010) proposed the concept of triangular intuitionistic fuzzy numbers by using the triangular fuzzy number to express the membership degree and the non-membership degree, and developed the weighted arithmetic averaging operator and the weighted geometric average operator. Then these operators are used to multiple attribute group decision making (MAGDM). Wang (2008) proposed the concept of intuitionistic trapezoidal fuzzy number and interval intuitionistic trapezoidal fuzzy number. Wang and Zhang (2008) proposed the expected values of intuitionistic trapezoidal fuzzy number and developed the programming method to solve the attribute weight for the MADM problems in which the attribute values are intuitionistic trapezoidal fuzzy numbers and attribute weight is incomplete certain information. Wang and Zhang (2009a) proposed the Hamming distance between two intuitionistic trapezoidal fuzzy numbers and developed the intuitionistic trapezoidal fuzzy weighted arithmetic averaging (ITFWAA) operator, then applied it to the MADM problems in which the attribute values are intuitionistic trapezoidal fuzzy number and attribute weight is with incomplete certain information. Wang and Zhang (2009b) proposed intuitionistic trapezoidal fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator, and applies then to the MADM problems. Wan (2010) proposed the expectation and expectant score by the gravity center of intuitionistic trapezoidal fuzzy number, and defined ordered weighted aggregation operator and hybrid aggregation operator for intuitionistic trapezoidal fuzzy numbers.

In real decision making, for some qualitative information, they are easily expressed by the linguistic variables. The research on multi-attribute decision making based on the linguistic variables has made great achievements (Alonso et al. 2009; Cabrerizo et al. 2010a, 2010b; Herrera et al.2009; Martínez et al.2009). However, for a linguistic assessment value, it usually means that membership degree is 1, and non-membership degree and hesitation degree are zero. In fact, we can give a linguistic value “good” for an attribute, but we cannot be entirely sure this assessment result. Maybe we have 80% certain degree and 10% negative degrees. In view of this situation, Wang and Li (2009) combined the intuitionistic fuzzy sets and linguistic variables, and proposed intuitionistic fuzzy linguistic numbers, and the Hamming distance between them, and rank the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative. Liu and Jin (2012) proposed the intuitionistic uncertain linguistic variables, and developed some aggregation operators and applied them to the MAGDM problems. Liu (2013) proposed some generalized dependent aggregation operators with intuitionistic linguistic numbers and applied them to group decision making. Liu and Wang (2014) proposed intuitionistic linguistic power generalized aggregation operators and applied them to group decision making.

In summary, the study on the multiple attribute group decision making problems with intuitionistic fuzzy linguistic information has just started. In this paper, we investigate the MAGDM problems in which both the attribute weights and the expert weights take the form of real numbers, and attribute values take the form of intuitionistic fuzzy linguistic numbers. In order to do this, the remainder of this paper is organized as follows. Some basic concepts on intuitionistic fuzzy linguistic numbers and their operational laws, comparison method are introduced in the next section. In section 3, some intuitionistic fuzzy linguistic numbers geometric aggregation operators are developed, and the desirable properties, such as boundary, idempotency, commutativity and monotonicity, etc. are studied. In section 4, an approach to group decision making based on the intuitionistic fuzzy linguistic numbers was proposed. An example is presented in section 5 to illustrate the feasibility and efficiency of the proposed method. In the last section, we conclude the paper.

2. Preliminaries

2.1 The intuitionistic fuzzy set

Definition 1. Let $X = x_1, x_2, \cdots, x_n$ be a universe of discourse, then a fuzzy set is

$$A = \{< x, u_A(x) > x \in X\} \tag{1}$$

which is characterized by membership function $u_A : X \to [0,1]$, where $u_A(x)$ indicates the membership degree of the element x to the set A (Zadeh, 1965).

Atanassov (1986, 1989) extended the fuzzy set to the IFS and defined it as follows.

Definition 2. An IFS A in X is given by

$$A = \{< x, u_A(x), v_A(x) > x \in X\} \tag{2}$$

where $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$, with the condition $0 \le u_A(x) + v_A(x) \le 1$, $\forall x \in X$ .The numbers $u_A(x)$ and $v_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

For each IFS A in X, $\pi(x)=1-u_A(x)-v_A(x)$ is called the degree of indeterminacy of x to the set A (Atanassov, 1986, 1989). It is obvious that $0\le\pi(x)\le 1$ and $\forall x\in X$.

Let $A=\{<x,u_A(x),v_A(x)>x\in X\}$ and $B=\{<x,u_B(x),v_B(x)>x\in X\}$ be two IF sets in the set X and $n\ge 0$, then the operations of IFS are defined as follows (Atanassov, 1986; De, et al. 2000)

$$A+B=\{<x,u_A(x)+u_B(x)-u_A(x)u_B(x),v_A(x)v_B(x)>x\in X\} \tag{3}$$

$$AB=\{<x,u_A(x)u_B(x),v_A(x)+v_B(x)-v_A(x)v_B(x)>x\in X\} \tag{4}$$

$$nA=\{<x,1-(1-u_A(x))^n,(v_A(x))^n>x\in X\} \tag{5}$$

$$A^n=\{<x,(u_A(x))^n,1-(1-v_A(x))^n>x\in X\} \tag{6}$$

2.2 The linguistic set and its extension

Suppose that $S=(s_1,s_2,\cdots,s_l)$ is a finite and totally ordered discrete term set, where l is the odd value. In real situation, l is equal to 3,5,7,9 etc. For example, when l =7, a set S could be given as follows:

$S=(s_1,s_2,s_3,s_4,s_5,s_6,s_7)$ = {very poor, poor, slightly poor, fair, slightly good, good, very good}.

