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ON MULTIPERIOD PORTFOLIO SELECTION WITH DIFFERENT BORROWING AND LENDING RATES

Abstract. *This paper deals with the problem of multiperiod portfolio selection, where borrowing and lending are allowed with different rates. Indeed, this work is mainly based on a recently published paper with the same subject. In this paper the underlying problem of multiperiod portfolio selection with different borrowing and lending rates is reformulated. After a thorough discussion about both conceptual and mathematical points, some new notations, compared to previous studies, are introduced. Afterward, a fuzzy representation of the reformulated model is proposed and a numerical example is used for implementing the presented model. Finally, the computational results are discussed.*

Keywords: *Multiperiod portfolio selection, Mathematical programming, Fuzzy set theory, Borrowing, Lending.*

JEL Classification: C38, G32

1 Introduction

Multi-period portfolio selection is raising the attention of various researchers and practitioners. Due to the forthcoming situations in the market, in a long-term horizon, an investor modifies his/her portfolio composition. Hence, the multi-period portfolio selection problem captures a more realistic image of the capital market conditions than the single-period one. Mulvey et al. (2003) indicate that exploiting the multi-period paradigm in the portfolio management problem is indispensable, in particular when there are transaction costs, when returns exhibit temporal dependence, and when the investor is able to borrow for investment. Hence, development of multi-period mathematical programming models in the area of portfolio management is a matter of particular importance.

Here, we review some of the most important works mainly focusing on developing mathematical programming models to deal with portfolio management problems. Topaloglou et al. (2008) developed a multistage stochastic programming model to dynamically deal with the international portfolio management problem. The proposed framework was able to jointly determine the capital allocated to each international market, the assets selected in each market and the appropriate currency hedging levels. Pınar (2007) developed multistage portfolio selection models to maximize the expected terminal wealth. Also, the presented models sought to minimize one-sided deviation from a target wealth level to ensure stability of the investment policies in the face of market risk.

Edirisinghe & Patterson (2007) developed a multiperiod mathematical model for stock portfolio optimization. Their proposed model incorporated various risk and policy constraints leading to significant period-by-period linkage in the model. Zenios et al. (1998) used multistage stochastic programming with recourse to develop multi-period fixed-income portfolio management models under uncertainty in a dynamic setting. Their presented models integrated the prescriptive stochastic programs with descriptive Monte Carlo simulation models of the term structure of interest rates. Escudero et al. (2009) presented a multistage stochastic mixed 0-1 model with complete recourse to optimize a mean risk portfolio management problem. Their proposed model dealt with a fixed income asset portfolio restructuring in which the interest rates and the liabilities considered to be uncertain along a given time horizon. Lacagnina & Pecorella (2006) integrated stochastic and possibilistic programming to develop a multistage stochastic soft constraints fuzzy program with recourse for capturing both uncertainty and imprecision in portfolio management problem. Lucka et al. (2008) proposed a multistage model to allocate financial resources to bond indices denominated in different currencies. Their study utilized historical data of interest and exchange rates to compare a two-stage and a three-stage stochastic programming model from a financial performance viewpoint.

Consiglio & Staino (2010) presented a multistage stochastic programming model to select bond portfolios aiming to minimize the cost of the decisions that must be taken based on the key stochastic economic factors underneath the model. Raubenheimer & Kruger (2010) formulated a multistage dynamic stochastic programming model to deal with a liquid asset portfolio management problem. The aim of the proposed model was to shape an optimal liquid asset portfolio for a financial institution without violating the mandatory regulations, about the minimum required liquid assets, it has to comply with. Ferstl & Weissensteiner (2010) formulated a multi-stage stochastic linear program to deal with a cash management problem in which a company with a given financial endowment and future cash flows is to minimize the Conditional Value at Risk of the terminal wealth. In the proposed model, interest rates and equity returns were considered to be uncertain.

