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A FUZZY OPTIMIZATION MODEL

Abstract. *A fuzzy optimization model, mainly aiming the portfolio selection issue, is discussed. Instead of being modeled as piecewise linear fuzzy numbers (triangular, trapezoidal etc.), the input data are presented as quasi S shape fuzzy numbers. In this framework some theoretical and numerical results are given.*

Key words**:** *fuzzy number, possibilistic mean value, quasi S shape fuzzy data, portfolio optimization.*

JEL Classification: C020, C440, C610.

1. Introduction

The issue of fuzzy mathematical programming is a topic of great interest in literature ([13], [14], [15], [16]). Also the financial market analysis, including portfolio optimization, has numerous approaches ([3], [10], [14], [17], [19]). In fact, the problem of portfolio optimization can be viewed as a problem of decision-making. In this area, the theory of fuzzy sets is a very useful tool [20]. In this paper we deal with both subjects mentioned above. More precisely, we discuss a fuzzy optimization model from the theoretical point of view and we also show its applications to the problem of portfolio selection. It is included in a certain class of models, namely those based on some crisp characteristics of fuzzy data (such as possibilistic mean value or possibilistic variance - these concepts are rigorously defined in [1], [2]). Usually, this type of model has applications in the difficult issue of portfolio selection but it also can be seen as a standalone optimization problem with constraints. In this paper we use a generic model whose objective function is based on possibilistic mean value of a certain fuzzy number. It is used quite a lot in the literature but with various changes in the parameters or constraints (see [4], [5], [9], [14], [18]). However, there is one thing in common: the possibility of transforming the initial problem in some linear optimization problems. Generally, the input data (in the case of portfolio optimization we discuss about the possible return rates) are modeled as triangular or trapezoidal numbers ([4], [5]). Also some generalization in the form of polygonal numbers was given ([9], [18]). By contrast, in this work we propose the use of quasi *S* shape fuzzy numbers, which have already proven useful in the field of statistical regression (see also [13]). These numbers are partially linearized *S* shape ones (a presentation of the S shape membership functions can be found in [8]). The linearization of some parts of the membership function is necessary for the model proposed here. In this way the data fulfill the necessary conditions that allow us to apply for our case some general results about binary operations (as sum, scalar multiplication or even some comparisons between fuzzy numbers - see [6], [7], [11], [12], [15], [16]). In section 2 these theoretical aspects are emphasized in a more detailed presentation. We will also show the steps that must be taken in order to find a solution to the model described here. In section 3 the applicability of the proposed algorithm is verified for a set of numerical data.

2. Theoretical results

At first we give some results regarding quasi S shape fuzzy numbers [13], which are derived from S shape numbers [8]. Because the membership function of a S shape number has an asymptotic behavior regarding the lines $y=0$, $y=1$, it appears to be difficult to make certain operations with them. However, their partial linearization allows us to use the general results regarding sum or scalar (that is a real number) multiplication ([6], [11], [16]). So consider the numbers $\alpha,\beta\in\mathbf{R}$, $l_{21},l_{22}\in\left(0,\frac{1}{2}\right]$, $l_{11},l_{12}\in\left[\frac{1}{2},1\right)$ and $\alpha_{11}=\ln\frac{1-l_{11}}{l_{11}}+\alpha$, $\alpha_{21}=\ln\frac{1-l_{21}}{l_{21}}+\alpha$, $\beta_{12}=\ln\frac{1-l_{12}}{l_{12}}+\beta$, $\beta_{22}=\ln\frac{1-l_{22}}{l_{22}}+\beta$.