Usually, for any linguistic set S, it requires that s_i and s_j must satisfy the following additional characteristics (Herrera, et al. 1996; Herrera and Herrera-Viedma, 2000):

(1) The set is ordered: $s_i\prec s_j$, if and only if $i<j$;

(2) There is the negation operator: $neg(s_i)=s_{l+1-i}$;

(3) Maximum operator: $\max(s_i,s_j)=s_i$, if $i\ge j$;

(4) Minimum operator: $\min(s_i,s_j)=s_i$, if $i\le j$;

For any linguistic set $S=(s_1,s_2,\cdots,s_l)$, the relationship between the element s_i and its subscript i is strictly monotone increasing (Herrera, et al. 1996; Xu, 2006), so the function can be defined as follows: $f:s_i=f(i)$. Obviously, the function $f(i)$ is the strictly monotone increasing function about subscript i. To preserve all the given information, the discrete linguistic label $S=(s_1,s_2,\cdots,s_l)$ is extended to a continuous linguistic label $\bar{S}=\{s_\alpha\mid\alpha\in R^+\}$, where R^+ is the set of all positive real number, and it satisfied the above characteristics. The operations are defined as follows (Xu, 2006):

(1) $\beta s_i=s_{\beta\times i}\quad 0\le\beta\le 1$ (7)

(2) $s_i \oplus s_j = s_{i+j}$ (8)

(3) $s_i / s_j = s_{i/j}$ (9)

(4) $s_i{}^n = s_{i^n} \quad n \ge 0$ (10)

(5) $\lambda(s_i \oplus s_j) = \lambda s_i \oplus \lambda s_j \quad 0 \le \lambda \le 1$ (11)

(6) $(\lambda_1 + \lambda_2)s_i = \lambda_1 s_i \oplus \lambda_2 s_i \quad 0 \le \lambda_1 \le 1, 0 \le \lambda_2 \le 1, \lambda_1 + \lambda_2 \le 1$ (12)

2.3 The intuitionistic fuzzy linguistic set (ILS)

Definition 3. An ILS A in X is given by Wang and Li (see Wang and Li, 2009)

$$A = \ < x[h_{\theta(x)}, (u_A(x), v_A(x))] >| x \in X \quad (13)$$

where $h_{\theta(x)} \in \overline{S}$, $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$, with the condition $0 \le u_A(x) + v_A(x) \le 1$, $\forall x \in X$ .The numbers $u_A(x)$ and $v_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to linguistic index $h_{\theta(x)}$.

For each ILS A in X , if $\pi(x) = 1 - u_A(x) - v_A(x)$, $\forall x \in X$, then $\pi(x)$ is called the degree of indeterminacy of x to linguistic index $h_{\theta(x)}$. It is obvious that $0 \le \pi(x) \le 1$ and $\forall x \in X$.

For example, suppose we evaluate a car performance, if 6 of 10 experts give the evaluation information s_6, and 2 experts think it is less than s_6, then we can use an intuitionistic fuzzy linguistic number to express evaluation information of the car performance as $< s_6, (0.6, 0.2) >$.

Definition 4 (Wang and Li, 2009)**.** Let $A = \ < x[h_{\theta(x)}, (u_A(x), v_A(x))] >| x \in X$ be ILS, the ternary group $< h_{\theta(x)}, (u_A(x), v_A(x)) >$ is called an intuitionistic fuzzy linguistic number, and A can also be viewed as a collection of the intuitionistic fuzzy linguistic number. So, it can also be expressed as $A = \ < h_{\theta(x)}, (u_A(x), v_A(x)) >| x \in X$. In addition, $\pi_A(x) = 1 - u_A(x) - v_A(x)$ represents the degree of indeterminacy, and it can also be called the intuitionistic fuzzy linguistic fuzzy degree.

Let $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ and $\tilde{a}_2 =< s_{\theta(a_2)}, (u(a_2), v(a_2)) >$ be two IL sets and $\lambda \ge 0$, then the operations of ILS are defined as follows (Wang and Li, 2009)

$$\tilde{a}_1 + \tilde{a}_2 =< s_{\theta(a_1)+\theta(a_2)}, (1-(1-u(a_1))(1-u(a_2)), v(a_1)v(a_2)) > \quad (14)$$

$$\tilde{a}_1 \otimes \tilde{a}_2 =< s_{\theta(a_1)\times\theta(a_2)}, (u(a_1)u(a_2), v(a_1)+v(a_2)-v(a_1)v(a_2)) > \quad (15)$$

$$\lambda\tilde{a}_1 =< s_{\lambda\times\theta(a_1)}, (1-(1-u(a_1))^{\lambda}, (v(a_1))^{\lambda}) > \quad (16)$$

$$\tilde{a}_1^{\lambda} =< s_{(\theta(a_1))^{\lambda}}, ((u(a_1))^{\lambda}, 1-(1-v(a_1))^{\lambda}) > \quad (17)$$

For example, let $\tilde{a}_1 =< s_2, (0.8, 0.1) >$, $\tilde{a}_2 =< s_4, (0.6, 0.2) >$ and $\lambda = 0.5$, the related calculation operations are shown as follows