Osorio et al. (2008a) developed a multistage mean-variance portfolio allocation model to investigate the role of decisions that affect the way taxes are paid in a general portfolio investment. To attain this goal, their proposed multistage portfolio optimization model integrated a number of risky assets grouped in wrappers with special taxation rules. Osorio et al. (2008b) developed a mixed integer stochastic programming approach to deal with the mean-variance post-tax portfolio management. Their presented stochastic programming approach considered risk in a multistage setting and allowed general withdrawals from original capital.

Date et al. (2011) presented a stochastic optimization-based approach to build a portfolio issued over a series of government auctions for the fixed income debt. Their proposed mixed integer linear programming model that uses a receding horizon, sought to minimize the cost of servicing debt while controlling risk and maintaining market liquidity. Rasmussen & Clausen (2007) formulated multistage stochastic integer programs to deal with the mortgagor's choices in the Danish mortgage loan system and also his/her attitude towards risk in a dynamic setting.

Barro & Canestrelli (2009) utilized stochastic programming framework to develop a multistage stochastic tracking error model. Their study investigated different tracking error measures which are common in static models and also a number of problems arising in dynamic settings.

Bertsimas & Pachamanova (2008) developed robust optimization formulations to deal with multiperiod portfolio selection in the presence of transaction costs. They compared the performance of the presented robust formulations to the performance of the traditional single period mean-variance formulations.

As mentioned above, the focus of this paper is on studies in which the investor(s) can borrow and lend money with different rates to invest in a multiperiod portfolio management setting. Thus, this paper tries to focus on the studies conducted in this context.

Seyedhosseini et al. (2010) presented a mathematical programming model to deal with the multiperiod portfolio selection problem where the borrowing rate is greater than the lending rate. They considered a numerical example to illustrate their presented mathematical formulation. Sadjadi et al. (2011) presented a fuzzy linear programming model to address the multiperiod portfolio selection problem where the borrowing rate is greater than the lending rate. Due to the intrinsic uncertainty of rates of return for risky assets and rates of borrowing and lending, they considered these parameters as triangular fuzzy numbers rather than crisp numbers. Finally, they presented a numerical example and discussed about the output results. Seyedhosseini et al. (2011) presented a stochastic programming model to address the multiperiod portfolio selection problem where the borrowing rate is greater than the lending rate. To deal with the intrinsic uncertainty of the problem, chance constrained programming was utilized. Finally, genetic algorithm was used to solve the formulated problem. In both of the above-mentioned studies, transaction costs were ignored. Hassanlou (2012) compared the above-mentioned approaches for solving the multi-period portfolio selection problem with different borrowing and lending rates and concluded that the results pertaining to the fuzzy mathematical programming approach outperform those pertaining to the stochastic programming approach.

Albeit the authors' intention in the above-mentioned works is ambitious, there are some conceptual and mathematical points that mislead their works. These points will go through in the following section. After a thorough discussion about these imperative points, the multi-period portfolio selection problem with different borrowing and lending rates is formulated. Then, a fuzzy representation of the proposed model is presented. Afterward, a numerical example is used to implement the proposed fuzzy multi-period portfolio selection model with different borrowing and lending rates.

The remainder of this paper is organized as follows: In the next section, some important, conceptual and mathematical points about multi-period portfolio selection models with different borrowing and lending rates are thoroughly discussed. In the third section, regarding former discussions, the multi-period portfolio selection problem with different borrowing and lending rates is formulated. A fuzzy variant of the proposed model is presented in the fourth section. In the fifth section, the proposed fuzzy linear programming model is implemented using a numerical example, provided from the literature. Finally, the last section concludes the paper.

2 A more detailed discussion about some points of related studies in the literature

As mentioned above, there are a number of important, conceptual and mathematical points about multi-period portfolio selection models with different borrowing and lending rates that mislead the works conducted in this area. Here, we try to elaborate these points.