Also consider two bijective functions:

$$f:\left[-\frac{1}{1-l_{21}}-\alpha_{21},\frac{1}{l_{11}}-\alpha_{11}\right]\to[0,1],\quad g:\left[-\frac{1}{l_{12}}+\beta_{12},\frac{1}{1-l_{22}}+\beta_{22}\right]\to[0,1]$$

given by

$$f(x)=\begin{cases}\left(l_{21}-l_{21}^2\right)\left(x+\ln\frac{1-l_{21}}{l_{21}}+\alpha\right)+l_{21}, & x\in\left[-\frac{1}{1-l_{21}}-\alpha_{21},-\alpha_{21}\right]\\ \frac{1}{1+e^{-x-\alpha}}, & x\in\left[-\alpha_{21},-\alpha_{11}\right]\\ \left(l_{11}-l_{11}^2\right)\left(x+\ln\frac{1-l_{11}}{l_{11}}+\alpha\right)+l_{11}, & x\in\left[-\alpha_{11},\frac{1}{l_{11}}-\alpha_{11}\right]\end{cases}$$

and

$$
g(x)=\begin{cases}
\left(l_{12}-l_{12}^{2}\right)\left(-x+\ln\dfrac{1-l_{12}}{l_{12}}+\beta\right)+l_{12},\ x\in\left[-\dfrac{1}{l_{12}}+\beta_{12},\beta_{12}\right]\\[2ex]
\dfrac{1}{1+e^{x-\beta}},\ x\in\left[\beta_{12},\beta_{22}\right]\\[2ex]
\left(l_{22}-l_{22}^{2}\right)\left(-x\ln\dfrac{1-l_{22}}{l_{22}}+\beta\right)+l_{22},\ x\in\left[\beta_{22},\dfrac{1}{1-l_{22}}+\beta_{22}\right].
\end{cases}
$$

A quasi S shape fuzzy number, $\tilde{u}$, can be completely defined by its membership function. Let $u:\mathbf{R}\rightarrow[0,1]$ be this function, as follows:

$$
u(x)=\begin{cases}
0,\ x\in\left(-\infty,-\dfrac{1}{1-l_{21}}-\ln\dfrac{1-l_{21}}{l_{21}}-\alpha\right)\cup\left(\dfrac{1}{1-l_{22}}+\ln\dfrac{1-l_{22}}{l_{22}}+\beta,+\infty\right)\\[2ex]
f(x),\ x\in\left[-\dfrac{1}{1-l_{21}}-\ln\dfrac{1-l_{21}}{l_{21}}-\alpha,\dfrac{1}{l_{11}}-\ln\dfrac{1-l_{11}}{l_{11}}-\alpha\right]\\[2ex]
g(x),\ x\in\left[-\dfrac{1}{l_{12}}+\ln\dfrac{1-l_{12}}{l_{12}}+\beta,\dfrac{1}{1-l_{22}}+\ln\dfrac{1-l_{22}}{l_{22}}+\beta\right]\\[2ex]
1,\ x\in\left[\dfrac{1}{l_{11}}-\ln\dfrac{1-l_{11}}{l_{11}}-\alpha,-\dfrac{1}{l_{12}}+\ln\dfrac{1-l_{12}}{l_{12}}+\beta\right].
\end{cases}
$$

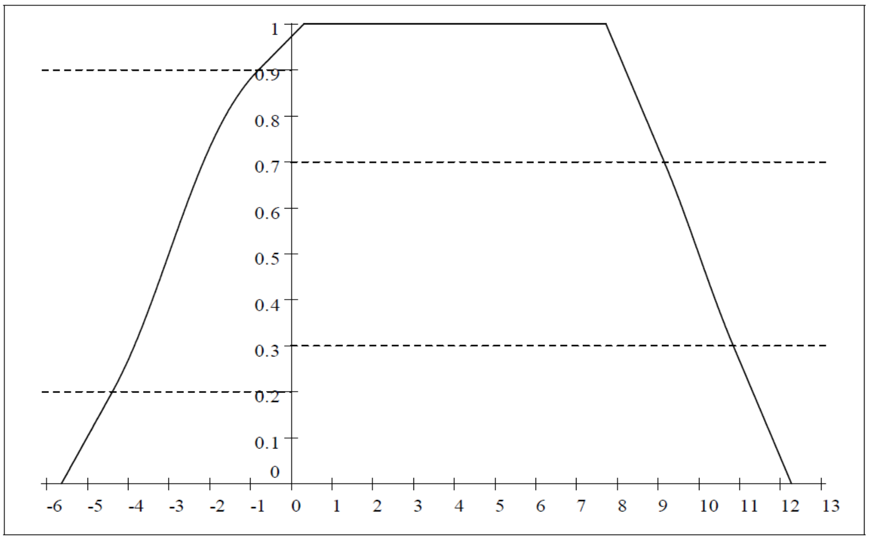

Figure 1. A quasi S shape fuzzy number

The condition for the existence of such a number is [13]:

$$\frac{1}{l_{11}} - \ln\frac{1-l_{11}}{l_{11}} - \alpha \le -\frac{1}{l_{12}} + \ln\frac{1-l_{12}}{l_{12}} + \beta$$

or

$$\frac{l_{11}+l_{12}}{l_{11}l_{12}} - \ln\frac{(1-l_{11})(1-l_{12})}{l_{11}l_{12}} \le \alpha + \beta .$$

In the case of equality we obtain a unique point, x_0, where the membership function is equal to 1. This is equivalent to

$$\frac{l_{11}+l_{12}}{l_{11}l_{12}} - \ln\frac{(1-l_{11})(1-l_{12})}{l_{11}l_{12}} = \alpha + \beta .$$

Remark that the parameters α and β control the left spreading and the right spreading, respectively. The parameters l_{21}, l_{11} determine the length of the linearization on the left side of the membership function. Similarly, the parameters l_{22}, l_{12} determine the length of the linearization on the right side of the membership function. We denote the quasi S shape fuzzy number $\tilde{u}$ by $\tilde{u} = (\alpha, \beta, l_{11}, l_{12}, l_{21}, l_{22})$. It can also be defined as a triplet $(\underline{u}(r), \overline{u}(r), r)$ (like any fuzzy number with certain properties, as it is shown in [6], [7], [11], [16]), where $\underline{u}(r), \overline{u}(r) : [0,1] \to \mathbf{R}$ and [13]:

$$\underline{u}(r) = \begin{cases} \frac{1}{l_{21}(1-l_{21})} r - \frac{1}{1-l_{21}} - \ln\frac{1-l_{21}}{l_{21}} - \alpha, & r \in [0, l_{21}] \\ -\ln\frac{1-r}{r} - \alpha, & r \in [l_{21}, l_{11}] \\ \frac{1}{l_{11}(1-l_{11})} r - \frac{1}{1-l_{11}} - \ln\frac{1-l_{11}}{l_{11}} - \alpha, & r \in [l_{11}, 1] \end{cases}$$

$$\overline{u}(r) = \begin{cases} -\frac{1}{l_{22}(1-l_{22})} r + \frac{1}{1-l_{22}} + \ln\frac{1-l_{22}}{l_{22}} + \beta, & r \in [0, l_{22}] \\ \ln\frac{1-r}{r} + \beta, & r \in [l_{22}, l_{12}] \\ -\frac{1}{l_{12}(1-l_{12})} r + \frac{1}{1-l_{12}} + \ln\frac{1-l_{12}}{l_{12}} + \beta, & r \in [l_{12}, 1] \end{cases}$$

Some crisp characteristics of a general fuzzy number $\tilde{a} = (\underline{a}(r), \overline{a}(r))$ are introduced in [1]: the lower possibilistic mean value ($\underline{M}(\tilde{a}) = 2\int_0^1 r\underline{a}(r)dr$), the upper possibilistic mean value ($\overline{M}(\tilde{a}) = 2\int_0^1 r\overline{a}(r)dr$) and the possibilistic mean value ($M(\tilde{a}) = \frac{\underline{M}(\tilde{a}) + \overline{M}(\tilde{a})}{2}$). By applying these definitions to $\tilde{u} = (\alpha, \beta, l_{11}, l_{12}, l_{21}, l_{22})$, we obtain the following results:

$$\underline{M}(\tilde{u}) = 2\int_0^1 r\underline{u}(r)dr = -\alpha - \frac{l_{21}^2}{3(1-l_{21})} + \frac{2-l_{11}-l_{11}^2}{3l_{11}} + (l_{11}-l_{21}) + \ln\frac{l_{11}}{1-l_{21}},$$