(1) $\tilde{a}_1 + \tilde{a}_2 =< s_{2+4}, (1-(1-0.8)(1-0.6), 0.1\times 0.2) >=< s_6, (0.92, 0.02) >$

(2) $\tilde{a}_1 \otimes \tilde{a}_2 =< s_{2\times 4}, (0.8\times 0.6, 0.1+0.2-0.1\times 0.2) >=< s_8, (0.48, 0.28) >$

(3) $\lambda\tilde{a}_1 =< s_{0.5\times 2}, (1-(1-0.8)^{0.5}, 0.1^{0.5}) >=< s_1, 0.553, 0.316) >$

(4) $\tilde{a}_1^{\lambda} =< s_{2^{0.5}}, 0.8^{0.5}, 1-(1-0.1)^{0.5}) >=< s_{1.414}, 0.894, 0.051) >$

Theorem 1: For any two intuitionistic fuzzy linguistic numbers $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ and $\tilde{a}_2 =< s_{\theta(a_2)}, (u(a_2), v(a_2)) >$, it can be proved the calculation Rules shown as follows

(1) $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$ (18)

(2) $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$ (19)

(3) $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda\tilde{a}_1 + \lambda\tilde{a}_2, 0 \le \lambda \le 1$ (20)

(4) $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1,\ 0 \le \lambda_1, \lambda_2\ , \lambda_1 + \lambda_2\ \le 1$ (21)

(5) $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_1^{\lambda_2} = (\tilde{a}_1)^{\lambda_1+\lambda_2},\ 0 \le \lambda_1, \lambda_2\ , \lambda_1 + \lambda_2\ \le 1$ (22)

(6) $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_2^{\lambda_1} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\lambda_1},\ \lambda_1 \ge 0$ (23)

2.4Comparison of two intuitionistic fuzzy linguistic numbers

Definition 5. Let $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ be an intuitionistic fuzzy linguistic number, an expected value $E(\tilde{a}_1)$ of $\tilde{a}_1$ can be represented as follows

$$E(\tilde{a}_1) = \tfrac{1}{2}\times(u(a_1)+1-v(a_1))\times s_{\theta(a_1)} = s_{(\theta(a_1)\times(u(a_1)+1-v(a_1)))/2} \quad (24)$$

where $E(\tilde{a}_1) \in [0, s_{\theta(a_1)}]$.

For example, suppose $\tilde{a}_1 =< s_2, (0.8, 0.1) >$, then the expected value $E(\tilde{a}_1)$ of $\tilde{a}_1$ can be calculated as follows:

$$E(\tilde{a}_1) = \tfrac{1}{2}\times(0.8+1-0.1)\times s_2 = s_{(2\times(0.8+1-0.1))/2} = s_{1.7}$$

Definition 6. Let $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ be an intuitionistic fuzzy linguistic number, a score function $S(\tilde{a}_1)$ of an intuitionistic fuzzy linguistic value $\tilde{a}_1$ can be represented as follows

$$S(\tilde{a}_1) = E(\tilde{a}_1)\times(u(a_1)-v(a_1)) \quad (25)$$

where $E(\tilde{a}_1)$ is the expected value of the intuitionistic fuzzy linguistic number $\tilde{a}_1$.

For example, suppose $\tilde{a}_1 =< s_2, (0.8, 0.1) >$, then the score function $S(\tilde{a}_1)$ of $\tilde{a}_1$ can be calculated as follows:

$$S(\tilde{a}_1) = E(\tilde{a}_1) \times (u(a_1) - v(a_1)) = s_{1.7} \times (0.8 - 0.1) = s_{1.19}$$

Definition 7. Let $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ be an intuitionistic fuzzy linguistic number, an accuracy function $H(\tilde{a}_1)$ of an intuitionistic fuzzy linguistic value $\tilde{a}_1$ can be represented as follows

$$H(\tilde{a}_1) = E(\tilde{a}_1) \times (u(a_1) + v(a_1)) \tag{26}$$

where $E(\tilde{a}_1)$ is the expected value of intuitionistic fuzzy linguistic number $\tilde{a}_1$.

For example, suppose $\tilde{a}_1 =< s_2, (0.8, 0.1) >$, then the accuracy function $H(\tilde{a}_1)$ of $\tilde{a}_1$ can be calculated as follows

$$H(\tilde{a}_1) = E(\tilde{a}_1) \times (u(a_1) + v(a_1)) = s_{1.7} \times (0.8 + 0.1) = s_{1.53}$$

Definition 8. If $\tilde{a}_1 =< s_{\theta(a_1)}, (u(a_1), v(a_1)) >$ and $\tilde{a}_2 =< s_{\theta(a_2)}, (u(a_2), v(a_2)) >$ are any two intuitionistic fuzzy linguistic number, then,

(1) If $E(\tilde{a}_1) > E(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$.

(2) If $E(\tilde{a}_1) = E(\tilde{a}_2)$ and $S(\tilde{a}_1) > S(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$.

(3) If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $S(\tilde{a}_1) = S(\tilde{a}_2)$ and $H(\tilde{a}_1) > H(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$.

(4) If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $S(\tilde{a}_1) = S(\tilde{a}_2)$ and $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

For example, suppose $\tilde{a}_1 =< s_2, (0.6, 0.2) >$ and $\tilde{a}_2 =< s_4, (0.4, 0.6) >$, because $E(\tilde{a}_1) = s_{1.4}$, $E(\tilde{a}_2) = s_{1.6}$, so $\tilde{a}_2 \succ \tilde{a}_1$. If $\tilde{a}_1 =< s_2, (0.8, 0.2) >$ and $\tilde{a}_2 =< s_4, (0.4, 0.6) >$, $S(\tilde{a}_1) = s_{0.96}$, $S(\tilde{a}_2) = s_{-0.32}$, so $\tilde{a}_1 \succ \tilde{a}_2$.