Sadjadi et al. (2011) and Seyedhosseini et al. (2010) mentioned that "*the transaction cost does not play an important role in the optimization results since many* *brokerage houses are planning to remove transaction costs in order to create a motivation to absorb more investment*". This statement does not make sense in the real word. We know that brokerage houses are parties in the capital market that facilitate the transactions between buyers and sellers. In return, they receive a commission fee for each transaction. These commission fees, referred to as "transaction costs", are the main sources of income for brokerage firms. Hence, removing transaction costs is an unrealistic assumption that casts doubt on the motivation of establishing these important entities of capital market. Moreover, as Mulvey et al. (2003) indicated, transaction costs are one of important reasons that necessitate exploiting multi-period portfolio selection models rather than iteratively solving single-period ones. In other words, in the absence of transaction costs, as well as some other conditions, one can consider the long-term investment process as a number of iterative single-period investment decisions. In fact, removing transaction costs not only is a practically unrealistic assumption, but also cast doubt on the necessity of utilizing multi-period paradigm for investment decisions. Thus, transaction costs are incorporated to make the presented model more conformed to the real world applications.

Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012) make some assumptions that do not seem to be reasonable in the real world conditions.

Their assumption on selling risky assets and investing the provided proceeds in the risk free asset with the lending rate makes sense for proceeds from selling only those ones held using the investor's own capital. In other words, investing the cash provided from borrowing and also proceeds from selling risky assets purchased using loans in the risk free asset is not affordable. This is due to the fact that the borrowing rate is assumed to be greater than the lending rate. Hence, it is not reasonable to borrow for investment with the lending rate. Consequently, defining a variable for investment in risk free asset using cash provided from loans as well as selling the risky assets purchased by loans and a corresponding balance equation does not seem reasonable in real world situations. We eliminate this variable and its corresponding balance constraint. Instead, we assume that the loan can merely be used to purchase risky assets. Also, we assume that the proceeds from selling risky assets purchased with loans are utilized to repay the principal of loans, rather than investment in risk free assets. This avoids paying additional interests to creditors. The interest payments to creditors are further considered in balance equations.

Even though, using loans for purchasing risk free asset does not sound reasonable, however, if one borrows in order to purchase the risk free asset, he/she should receive some proceeds with the lending rate. However, in spite of allowing such an action in Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou

(2012) authors ignore these proceeds in their models.

Another important point is that Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012) only discuss about the interests of loans and do not take the repayment of the principal of loans into consideration. To deal with, we define additional variables as well as constraints pertaining to the net liabilities in different periods. Net terminal liabilities are assumed to diminish the total utility of investor as well.

A modified model is suggested to cope with the above-mentioned points. In this regard, we reformulate the multi-period portfolio selection problem with different borrowing and lending rates. Also, as in Sadjadi et al. (2011), we utilize fuzzy set theory to present a fuzzy variant of this problem in which the rates of returns and rates of borrowing and lending are considered to be triangular fuzzy numbers rather than crisp numbers.

3 The formulated multi-period portfolio selection model

To formulate the multi-period portfolio selection model with different borrowing and lending rates, in addition to the notations utilized in Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012), some new variables and parameters must be introduced. This is due to the fact that, as mentioned above, the mathematical formulations presented in previous studies must be necessarily modified from both conceptual and mathematical points of view.

M the number of risky assets (stocks);

N the number of trading periods;

X_t^m the investor's dollar holdings in asset m at the beginning of period t (funded with his/her own capital), (m = 0,1,…,M), (t = 0,1,…,N), where, m = 0 denotes the risk free asset;

$X_t'^m$ the investor's dollar holdings in risky asset m at the beginning of period t (funded with borrowing), (m = 1,…,M), (t = 0,1,…,N);

r_t^m the rate of return for risky asset m over time period (t, t + 1), (m = 1, 2,…,M),

(t = 0,1,…,N-1);

r_t^b the riskless borrowing rate over time period (t, t + 1), (t = 0,1,…,N-1);

r_t^l the riskless lending rate over time period (t, t + 1), (t = 0,1,…,N-1);

u_t^m the amount of risky asset m funded with the investor's own capital which is sold in period t, (m = 1,…,M), (t = 1,…,N-1);

v_t^m the amount of risky asset m funded with the investor's own capital which is purchased in period t, (m = 1,…,M), (t = 1,…,N-1);

$u_t'^m$ the amount of $X_{t-1}'^m$ which is sold in period t, (m = 1,…,M), (t = 1,…,N-1);