$$\overline{M}(\tilde{u}) = 2\int_0^1 r\overline{u}(r)dr = \beta + \frac{l_{22}^2}{3(1-l_{22})} - \frac{2-l_{12}-l_{12}^2}{3l_{12}} + (l_{22}-l_{12}) + \ln\frac{1-l_{22}}{l_{12}}$$

and (see also [13])

$$\begin{aligned} M(\tilde{u}) &= -\frac{\alpha}{2} - \frac{l_{21}^2}{6(1-l_{21})} + \frac{2-l_{11}-l_{11}^2}{6l_{11}} + \frac{l_{11}-l_{21}}{2} + \frac{1}{2}\ln\frac{l_{11}}{1-l_{21}} + \\ &+ \frac{\beta}{2} + \frac{l_{22}^2}{6(1-l_{22})} - \frac{2-l_{12}-l_{12}^2}{6l_{12}} + \frac{l_{22}-l_{12}}{2} + \frac{1}{2}\ln\frac{1-l_{22}}{l_{12}} = \\ &= \frac{\beta-\alpha}{2} + \frac{1}{6}\left(-\frac{l_{21}^2}{1-l_{21}} + \frac{2-l_{11}-l_{11}^2}{l_{11}} + \frac{l_{22}^2}{1-l_{22}} - \frac{2-l_{12}-l_{12}^2}{l_{12}}\right) + \\ &+ \frac{l_{11}-l_{21}+l_{22}-l_{12}}{2} + \frac{1}{2}\ln\frac{l_{11}(1-l_{22})}{l_{12}(1-l_{21})}. \end{aligned}$$

For example, if we analyze the number $\tilde{u} = (3, 10, 0.9, 0.7, 0.2, 0.3)$ (Figure 1) we obtain: $\underline{M}(\tilde{u}) = -2.0915$, $\overline{M}(\tilde{u}) = 9.2571$ and $M(\tilde{u}) = 3.5828$. We also remark that if $l_{11} = l_{12}$ and $l_{21} = l_{22}$ (i.e. same linearization levels on both sides) we obtain $M(\tilde{u}) = \frac{\beta-\alpha}{2}$. Taking into account some previous results from literature ([4], [14], [18]), we consider the fuzzy optimization model (S), as below. Notice that in our problem the input data are modeled as quasi S shape numbers instead of polygonal ones. So the initial problem is formulated as:

$$(6)\quad \begin{cases} \max\limits_{\mathbf{p}\in\mathbf{R}^n}\left[M\left(\sum\limits_{i=1}^{n} p_i\tilde{u}_i\right)\right] \\ \text{s.t.} \\ \mathrm{Pos}\left(\sum\limits_{i=1}^{n} p_i\tilde{u}_i \le t\right) \le \delta \\ \sum\limits_{i=1}^{n} p_i = 1 \\ \theta_i \le p_i \le \rho_i,\ i=\overline{1,n} \end{cases}$$

where $\mathbf{p}=(p_1,\ldots,p_n)$ is a vector for which $\sum\limits_{i=1}^{n} p_i = 1$, $\delta\in(0,1)$, $t\in\mathbf{R}$ and $\tilde{u}_i$ are quasi S shape fuzzy numbers. When such a model is used for improving a portfolio selection, the parameters gain some concrete meanings: for example $\mathbf{p}$ show how an investor must share this capital between n (possibly risky) assets. In order to solve (6), in the next steps we change the form of this optimization model.

In [1] it is proved that $M\left(\sum\limits_{i=1}^{n}\tilde{a}_i\right)=\sum\limits_{i=1}^{n}M(\tilde{a}_i)$ and $M(b\tilde{a})=bM(\tilde{a})$ (where $b\in\mathbf{R}$ and $\tilde{a}_1,\ldots,\tilde{a}_n,\tilde{a}$ are fuzzy numbers). Applying these results for our case, we conclude that the objective function can be transformed as follows:

$$M\left(\sum_{i=1}^{n} p_i\tilde{u}_i\right)=\frac{1}{2}\sum_{i=1}^{n}(\beta_i-\alpha_i)p_i+$$

