Professor S. S. CHADHA, PhD
University of Wisconsin, Eau Claire, USA
E-mail: schadha@uwec.edu
Professor Veena CHADHA
University of Wisconsin, Eau Claire, USA
E-mail: chadhav@uwec.edu

GENERALIZED INTEGER PROGRAMMING

Abstract . *This paper considers a mathematical programming problem whose objective function happens to be the sum of linear and linear fractional functions. The constraint set consists of linear inequalities with non-negative and integer requirements on the variables. A numerical example illustrates the steps of the proposed algorithm.*

Keywords**:** *Linear programming; Linear fractional programming; Integer programming.*

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1. Introduction

A wide class of problems where one seeks to optimize a combination of linear and linear fractional targets subject to linear constraints do arise if one wishes to optimize a linear combination of income and profitability. Problems of optimizing enterprise capital, the production development fund and the social, cultural and construction fund also fall in this class. These problems have the following structure:

Maximize + ——— (1)

subject to .

Here = (; is m by n matrix; are n-component row vectors, and are n and m components column vectors respectively, is a scalar.

Under the following set of assumptions, Teterev [10] has suggested a simplex type (ST) approach to solve such a class of problems:

(a) the set S is regular i.e. is non-empty and bounded,
(b) > 0 over ,
(c) , over .

The objective function assumes a unique maximum at ; symbol ' over a matrix denotes its transpose; if

+ (2)

for j = 1,2,…n.

Here and are the jth-elements of the vectors C, P, and Q respectively; and are the sub-vectors of C, P, and Q respectively, corresponding to the basis matrix B of A; , , and . Matrix A is portioned as [B,N].

However, for such a family of problems we have following observations:

(a) A relative optimum solution need not be an absolute optimal solution, Hirche [6]. However, in the event of the objective function being pseudoconvex the relative optima will turn out to be absolute optima as well.
(b) Moreover, if the objective function is pseudolinear then the optimal solution is attained at an extreme point of the feasible set S, Schible [8].
(c) The objective function is pseudolinear if and only if one of the followings hold true
    (i) there exists such that and there exists such that and p =
    (ii) there exists such that and there exists such that and p =

However, the present work is an attempt to find an integer solution to a problem represented by (1). Vector will lead to an integer solution of a linear programming problem, were as vector will solve an integer-linear fractional programming problem. A numerical example illustrates the steps of the algorithm.

2. The Algorithm

Let us refer to problem (1) with integer requirements, is an integer for j= 1,2,…n, as generalized integer programming (GIP) problem. If an optimum solution of problem (1) is not a feasible solution to (GIP) problem, because some of the integrality restrictions are violated, new linear constraints are added to problem (1), one at a time, to yield a sequence of new problems of type (1). The first linear constraint added to problem (1) has the property that the set of feasible solutions to the new problem of type (1) does not include the optimal solution to problem (1), but does include every feasible solution to (GIP) problem. This new problem of type (1) with this one additional constraint is then solved (by using the ST approach). If the solution is a feasible solution to (GIP), it is an optimal solution to (GIP). If not, one adds another constraint with the property that the set of feasible solutions to the new problem includes every feasible solution to (GIP) problem, but excludes the optimal solution obtained to the previous programming problem of type (1). These additional linear constraints that are added as one moves from one problem to the next are referred to as cuts. Geometrically, each of these cuts eliminates part of the set of feasible solutions to problem (1). These cuts were introduced by Dantzig [3], Gomory [4,5]and by others. This paper adepts the cuts suggested by Gomory to find an optimal solution for (GIP) problem. A brief description for generating such a cut is given below.
Assume that we have solved problem (1) and B is the basis matrix associated with the optimal solution. Let = u be a vector containing the values of the basic variables for the optimal solution to problem (1). Any feasible solution x must satisfy

(3)

Suppose that not all components of u are integers. Let us suppose that is not integral. Then consider the ith equation of (3), which reads

(4)

Now write

, ; , (5)

Here is the largest integer less than or equal to , and is the largest integer less than or equal to .