$v_t'^m$ the amount of risky asset m which is purchased using loan in period t, (m = 1,…,M), (t = 1,…,N-1);

V the maximum permitted amount of purchasing each risky asset in each period

$U(X)$ the investor's utility function

η the proportional transaction cost for selling risky assets

η' the proportional transaction cost for purchasing risky assets

L_t^m the net borrowed capital invested in risky asset m up to the beginning of period t, (m = 1,…,M), (t = 0,1,…,N), Note that this notation is different from $X_t'^m$;

Note that, in addition to introducing some new parameters and variables, some unnecessary variables used in Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012) have been removed and ranges of some indices have been modified. The investor can invest in M risky assets, i.e. stocks, and one risk free asset. Recall that the borrowing rate is greater than the lending rate, $r_t^b \geq r_t^l$, the proposed model would be as follows:

$$\max U(\sum_{m=0}^{M} X_N^m + \sum_{m=1}^{M} X'^m_N - \sum_{m=1}^{M} L_N^m) = \sum_{m=0}^{M} X_N^m + \sum_{m=1}^{M} X'^m_N - \sum_{m=1}^{M} L_N^m \tag{1}$$

$$X_t^m = (1 + r_{t-1}^m)(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m),\ t = (1,\ldots,N), m = (1,\ldots,M) \tag{2}$$

$$X_t^0 = (1 + r_{t-1}^l)(X_{t-1}^0 + \sum_{m=1}^{M}(u_{t-1}^m)(1-\eta) - \sum_{m=1}^{M} v_{t-1}^m(1+\eta')),\quad t = (1,\ldots,N) \tag{3}$$

$$L_0^m = X'^m_0,\quad m = (1,\ldots,M) \tag{4}$$

$$L_t^m = L_{t-1}^m - u'^m_{t-1}(1-\eta) + v'^m_{t-1}(1+\eta'),\quad t = (1,\ldots,N),\ m = (1,\ldots,M) \tag{5}$$

$$X'^m_t = (X'^m_{t-1} - u'^m_{t-1} + v'^m_{t-1})(1 + r_{t-1}^m) - L_t^m(r_{t-1}^b),\quad t = (1,\ldots,N),\ m = (1,\ldots,M) \tag{6}$$

$$\sum_{m=0}^{M} X_t^m \geq \beta(\sum_{m=1}^{M} X'^m_t),\quad t = (1,\ldots,N),\quad \beta \in [0,1] \tag{7}$$

$$v_t^m \leq V,\quad t = (0,1,\ldots,N-1),\quad m = (1,\ldots,M) \tag{8}$$

$$v_t^m, u_t^m, v'^m_t, u'^m_t \geq 0,\quad t = (0,1,\ldots,N-1),\quad m = (1,\ldots,N) \tag{9}$$

$$X_t^m \geq 0,\quad t = (0,1,\ldots,N),\quad m = (0,1,\ldots,N) \tag{10}$$

$$X'^m_t, L_t^m \geq 0,\quad t = (0,1,\ldots,N),\quad m = (1,\ldots,N) \tag{11}$$

where, eq. (1), the objective function, computes the terminal value of total risky and risk free asset holdings minus the terminal liability that must be repaid to creditors. The part pertaining to liability has not been considered in Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012). Note that the investor is not allowed to use borrowing for investing in risk free assets.

The balance of investor's dollar holdings, funded with his/her own capital, in risky assets at each period is considered in eq. (2). Also, eq. (3) considers this balance for investment in the risk free asset. Note that eq. (3), additionally, considers purchasing and selling transaction costs whose importance were formerly discussed.

Eq.'s (4) and (5) denote the balance of net borrowed capital invested in risky assets up to different time periods. Note that eq. (5) assumes that the proceeds from selling risky assets purchased with borrowing are utilized to repay the principals of liabilities. Besides, eq. (5) takes the purchasing and selling transaction costs into account.

Eq. (6) denotes the balance of investor's dollar holdings, funded with borrowing, in risky assets at each period. Returns of stocks and payments of interests pertaining to the borrowed capital over time periods are also considered in eq. (6).