$$+\frac{1}{6}\left(-\frac{l_{21}^2}{1-l_{21}}+\frac{2-l_{11}-l_{11}^2}{l_{11}}+\frac{l_{22}^2}{1-l_{22}}-\frac{2-l_{12}-l_{12}^2}{l_{12}}\right)\sum_{i=1}^{n}p_i+$$

$$+\frac{1}{2}(l_{11}-l_{21}+l_{22}-l_{12})\sum_{i=1}^{n}p_i+\frac{1}{2}\ln\frac{l_{11}(1-l_{22})}{l_{12}(1-l_{21})}\sum_{i=1}^{n}p_i\ .$$

Moreover, by using the substitutions

$$L_1=\frac{1}{6}\left(-\frac{l_{21}^2}{1-l_{21}}+\frac{2-l_{11}-l_{11}^2}{l_{11}}+\frac{l_{22}^2}{1-l_{22}}-\frac{2-l_{12}-l_{12}^2}{l_{12}}\right),$$

$$L_2=\frac{1}{2}(l_{11}-l_{21}+l_{22}-l_{12}),$$

$$L_3=\frac{1}{2}\ln\frac{l_{11}(1-l_{22})}{l_{12}(1-l_{21})}$$

and

$$\Delta_i=\frac{\beta_i-\alpha_i}{2}\ \text{(for } i=\overline{1,n}\text{)}$$

we can write:

$$M\left(\sum_{i=1}^{n} p_i \tilde{u}_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{3} L_j + \Delta_i\right) p_i \ .$$

Taking into account the definition of the possibility of $\tilde{a} \le b$ (where $\tilde{a}$ is a fuzzy number and $b \in \mathbf{R}$) [15], we can find some characteristics of a quasi S shape fuzzy number $\tilde{u}$. So, for $t \in \mathbf{R}$, we have:

1)
$$\mathrm{Pos}(\tilde{u} \le t) = 0$$
when
$$t \le -\frac{1}{1-l_{21}} - \ln\frac{1-l_{21}}{l_{21}} - \alpha \ ;$$

2)
$$\mathrm{Pos}(\tilde{u} \le t) = l_{21}(1-l_{21})t + l_{21} + l_{21}(1-l_{21})\alpha + l_{21}(1-l_{21})\ln\frac{1-l_{21}}{l_{21}}$$
when
$$-\frac{1}{1-l_{21}} - \ln\frac{1-l_{21}}{l_{21}} - \alpha \le t \le -\ln\frac{1-l_{21}}{l_{21}} - \alpha \ ;$$

3)
$$\mathrm{Pos}(\tilde{u} \le t) = \frac{1}{1+e^{-t-\alpha}}$$
when
$$-\ln\frac{1-l_{21}}{l_{21}} - \alpha \le t \le -\ln\frac{1-l_{11}}{l_{11}} - \alpha \ ;$$

4)
$$\mathrm{Pos}(\tilde{u} \le t) = l_{11}(1-l_{11})t + l_{11} + l_{11}(1-l_{11})\alpha + l_{11}(1-l_{11})\ln\frac{1-l_{11}}{l_{11}}$$
when
$$-\ln\frac{1-l_{11}}{l_{11}} - \alpha \le t \le \frac{1}{l_{11}} - \ln\frac{1-l_{11}}{l_{11}} - \alpha \ ;$$

5)
$$\mathrm{Pos}(\tilde{u} \le t) = 1$$
when
$$t \ge \frac{1}{l_{11}} - \ln\frac{1-l_{11}}{l_{11}} - \alpha \ .$$

Remark that the subset of quasi S shape fuzzy numbers, with the usual two binary operations (addition and scalar multiplications), is not necessarily closed. For a convenient conversion of the first constraint, we need to discuss the inequality $\mathrm{Pos}\left(\sum_{i=1}^{n} p_i \tilde{u}_i \le t\right) \le \delta$. We see that there are several possibilities, as follows:

1) If

$$t \le \sum_{i=1}^{n}\left(-\frac{1}{1-l_{21}}-\ln\frac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i$$

then

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right)=0\,.$$

2) If

$$\sum_{i=1}^{n}\left(-\frac{1}{1-l_{21}}-\ln\frac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i \le t \le \sum_{i=1}^{n}\left(-\ln\frac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i$$

then

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right)=\frac{\left(l_{21}-l_{21}^2\right)\left[t+\sum_{i=1}^{n}\left(\frac{1}{1-l_{21}}+\ln\frac{1-l_{21}}{l_{21}}+\alpha_i\right)p_i\right]}{\sum_{i=1}^{n} p_i}\,.$$

Thus

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right)\le\delta$$

is equivalent to

$$\sum_{i=1}^{n}\left(\frac{1}{1-l_{21}}-\frac{\delta}{l_{21}-l_{21}^2}+\ln\frac{1-l_{21}}{l_{21}}+\alpha_i\right)p_i \le -t\,.$$

3) If

$$\sum_{i=1}^{n}\left(-\ln\frac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i \le t \le \sum_{i=1}^{n}\left(-\ln\frac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i$$

then

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right)=\frac{1}{1+e^{-\frac{t+\sum_{i=1}^{n}\alpha_i p_i}{\sum_{i=1}^{n} p_i}}}\,.$$

Thus

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right)\le\delta$$

is equivalent to

$$\sum_{i=1}^{n}\left(\ln\frac{1-\delta}{\delta}+\alpha_i\right)p_i \le -t\,.$$

4) Assume that

$$\sum_{i=1}^{n}\left(-\ln\frac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i \le t \le \sum_{i=1}^{n}\left(\frac{1}{l_{11}}-\ln\frac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i \;.$$

It follows that

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right) \le \delta$$

is equivalent to

$$\sum_{i=1}^{n}\left(\frac{1}{1-l_{11}}-\frac{\delta}{l_{11}-l_{11}^2}+\ln\frac{1-l_{11}}{l_{11}}+\alpha_i\right)p_i \le -t \,.$$

5) If

$$t \ge \sum_{i=1}^{n}\left(\frac{1}{l_{11}}-\ln\frac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i$$

then

$$\mathrm{Pos}\left(\sum_{i=1}^{n} p_i\tilde{u}_i \le t\right) = 1 \,.$$

Taking into account the previous lines, we can conclude that the solution for the initial model can be obtained by solving three linear models (that will be denoted by (S1), (S2), (S3)), which have the same objective function. The solution for (S) is chosen such that the common objective function reaches its maximum. In other words, if $\mathbf{p}_{(1)}$, $\mathbf{p}_{(2)}$ and $\mathbf{p}_{(3)}$ are the solution vectors for (S1), (S2), (S3) then the solution for the initial problem is one of these three but so that the objective function takes the greatest possible value. Therefore $\mathbf{p}_{(1)}$, $\mathbf{p}_{(2)}$ and $\mathbf{p}_{(3)}$ can be viewed only as preliminary results for the initial model, only one being the optimal solution. On the other hand, there is the possibility that one of these problems has no admissible solutions, as we shall see in the next section. In this case we compare only the existing solutions. We have:

$$
(1\;\begin{cases}
\max\limits_{\mathbf{p}\in\mathbf{R}^n}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{3}L_j+\Delta_i\right)p_i\right] \\
\text{s.t.} \\
\sum\limits_{i=1}^{n}\left(\dfrac{1}{1-l_{21}}-\dfrac{\delta}{l_{21}-l_{21}^2}+\ln\dfrac{1-l_{21}}{l_{21}}+\alpha_i\right)p_i\le -t \\
\sum\limits_{i=1}^{n}\left(-\dfrac{1}{1-l_{21}}-\ln\dfrac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i\le t \\
\sum\limits_{i=1}^{n}\left(-\ln\dfrac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i\ge t \\
\sum\limits_{i=1}^{n}p_i=1 \\
\theta_i\le p_i\le\rho_i,\; i=\overline{1,n},
\end{cases}
\qquad
(2\;\begin{cases}
\max\limits_{\mathbf{p}\in\mathbf{R}^n}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{3}L_j+\Delta_i\right)p_i\right] \\
\text{s.t.} \\
\sum\limits_{i=1}^{n}\left(\ln\dfrac{1-\delta}{\delta}+\alpha_i\right)p_i\le -t \\
\sum\limits_{i=1}^{n}\left(-\ln\dfrac{1-l_{21}}{l_{21}}-\alpha_i\right)p_i\le t \\
\sum\limits_{i=1}^{n}\left(-\ln\dfrac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i\ge t \\
\sum\limits_{i=1}^{n}p_i=1 \\
\theta_i\le p_i\le\rho_i,\; i=\overline{1,n}
\end{cases}
$$