3. The intuitionistic fuzzy linguistic numbers Hybrid weighted geometric operator

Definition 9. Let $\tilde{a}_j =< s_{\theta(a_j)}, (u(a_j), v(a_j)) >$ $(j = 1, 2 \cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, and $ILNWGA : \Omega^n \to \Omega$, if

$$ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \prod_{j=1}^{n} (\tilde{a}_j)^{\omega_j} \tag{27}$$

where Ω is the set of all intuitionistic fuzzy linguistic numbers, and

$\omega = \omega_1, \omega_2, \cdots, \omega_n{}^T$ is the weight vector of $\tilde{a}_j (j = 1, 2, \cdots, n)$, $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$, then $ILNWGA$ is called the intuitionistic fuzzy linguistic numbers weighted geometric average operator. Specially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, $ILNWGA$ is called the intuitionistic fuzzy linguistic numbers geometric average operator ($ILNGA$).

Theorem 2: Let $\tilde{a}_j = < s_{\theta(a_j)}, (u(a_j), v(a_j)) >$ $(j = 1, 2\cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, then, the result aggregated from Definition 9 is still intuitionistic fuzzy linguistic number, and even

$$ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = < s_{\prod_{j=1}^{n}\left(\theta(a_j)^{\omega_j}\right)}, \left(\prod_{j=1}^{n}(u(a_j))^{\omega_j}, (1 - \prod_{j=1}^{n}(1 - v(a_j))^{\omega_j})\right) > \tag{28}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of $\tilde{a}_j (j = 1, 2, \cdots, n)$, $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$.

Theorem 2 can be proved by mathematical induction, the steps is shown as follows

Proof. (1) When $n = 1$, obviously, it is right.

(2) When $n = 2$,

$\because$ $\tilde{a}_1^{\omega_1} = < s_{\theta(a_1)^{\omega_1}}, (u(a_1))^{\omega_1}, (1 - (1 - v(a_1))^{\omega_1}) >$

$\tilde{a}_2^{\omega_2} = < s_{\theta(a_2)^{\omega_2}}, (u(a_2))^{\omega_2}, (1 - (1 - v(a_2))^{\omega_2}) >$

$\therefore$

$$ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2) = \tilde{a}_1^{\omega_1} \times \tilde{a}_2^{\omega_2} = < s_{\theta(a_1)^{\omega_1}}, (u(a_1))^{\omega_1}, (1 - (1 - v(a_1))^{\omega_1}) >$$

$$\times < s_{\theta(a_2)^{\omega_2}}, (u(a_2))^{\omega_2}, (1 - (1 - v(a_2))^{\omega_2}) >$$

$$= < s_{\theta(a_1)^{\omega_1} \times \theta(a_2)^{\omega_2}}, (u(a_1))^{\omega_1} \times (u(a_2))^{\omega_2}, (1 - (1 - v(a_1))^{\omega_1}(1 - v(a_2))^{\omega_2}) >$$

$$= < s_{\prod_{i=1}^{2}\theta(a_j)^{\omega_j}}, \left(\prod_{j=1}^{2}(u(a_j))^{\omega_j}, (1 - \prod_{j=1}^{2}(1 - v(a_j))^{\omega_j})\right) >$$

So, when $n = 2$, formula (28) is right, too.

(3) Suppose when $n = k$, formula (28) is right, i.e.

$$ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = < s_{\prod_{j=1}^{k}\left(\theta(a_j)^{\omega_j}\right)}, \left(\prod_{j=1}^{k}(u(a_j))^{\omega_j}, (1 - \prod_{j=1}^{k}(1 - v(a_j))^{\omega_j})\right) >$$

Then, when $n = k + 1$,

$$ILNWGA_{\omega}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_k,\tilde{a}_{k+1}) = ILNWGA_{\omega}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_k)\times\ \tilde{a}_{k+1}^{\ \omega_{k+1}}$$

$$=\left(< s_{\prod_{j=1}^{k}\left(\theta(a_j)^{\omega_j}\right)}, \left(\prod_{j=1}^{k}(u(a_j))^{\omega_j}, (1-\prod_{j=1}^{k}(1-v(a_j))^{\omega_j}) \right) > \right)$$

$$\times\ < s_{\theta(a_{k+1})^{\omega_{k+1}}},\ u(a_{k+1}))^{\omega_{k+1}}, (1-(1-v(a_{k+1}))^{\omega_{k+1}})\ >$$

$$=< s_{\prod_{j=1}^{k}\left(\theta(a_j)^{\omega_j}\right)\times\left(\theta(a_{k+1})^{\omega_{k+1}}\right)}, \left(\prod_{j=1}^{k}(u(a_j))^{\omega_j}\times(u(a_{k+1}))^{\omega_{k+1}}, (1-\prod_{j=1}^{k}(1-v(a_j))^{\omega_j}\times(1-v(a_{k+1}))^{\omega_{k+1}}) \right) >$$

$$=< s_{\prod_{j=1}^{k+1}\left(\theta(a_j)^{\omega_j}\right)}, \left(\prod_{j=1}^{k+1}(u(a_j))^{\omega_j}, (1-\prod_{j=1}^{k+1}(1-v(a_j))^{\omega_j}) \right) >$$

So, when $n = k+1$, formula (28) is right, too.