Then by assumption, , since is not integral. Furthermore, .
Substitution of (5) into (4) yields

(6)

From (6), for any integer solution to (3),

(7)

must be an integer. Now cannot be negative. Thus, since , (7) cannot be a positive integer. Therefore, every feasible solution to (GIP) problem must satisfy

(8)

Obviously, the optimal solution obtained to problem (1) does not satisfy (8), since Thus if we add (8) to the constraints set of problem (1), the new set of feasible solutions will be smaller than that for problem (1), but will still contain all feasible solution to (GIP) problem. Inequality (8) is the form of the cuts introduced by Gomory.

Followings are the steps for the proposed iterative procedure for solving a Generalized Integer Programming problem:

(i) We solve problem (1) by using the ST procedure of Tetertev [10]. If the solution so obtained is feasible to (GIP) problem, halt; the optimal solution to (GIP) problem is at hand. Otherwise, go to step (ii).

(ii) Determine a cut (8). Gomory, gave a rule for deciding which equation of (3) should be selected for use in determining the cut (8) in cases where more than a single component of u is nonintegral. Intuitively, it would seem desirable to select a cut which cuts as deeply as possible. A rule which has been adopted in practice is simply to select the equation from (3) for which is largest, in case of a tie select any one among the tied values.

(iii) Annex the cut, determined in step (ii), to problem (1). This new problem has one additional linear constraint and one more variable than the original problem (1).

(iv) Use the Phase-I (artificial variable) technique to find the basic feasible solution for the new constraint set obtained in step (iii) and go to step (v).

(v) Use the solution found in step (iv) to determine an optimum solution, by using (ST) approach, for the new problem (1) as formulated in step (iii).

(vi) If the solution so obtained is feasible to (GIP) problem, it is an optimal solution to (GIP) problem. If not, repeat steps (ii) – (v) till the desired solution is reached.

3. Numerical Example

Maximize z = 5 + ———

Subject to

(9)

,

integers. (10)

Optimal solution to (9) , using (ST) method is given by the following table

			Basic variables					b
5	1.5	1		1	0	—	—	–
1	1	1		0	1	0	1	2

– , , z = 11.81, , and .

This solution does not meet the integer requirements. We introduce our 1st cut as

— — – or

— — – , .

We make use of phase-1 technique of the simplex method.

Maximize –y

Subject to

— — –

— — –

Basic feasible solution of the new set of equations can be read from the following table

Basic variables						b
	1	0	–	0	—	—
	0	1	—	0	—	—
	0	0	–	1	——	–

We use this solution as a basic feasible solution for problem (9). The optimum solution for problem (9) is given by the following table.

			Basic variables						b
5	1.5	1		1	0	–	0	—	—
1	1	1		0	1	—	0	—	—
0	0	0		0	0	–	1	——	–

— , —, – , z = 11.34, , and .

This does not solve problem (9)-((10). We introduce our 2nd cut as

– – – or

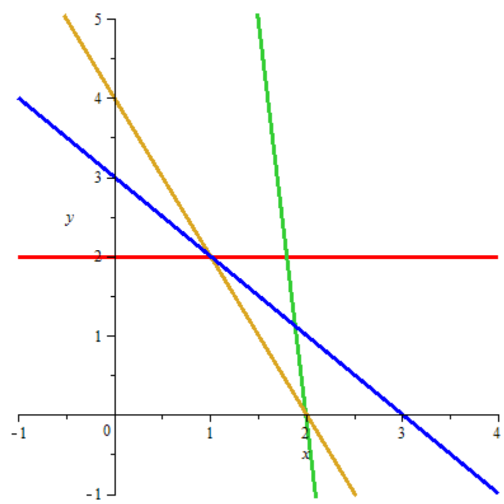
– – – ,

We again make use of the phase-1 technique to find the basic feasible solution of this new system of equations. The optimum solution to problem (9)-(10) is given in the next table.

			Basic variables							b
5	1.5	1		1	0	_	0	0	__	2
1	1	1		0	1	__	0	0	_	0
0	0	0		0	0	_	1	0	__	2
0	0	0		0	0	_	0	1	__	1

Current solution, , , with z = 10.75 is the desired solution to problem (9)-(10).
The 1st and 2nd cuts introduced in terms of , are , and respectively.

The geometric interpretation of the cuts introduced is given in Figure below:



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