and

$$
(3\;\begin{cases}
\max\limits_{\mathbf{p}\in\mathbf{R}^n}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{3}L_j+\Delta_i\right)p_i\right] \\
\text{s.t.} \\
\sum\limits_{i=1}^{n}\left(\dfrac{1}{1-l_{11}}-\dfrac{\delta}{l_{11}-l_{11}^2}+\ln\dfrac{1-l_{11}}{l_{11}}+\alpha_i\right)p_i\le -t \\
\sum\limits_{i=1}^{n}\left(-\ln\dfrac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i\le t \\
\sum\limits_{i=1}^{n}\left(\dfrac{1}{l_{11}}-\ln\dfrac{1-l_{11}}{l_{11}}-\alpha_i\right)p_i\ge t \\
\sum\limits_{i=1}^{n}p_i=1 \\
\theta_i\le p_i\le\rho_i,\; i=\overline{1,n}.
\end{cases}
$$

3. Numerical test

We will study the case of a virtual portfolio with five assets. Assume that the return rates (including the possibility of some losses) are modeled as five quasi S shape fuzzy data (see Fig. 2), as follows: $\tilde{u}_1=(3,10,0.9,0.7,0.2,0.3)$, $\tilde{u}_2=(,9,0.9,0.7,0.2,0.3)$, $\tilde{u}_3=(2,6,0.9,0.7,0.2,0.3)$, $\tilde{u}_4=(0.5,7,0.9,0.7,0.2,0.3)$, $\tilde{u}_5=(.5,6.5,0.9,0.7,0.2,0.3)$. For each number $\tilde{u}_i$ ($i=\overline{1,5}$) we consider the pair

$(\lambda_i, \mu_i) \in \mathbf{R} \times \mathbf{R}$, which gives the left point and the right point where the membership function intersects the horizontal axis. For our numbers, we obtain: $(-5.6363, 12.276)$, $(-3.6363, 11.276)$, $(-4.6363, 8.2759)$, $(-3.1363, 9.2759)$ and $(-4.1363, 8.7759)$, respectively. The level $t \in \mathbf{R}$ is taken equal to $\max_{i=\overline{1,5}} \lambda_i$ i.e. $t = -3.1363$. The other parameters can be chosen depending on the level of risk acceptable to the investor. So we consider $\delta = 0.5$ and $\theta_i = 0.1$, $\rho_i = 0.5$, for all $i = \overline{1,5}$. Taking into account those stated in section 2, we proceed directly to the formulation of problems (S1), (S2) and (S3). After calculations, we obtain:

$$(S1)\begin{cases} \max\limits_{\mathbf{p}\in\mathbf{R}^5} (3.5828p_1 + 4.0828p_2 + 2.0828p_3 + 3.3328p_4 + 2.5828p_5) \\ \text{s.t.} \\ 2.5113p_1 + 0.51129p_2 + 1.5113p_3 + 0.01129p_4 + 1.0113p_5 \le 3.1363 \\ -5.6363p_1 - 3.6363p_2 - 4.6363p_3 - 3.1363p_4 - 4.1363p_5 \le -3.1363 \\ -4.3863p_1 - 2.3863p_2 - 3.3863p_3 - 1.8863p_4 - 2.8863p_5 \ge -3.1363 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \\ 0.1 \le p_i \le 0.5, \forall i = \overline{1,5} \end{cases}$$