According to steps (1), (2) and (3), we can conclude that, equation (28) is right for all n.

For example, suppose three attributes of a decision making problem are expressed as $\tilde{a}_1 = \langle s_4, (0.4, 0.6)\rangle$, $\tilde{a}_2 = \langle s_5, (0.4, 0.5)\rangle$ and $\tilde{a}_3 = \langle s_6, (0.5, 0.5)\rangle$, the attribute weight ω is $(0.2, 0.5, 0.3)$, then

$$ILNWGA_{\omega}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3) =< s_{\prod_{j=1}^{3}\left(\theta(a_j)^{\omega_j}\right)}, \left(\prod_{j=1}^{3}(u(a_j))^{\omega_j}, (1-\prod_{j=1}^{3}(1-v(a_j))^{\omega_j}) \right) >$$

$$=< s_{4^{0.2}\times 5^{0.5}\times 6^{0.3}},\ 0.4^{0.2}\times 0.4^{0.5}\times 0.5^{0.3}, (1-(1-0.6)^{0.2}\times(1-0.5)^{0.5}\times(1-0.5)^{0.3})\ >$$

$$=< s_{5.05},\ 0.43, 0.52\ >$$

Definition 10. Let $\tilde{a}_j =< s_{\theta(a_j)}, (u(a_j), v(a_j)) >$ $(j = 1, 2\cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, and $ILNOWG : \Omega^n \to \Omega$, if

$$ILNOWG_W(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \prod_{j=1}^{n}(\tilde{a}_{\sigma_j})^{w_j} \tag{29}$$

where Ω is the set of all intuitionistic fuzzy linguistic numbers, and $W = (w_1, w_2, \cdots, w_n)^T$ is an associated weight vector with $ILNOWG$, $w_j \in [0,1]$,

$\sum_{j=1}^{n} w_j = 1$. $\sigma_1, \sigma_2, \cdots, \sigma_n$ is a permutation of $1, 2, \cdots, n$, such that $\tilde{a}_{\sigma_{j-1}} \geq \tilde{a}_{\sigma_j}$ for all $j = 1, 2, \cdots, n$, then $ILNOWG$ is called the intuitionistic fuzzy linguistic numbers ordered weighted geometric operator.

Theorem 3: Let $\tilde{a}_j =< s_{\theta(a_j)}, (u(a_j), v(a_j)) > \ (j = 1, 2 \cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, then the result aggregated from Definition 10 is still intuitionistic fuzzy linguistic number, and even

$$ILNOWG_W(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) =< s_{\prod_{j=1}^{n}\left(\theta(a_{\sigma_j})^{w_j}\right)}, \left(\prod_{j=1}^{n}(u(a_{\sigma_j}))^{w_j}, (1 - \prod_{j=1}^{n}(1 - v(a_{\sigma_j}))^{w_j})\right) > \tag{30}$$

$W = (w_1, w_2, \cdots, w_n)^T$ is an associated weight vector with $ILNOWG$, $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$. $\sigma_1, \sigma_2, \cdots, \sigma_n$ is a permutation of $1, 2, \cdots, n$, such that $\tilde{a}_{\sigma_{j-1}} \geq \tilde{a}_{\sigma_j}$ for all $j = 1, 2, \cdots, n$.

Similar to the proof of Theorem 2, Theorem 3 can be proved by mathematical induction, and proof steps are omitted here.

It is easy to prove that the operators $ILNWGA$ and $ILNOWG$ have the following properties:

(1) Theorem 4 (Commutativity)

Let $(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n')$ be any a permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, then

$$ILNWGA_\omega(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n') = ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$ILNOWG_W(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n') = ILNOWG_W(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

(2) Theorem 5 (Idempotency)

Let $\tilde{a}_j = \tilde{a}, j = 1, 2, \cdots, n$, then

$$ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}, \ ILNOWG_W(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}$$

(3) Theorem 6 (Monotonicity)

Let $(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n')$ and $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ be two intuitionistic fuzzy linguistic numbers, if $\tilde{a}_j' \leq \tilde{a}_j$ for all $j = 1, 2, \cdots, n$, then

$$ILNWGA_\omega(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n') \leq ILNWGA_\omega(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$ILNOWG_W(\tilde{a}_1', \tilde{a}_2', \cdots, \tilde{a}_n') \leq ILNOWG_W(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

The proof steps are similar to literature (Xu, 2007b), and they are omitted here.

From Definitions 9 and 10, we know that the $ILNWGA$ operator weights the intuitionistic fuzzy linguistic arguments while the $ILNOWG$ operator weights the ordered positions of the intuitionistic fuzzy linguistic arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the $ILNWGA$ operator and the $ILNOWG$ operator. However, both the operators consider only one of them. To overcome this drawback, in the following we shall propose an intuitionistic fuzzy linguistic numbers hybrid geometric operator ($ILNHG$).