The balance between total investment using the investor's own capital and borrowing in each period is considered in eq. (7). Eq. (8) defines an upper bound for investor to use his/her own capital for purchasing each risky asset in each period. In addition, eq. (9) ensures that the amounts of purchase and sale for risky assets are non-negative. Moreover, eq.'s (10) and (11) guarantee that the investor's dollar holdings in various assets and the amounts of his/her liabilities in different periods are non-negative. Note that these non-negativity constraints have been ignored in Sadjadi et al. (2011), Seyedhosseini et al. (2011) and Hassanlou (2012).

4 The presented fuzzy multi-period portfolio selection model

The rates of return for risky assets as well as borrowing and lending rates are considered triangular fuzzy numbers, $\tilde{r} = (l, m, n)$, whose membership function is illustrated in fig. 1. This helps make a fair comparison between the provided results with those provided in Sadjadi et al. (2011).

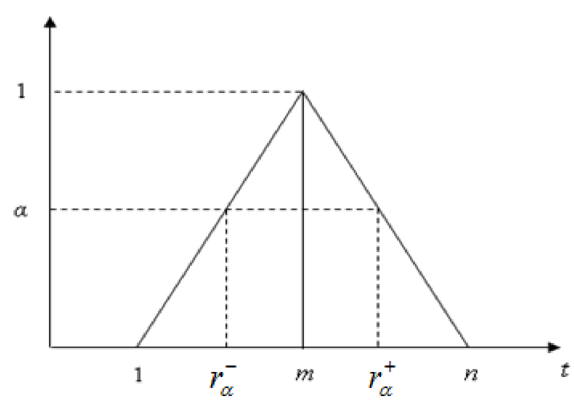

Figure 1 The membership function of $\tilde{r}$

Similarly, the α-cut on membership functions is implemented to provide the α-level confidence of $\tilde{r}$ in terms of interval values corresponding to the triangular fuzzy number $\tilde{r}=(l,m,n)$ as follows:

$$\tilde{r}_{\alpha}=[r_{\alpha}^{-},r_{\alpha}^{+}]=[(m-l)\alpha+l,n-(n-m)\alpha],\forall\alpha\in[0,1] \tag{12}$$

Thus, lower and upper bounds for α-level confidence can be simply provided. The fuzzy variant of the proposed multi-period portfolio selection model with different borrowing and lending rates is as follows:

$$\max U(\sum_{m=0}^{M}X_{N}^{m}+\sum_{m=1}^{M}X_{N}'^{m}-\sum_{m=1}^{M}L_{N}^{m})=\sum_{m=0}^{M}X_{N}^{m}+\sum_{m=1}^{M}X_{N}'^{m}-\sum_{m=1}^{M}L_{N}^{m} \tag{1$'$}$$

$$X_{t}^{m}=(1+\tilde{r}_{t-1}^{m})(X_{t-1}^{m}-u_{t-1}^{m}+v_{t-1}^{m}),\ t=(1,...,n),m=(1,...,M) \tag{2$'$}$$

$$X_{t}^{0}=(1+\tilde{r}_{t-1}^{l})(X_{t-1}^{0}+\sum_{m=1}^{M}(u_{t-1}^{m})(1-\eta)-\sum_{m=1}^{M}v_{t-1}^{m}(1+\eta')),\quad t=(1,...,N) \tag{3$'$}$$

$$L_{0}^{m}=X_{0}'^{m},\quad m=(1,...,M) \tag{4$'$}$$

$$L_{t}^{m}=L_{t-1}^{m}-u_{t-1}'^{m}(1-\eta)+v_{t-1}'^{m}(1+\eta'),\quad t=(1,...,N),\ m=(1,...,M) \tag{5$'$}$$

$$X_{t}'^{m}=(X_{t-1}'^{m}-u_{t-1}'^{m}+v_{t-1}'^{m})(1+\tilde{r}_{t-1}^{m})-L_{t}^{m}(\tilde{r}_{t-1}^{b}),\quad t=(1,...,N),\ m=(1,...,M) \tag{6$'$}$$