$$(S2)\begin{cases} \max\limits_{\mathbf{p}\in\mathbf{R}^5} (3.5828p_1 + 4.0828p_2 + 2.0828p_3 + 3.3328p_4 + 2.5828p_5) \\ \text{s.t.} \\ 3p_1 + p_2 + 2p_3 + 0.5p_4 + 1.5p_5 \le 3.1363 \\ -4.3863p_1 - 2.3863p_2 - 3.3863p_3 - 1.8863p_4 - 2.8863p_5 \le -3.1363 \\ -0.8028p_1 + 1.1972p_2 + 0.1972p_3 + 1.6972p_4 + 0.6972p_5 \ge -3.1363 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \\ 0.1 \le p_i \le 0.5, \forall i = \overline{1,5} \end{cases}$$

$$(S3)\begin{cases} \max\limits_{\mathbf{p}\in\mathbf{R}^5} (3.5828p_1 + 4.0828p_2 + 2.0828p_3 + 3.3328p_4 + 2.5828p_5) \\ \text{s.t.} \\ 5.2472p_1 + 3.2472p_2 + 4.2472p_3 + 2.7472p_4 + 3.7472p_5 \le 3.1363 \\ -0.8028p_1 + 1.1972p_2 + 0.1972p_3 + 1.6972p_4 + 0.6972p_5 \le -3.1363 \\ 0.3083p_1 + 2.3083p_2 + 1.3083p_3 + 2.8083p_4 + 1.8083p_5 \ge -3.1363 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \\ 0.1 \le p_i \le 0.5, \forall i = \overline{1,5} \end{cases}$$

The solution for (S1) is $\mathbf{p}_{(S1)} = (0.2, 0.5, 0.1, 0.1, 0.1)$, for which the maximum of the objective function is 3.5578. The solution for the problem (S2) is $\mathbf{p}_{(S2)} = (0.325, 0.375, 0.1, 0.1, 0.1)$, for which the maximum of the common objective function is 3.4953. The problem (S3) has no admissible solutions. Due to the fact that $3.4953 < 3.5578$, the final result for the initial model (S) is $\mathbf{p}_{(S1)}$.

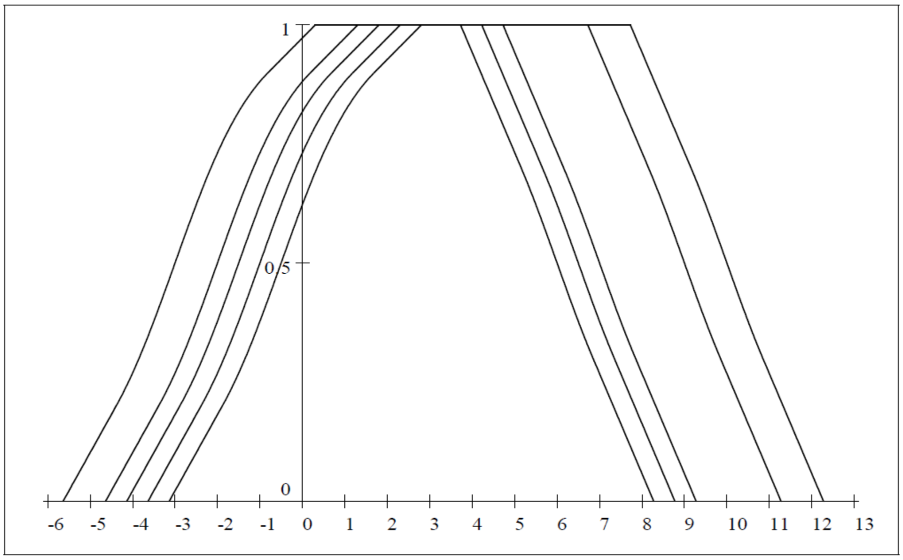

Figure 2. Five data of quasi S shape type

4. Conclusion

From the general set of fuzzy numbers, the subset of quasi S shape elements was considered. Some theoretical features (such as possibilistic mean value, some binary operations) of these elements were emphasized. An optimization model in which the data are presented in the form of quasi S shape fuzzy numbers was discussed. A numerical application, in order to test the applicability of the model, was given.

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