Definition 11. Let $\tilde{a}_j = < s_{\theta(a_j)}, (u(a_j), v(a_j)) >$ $(j = 1, 2 \cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, and $ILNHG : \Omega^n \to \Omega$, if

$$ILNHG_{\omega,W}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \prod_{j=1}^{n} (\tilde{b}_{\sigma_j})^{w_j} \quad (31)$$

where Ω is the set of all intuitionistic fuzzy linguistic numbers, and $W = (w_1, w_2, \cdots, w_n)^T$ is an associated weight vector with $ILNHG$, $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$; $\tilde{b}_{\sigma_j}$ is j th largest of the intuitionistic fuzzy linguistic weighted argument $\tilde{b}_k (\tilde{b}_k = \tilde{a}_k^{\;n\omega_k}, k = 1, 2, \cdots, n)$, $\omega = \omega_1, \omega_2, \cdots, \omega_n{}^T$ is the weight vector of $\tilde{a}_i (i = 1, 2, \cdots, n)$, $\omega_i \in [0,1], \sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient, then $ILNOWG$ is called the intuitionistic fuzzy linguistic numbers hybrid geometric operator.

Theorem 7. Let $\tilde{a}_j = < s_{\theta(a_j)}, (u(a_j), v(a_j)) >$ $(j = 1, 2 \cdots, n)$ be a collection of the intuitionistic fuzzy linguistic numbers, then the result aggregated from Definition 11 is still intuitionistic fuzzy linguistic number, and even

$$ILNHG_{\omega,W}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = < s_{\prod_{j=1}^{n}\left(\theta(b_{\sigma_j})^{w_j}\right)}, \left(\prod_{j=1}^{n} (u(b_{\sigma_j}))^{w_j}, (1 - \prod_{j=1}^{n} (1 - v(b_{\sigma_j}))^{w_j})\right) > \quad (32)$$

where $W = (w_1, w_2, \cdots, w_n)^T$ is an associated weight vector with $ILNHG$, $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$; $\tilde{b}_{\sigma_j}$ is j th largest of the intuitionistic fuzzy linguistic weighted argument $\tilde{b}_k (\tilde{b}_k = \tilde{a}_k^{\;n\omega_k}, k = 1, 2, \cdots, n)$, $\omega = \omega_1, \omega_2, \cdots, \omega_n{}^T$ is the weight vector of $\tilde{a}_i (i = 1, 2, \cdots, n)$, $\omega_i \in [0,1], \sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient.

Similar to the proof of Theorem 2, Theorem 7 can be proved by mathematical induction, and proof steps are omitted here.

Theorem 8. The *ILNWGA* operator is a special case of the *ILNHG* operator.

Proof. Let $W = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T$, then

$$ILNHG_{\omega,W}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \prod_{j=1}^{n} (\tilde{b}_{\sigma_j})^{w_j} = \prod_{j=1}^{n} \tilde{a}_j^{\;n\omega_j \; \frac{1}{n}} = \prod_{j=1}^{n} \tilde{a}_j^{\;\omega_j}$$

which completes the proof of Theorem 8.

Theorem 9. The *ILNOWG* operator is a special case of the *ILNHG* operator.

Proof. Let $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T$, then $\tilde{b}_k = \tilde{a}_k, k = 1, 2, \cdots, n$. This completes the proof of Theorem 9.

From Theorems 8 and 9, we know that the *ILNHG* operator generalizes both the *ILNWGA* operator and the *ILNOWG* operator, and reflects the importance degrees of both the given arguments and their ordered positions.

4. An approach to group decision making based on the intuitionistic fuzzy linguistic numbers

Consider a multiple attribute group decision making with intuitionistic fuzzy linguistic information: let $A = A_1, A_2, \cdots, A_m$ be a discrete set of alternatives, and $C = C_1, C_2, \cdots, C_n$ be the set of attributes, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighting vector of the attribute $C_j (j = 1, 2, \cdots, n)$, where $\omega_j \geq 0, j = 1, 2, \cdots, n, \sum_{j=1}^{n} \omega_j = 1$. Let $D = D_1, D_2, \cdots, C_p$ be the set of decision makers, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T$ be the weight vector of decision makers, where $\lambda_k \geq 0, k = 1, 2, \cdots, p, \sum_{k=1}^{p} {}_k = 1$. Suppose that $\tilde{R}^k = \left[\tilde{r}_{ij}^k \right]_{m \times n}$ is the decision matrix, where $\tilde{r}_{ij}^k = \left\langle a_{ij}^k,\ u_{ijk}, v_{ijk} \right\rangle$ takes the form of the intuitionistic fuzzy linguistic number, given by the decision maker D_k, for alternative A_i with respect to attribute C_j, and $0 \le u_{ijk} \le 1$, $0 \le v_{ijk} \le 1$, $u_{ijk} + v_{ijk} \le 1$, $a_{ij}^k \in S$ .Then, the ranking of alternatives is required.

In the following, we apply the *ILNWGA* operator and the *ILNHG* operator to multiple attribute group decision making based on intuitionistic fuzzy linguistic information. The method involves the following steps:

Step 1. Utilize the *ILNWGA* operator

$$\tilde{r}_{ij} = ILNWGA_\lambda(\tilde{r}_{ij}^1, \tilde{r}_{ij}^2, \cdots, \tilde{r}_{ij}^p) = \prod_{k=1}^{p} (\tilde{r}_{ij}^k)^{\lambda_k} \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, n \qquad (33)$$

to aggregate all the decision matrices $\tilde{R}^k (k = 1, 2, \cdots, p)$ into a collective decision matrix $\tilde{R} = \left[\tilde{r}_{ij} \right]_{m \times n}$, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T$ is the weight vector of decision makers.