$$\sum_{m=0}^{M}X_{t}^{m}\geq\beta(\sum_{m=1}^{M}X_{t}'^{m}),\quad t=(1,...,N),\quad \beta\in[0,1] \tag{7$'$}$$

$$v_{t}^{m}\leq V,\quad t=(0,1,...,N-1),\quad m=(1,...,M) \tag{8$'$}$$

$$v_{t}^{m},u_{t}^{m},v_{t}'^{m},u_{t}'^{m}\geq 0,\quad t=(0,1,...,N-1),\quad m=(1,...,N) \tag{9$'$}$$

$$X_t^m \geq 0, \qquad t=(0,1,\ldots,N), \quad m=(0,1,\ldots,N) \tag{10'}$$

$$X_t'^m, L_t^m \geq 0, \qquad t=(0,1,\ldots,N), \quad m=(1,\ldots,N) \tag{11'}$$

Now, the α-level confidence of fuzzy numbers can be used to reformulate the fuzzy linear programming model as follows:

$$\max U(\sum_{m=0}^{M} X_N^m + \sum_{m=1}^{M} X_N'^m - \sum_{m=1}^{M} L_N^m) = \sum_{m=0}^{M} X_N^m + \sum_{m=1}^{M} X_N'^m - \sum_{m=1}^{M} L_N^m \tag{1''}$$

$$X_t^m = (1+[\tilde{r}_{\alpha,t-1}^{-,m}, \tilde{r}_{\alpha,t-1}^{+,m}])(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m),\ t=(1,\ldots,n), m=(1,\ldots,M) \tag{2''}$$

$$X_t^0 = (1+[\tilde{r}_{\alpha,t-1}^{-,l}, \tilde{r}_{\alpha,t-1}^{+,l}])(X_{t-1}^0 + \sum_{m=1}^{M}(u_{t-1}^m)(1-\eta) - \sum_{m=1}^{M} v_{t-1}^m(1+\eta')), \quad t=(1,\ldots,N) \tag{3''}$$

$$L_0^m = X_0'^m, \quad m=(1,\ldots,M) \tag{4''}$$

$$L_t^m = L_{t-1}^m - u_{t-1}'^m(1-\eta) + v_{t-1}'^m(1+\eta'), \quad t=(1,\ldots,N),\ m=(1,\ldots,M) \tag{5''}$$

$$X_t'^m = (X_{t-1}'^m - u_{t-1}'^m + v_{t-1}'^m)(1+[\tilde{r}_{\alpha,t-1}^{-,m}, \tilde{r}_{\alpha,t-1}^{+,m}]) - L_t^m([\tilde{r}_{\alpha,t-1}^{-,b}, \tilde{r}_{\alpha,t-1}^{+,b}]), \quad t=(1,\ldots,N),\ m=(1,\ldots,M) \tag{6''}$$

$$\sum_{m=0}^{M} X_t^m \geq \beta(\sum_{m=1}^{M} X_t'^m), \quad t=(1,\ldots,N), \quad \beta \in [0,1] \tag{7''}$$

$$v_t^m \leq V, \qquad t=(0,1,\ldots,N-1), \quad m=(1,\ldots,M) \tag{8''}$$

$$v_t^m, u_t^m, v_t'^m, u_t'^m \geq 0, \qquad t=(0,1,\ldots,N-1), \quad m=(1,\ldots,N) \tag{9''}$$

$$X_t^m \geq 0, \qquad t=(0,1,\ldots,N), \quad m=(0,1,\ldots,N) \tag{10''}$$

$$X_t'^m, L_t^m \geq 0, \qquad t=(0,1,\ldots,N), \quad m=(1,\ldots,N) \tag{11''}$$

Using the lower bound for the rate of borrowing and the upper bound for the rate of lending as well as rates of return pertaining to the risky assets, the upper bound of investor's utility function can be provided. Similarly, using the upper bound for the rate of borrowing and the lower bound for the rate of lending as well as rates of return pertaining to the risky assets, we can provide the lower bound of investor's utility function. Hence, to achieve an interval associated with the investor's utility for any α, it is enough to solve the crisp model twice with appropriate bounds of intervals.