Step 2. Utilize the decision information given in matrix $\tilde{R}$, and the *ILNHG* operator

$$\tilde{r}_i = ILNHG_{\omega,W}(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) = \prod_{j=1}^{n} (\tilde{b}_{i\sigma_j})^{w_j} \quad i = 1, 2, \cdots, m \qquad (34)$$

to derive the collective overall preference values $\tilde{r}_i (i = 1, 2, \cdots, m)$ of the alternative A_i, where $W = (w_1, w_2, \cdots, w_n)^T$ is an associated weight vector with *ILNHG*, $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$; $\tilde{b}_{i\sigma_j}$ is j th largest of the intuitionistic fuzzy linguistic weighted argument $\tilde{b}_{ik} (\tilde{b}_{ik} = \tilde{r}_{ik}^{\ n\omega_k}, k = 1, 2, \cdots, n)$, $\omega = \omega_1, \omega_2, \cdots, \omega_n{}^T$ is the weight vector of $\tilde{r}_{ij} (j = 1, 2, \cdots, n)$, $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient.

Step 3. Calculate the expected values $E(\tilde{r}_i)$ $(i = 1, 2, \cdots, m)$ of the collective overall values $\tilde{r}_i (i = 1, 2, \cdots, m)$ to rank all the alternatives $A_i (i = 1, 2, \cdots, m)$ and then to select the best one(s). If there is no difference between two expected values $E(\tilde{r}_i)$ and $E(\tilde{r}_j)$, then we need to calculate the scores $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$, even the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$ of the collective overall preference values $\tilde{r}_i$ and $\tilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the scores $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$,or the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$.

Step 4. Rank all the alternatives $A_i (i = 1, 2, \cdots, m)$ and select the best one(s) in accordance with $E(\tilde{r}_i)$, $S(\tilde{r}_i)$ or $H(\tilde{r}_i)$ $(i = 1, 2, \cdots, m)$.

Step 5. End.

5. Example

Let us suppose an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money:

(1) A_1 is a car company;

(2) A_2 is a computer company;

(3) A_3 is a TV company;

(4) A_4 is a food company.

The investment company must make a decision according to the following four attributes (suppose that the weight vector of four attributes is $\omega = (0.32, 0.26, 0.18, 0.24)^T$):

(1) C_1 is the risk analysis;

(2) C_2 is the growth analysis;

(3) C_3 is the social–political impact analysis;

(4) C_4 is the environmental impact analysis.

The four possible alternatives A_1, A_2, A_3, A_4 are evaluated using the linguistic term set $S = (s_1, s_2, s_3, s_4, s_5, s_6, s_7)$ by three decision makers (their weight vector $\lambda = (0.4, 0.32, 0.28)^T$) under the above four attributes, and construct, respectively, the decision matrices $\tilde{R}^k = \left[\tilde{r}_{ij}^k\right]_{4\times4}$ $(k = 1, 2, 3)$ as listed in Tables 1–3.

Table 1: Decision matrix $\tilde{R}^1$

	C_1	C_2	C_3	C_4
A_1	$\langle s_5, (0.2, 0.7)\rangle$	$\langle s_2, (0.4, 0.6)\rangle$	$\langle s_5, (0.5, 0.5)\rangle$	$\langle s_3, (0.2, 0.6)\rangle$
A_2	$\langle s_4, (0.4, 0.6)\rangle$	$\langle s_5, (0.4, 0.5)\rangle$	$\langle s_3, (0.1, 0.8)\rangle$	$\langle s_4, (0.5, 0.5)\rangle$
A_3	$\langle s_3, (0.2, 0.7)\rangle$	$\langle s_4, (0.2, 0.7)\rangle$	$\langle s_4, (0.3, 0.7)\rangle$	$\langle s_5, (0.2, 0.7)\rangle$
A_4	$\langle s_6, (0.5, 0.4)\rangle$	$\langle s_2, (0.2, 0.8)\rangle$	$\langle s_3, (0.2, 0.6)\rangle$	$\langle s_3, (0.3, 0.6)\rangle$

Table 2: Decision matrix $\tilde{R}^2$

	C_1	C_2	C_3	C_4
A_1	$\langle s_4,(0.1,0.7)\rangle$	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.2,0.8)\rangle$	$\langle s_6,(0.4,0.5)\rangle$
A_2	$\langle s_5,(0.4,0.5)\rangle$	$\langle s_3,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_3,(0.2,0.7)\rangle$
A_3	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_2,(0.4,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$
A_4	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_2,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$

Table 3: Decision matrix $\tilde{R}^3$

	C_1	C_2	C_3	C_4
A_1	$\langle s_5,(0.2,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_2	$\langle s_4,(0.3,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_2,(0.1,0.8)\rangle$	$\langle s_3,(0.1,0.8)\rangle$
A_3	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_1,(0.1,0.8)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_4	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.1,0.7)\rangle$	$\langle s_4,(0.3,0.6)\rangle$	$\langle s_5,(0.4,0.5)\rangle$

To get the best alternative(s), the following steps are involved:

Step 1. Utilize the decision information given in matrix $\tilde{R}^k$ $(k=1,2,3)$, and the $ILNWGA$ operator

$$\tilde{r}_{ij} = ILNWGA_{\lambda}(\tilde{r}_{ij}^1,\tilde{r}_{ij}^2,\tilde{r}_{ij}^3) = ILNWGA_{\lambda}\ \left\langle a_{ij}^1,\ u_{ij1},v_{ij1}\right\rangle,\left\langle a_{ij}^2,\ u_{ij2},v_{ij2}\right\rangle,\left\langle a_{ij}^3,\ u_{ij3},v_{ij3}\right\rangle$$

$$i=1,2,\cdots,4;\ j=1,2,\cdots,4$$

to aggregate all the decision matrices $\tilde{R}^k$ $(k=1,2,3)$ into a collective decision matrix $\tilde{R}=\left[\tilde{r}_{ij}\right]_{4\times 4}$ (Table 4).

Table 4: The collective decision matrix $\tilde{R}$

	C_1	C_2	C_3	C_4
A_1	$\langle s_{4.66},(0.16,0.33)\rangle$	$\langle s_{2.55},(0.30,0.34)\rangle$	$\langle s_{3.99},(0.35,0.42)\rangle$	$\langle s_{4.06},(0.25,0.41)\rangle$
A_2	$\langle s_{4.30},(0.37,0.41)\rangle$	$\langle s_{4.25},(0.34,0.44)\rangle$	$\langle s_{2.94},(0.12,0.27)\rangle$	$\langle s_{3.37},(0.24,0.36)\rangle$
A_3	$\langle s_{3.57},(0.20,0.33)\rangle$	$\langle s_{4.26},(0.22,0.33)\rangle$	$\langle s_{2.17},(0.24,0.31)\rangle$	$\langle s_{3.99},(0.23,0.30)\rangle$
A_4	$\langle s_{4.66},(0.33,0.47)\rangle$	$\langle s_{2.80},(0.21,0.34)\rangle$	$\langle s_{2.86},(0.26,0.40)\rangle$	$\langle s_{3.80},(0.29,0.43)\rangle$

Step 2. Utilize the weight vector of attributes $\omega = (0.32, 0.26, 0.18, 0.24)^T$, and the $ILNHG$ operator (let $W = (0.2, 0.3, 0.3, 0.2)^T$):

$$\tilde{r}_i = ILNHG_{\omega,W}(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) = \prod_{j=1}^{4} (\tilde{b}_{i\sigma_j})^{w_j} \quad (i = 1, 2, 3, 4)$$

to derive the collective overall preference values $\tilde{r}_i (i = 1, 2, 3, 4)$ of alternative A_i, where $\tilde{b}_{i\sigma_j}$ is j th largest of the intuitionistic fuzzy linguistic weighted argument $\tilde{b}_{ik} (\tilde{b}_{ik} = (\tilde{r}_{ik})^{4\omega_k}, k = 1, 2, 3, 4)$.

$$\tilde{r}_1 = \langle s_{4.01}, (0.22, 0.38) \rangle, \; \tilde{r}_2 = \langle s_{4.05}, (0.27, 0.40) \rangle$$
$$\tilde{r}_3 = \langle s_{3.73}, (0.21, 0.33) \rangle, \; \tilde{r}_4 = \langle s_{3.82}, (0.27, 0.43) \rangle$$

Step 3. Calculate the expected values $E(\tilde{r}_i)$ $(i = 1, 2, 3, 4)$ of the collective overall intuitionistic fuzzy linguistic preference values $\tilde{r}_i (i = 1, 2, 3, 4)$

$$E(\tilde{r}_1) = s_{1.696}, \; E(\tilde{r}_2) = s_{1.761}, \; E(\tilde{r}_3) = s_{1.636}, \; E(\tilde{r}_4) = s_{1.601}$$

Step 4. Rank all the alternatives $A_i (i = 1, 2, 3, 4)$ in accordance with expected values $E(\tilde{r}_i)$ $(i = 1, 2, 3, 4)$ of the collective overall intuitionistic fuzzy linguistic preference values $\tilde{r}_i (i = 1, 2, 3, 4)$, we can get $A_2 \succ A_1 \succ A_3 \succ A_4$, and thus the most desirable alternative is A_2.

In addition, in order to verify the validity of the method proposed in this paper, we adopt the method proposed by Wang's (see Wang and Zhang, 2009b) to verify this example. Firstly, we transform the intuitionistic fuzzy linguistic number into an intuitionistic trapezoidal fuzzy number by converting the linguistic value of the intuitionistic fuzzy linguistic number in this example into trapezoidal fuzzy number. Then we can use the method based on the trapezoidal fuzzy number, which proposed by Wang's (see Wang and Zhang, 2009b). By calculating re-ranking the alternatives, the result is shown as follows:

$$A_2 \succ A_1 \succ A_3 \succ A_4.$$

Obviously, two methods have the same ranking results; this verifies the validity of the method in this paper.

6. Conclusions and limitations

The aggregation operators are widely used in the multi-attribute decision making, and intuitionistic fuzzy linguistic numbers can more effectively express the fuzzy information. In this paper, we applied the intuitionistic fuzzy linguistic numbers to the geometric aggregating operators, and proposed an intuitionistic fuzzy linguistic numbers weighted geometric average (ILNWGA) operator, an intuitionistic fuzzy linguistic numbers ordered weighted geometric (ILNOWG) operator, and an intuitionistic fuzzy linguistic numbers hybrid geometric (ILNHG) operator. Furthermore, we have studied some properties of these operators, and proposed a new method for the multiple attribute group decision making problems with intuitionistic fuzzy linguistic information. Finally, an illustrative example has been given to show the steps of the proposed method. In the future, we shall continue researching the decision making methods for the decision making problems in which decision information is expressed by intuitionistic fuzzy linguistic numbers, for example, TOPSIS, VIKOR, and etc. in addition, we also continue working in the extension and application of the proposed operators to other domains.

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