5 Computational results and discussion

Here, we consider the numerical example used in Sadjadi et al. (2011), to implement the proposed model. This example considers one risk free and four risky assets ($M = 4$). Also, the problem has four periods ($T = 4$). The borrowing and lending rates in these four periods are respectively as follows:

$$\tilde{r}_b = [0.0\tilde{8}, 0.0\tilde{7}, 0.0\tilde{8}, 0.0\tilde{9}]$$
$$\tilde{r}_l = [0.0\tilde{6}, 0.0\tilde{7}, 0.0\tilde{5}, 0.0\tilde{7}]$$

Also, rates of return for risky assets in these four periods are as follows:

$$\tilde{r} = \begin{bmatrix} 0.0\tilde{9} & 0.\tilde{1}0 & 0.0\tilde{8} & 0.0\tilde{9} \\ 0.0\tilde{9} & 0.0\tilde{9} & 0.\tilde{1}0 & 0.0\tilde{8} \\ 0.0\tilde{8} & 0.0\tilde{9} & 0.0\tilde{9} & 0.\tilde{1}0 \\ 0.\tilde{1}0 & 0.0\tilde{8} & 0.0\tilde{9} & 0.0\tilde{8} \end{bmatrix}$$

where, $\tilde{r}_{ij}$ denotes the fuzzy rate of return for risky asset *i* in period *j*.

To implement the proposed model using the data, the parameter β has been set to 1. Furthermore, the initial values of investor's holdings have been substituted with the values used in Sadjadi et al. (2011). Now, eq. (12) can be utilized to determine the confidence interval of each triangular fuzzy number for $\alpha \in [0,1]$. In all triangular fuzzy numbers, we have $n - m = m - l = 0.01$. The confidence intervals for all triangular fuzzy numbers have been illustrated in table 1. Note that table 1 is a modified version of that in Sadjadi et al. (2011).

Table 1 α-level confidence of fuzzy numbers in each period

N	1	2	3	4
$\tilde{r}_t^1$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .09, -.01\alpha + .11]$	$[.01\alpha + .07, -.01\alpha + .09]$	$[.01\alpha + .08, -.01\alpha + .10]$
$\tilde{r}_t^2$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .09, -.01\alpha + .11]$	$[.01\alpha + .07, -.01\alpha + .09]$
$\tilde{r}_t^3$	$[.01\alpha + .07, -.01\alpha + .09]$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .09, -.01\alpha + .11]$
$\tilde{r}_t^4$	$[.01\alpha + .09, -.01\alpha + .11]$	$[.01\alpha + .07, -.01\alpha + .09]$	$[.01\alpha + .08, -.01\alpha + .10]$	$[.01\alpha + .07, -.01\alpha + .09]$
$\tilde{r}_t^l$	$[.01\alpha + .05, -.01\alpha + .07]$	$[.01\alpha + .06, -.01\alpha + .08]$	$[.01\alpha + .04, -.01\alpha + .06]$	$[.01\alpha + .06, -.01\alpha + .08]$
$\tilde{r}_t^b$	$[.01\alpha + .07, -.01\alpha + .09]$	$[.01\alpha + .06, -.01\alpha + .08]$	$[.01\alpha + .07, -.01\alpha + .09]$	$[.01\alpha + .08, -.01\alpha + .10]$

Source: Authors' computations

The reformulated model has been implemented using GAMS 22.2 considering three different confidence levels namely $\alpha = 0$, 0.7 and 1. Table 2 illustrates the computational results for these three confidence levels. Since rates of return for risky assets as well as borrowing and lending rates have been considered to be triangular fuzzy numbers, a confidence interval of the objective function for each α can be provided. Table 2 shows that the optimal utility function of the investor for $\alpha = 1$ is 21701.495. Also, it provides the confidence intervals of investor's utility for $\alpha = 0$ as [19739.762, 24077.12] and for $\alpha = 0.7$ as [21061.058, 22403.498]. As α increases, the associated confidence interval becomes tighter. Even though the trend of changing objective values is the same as that in Sadjadi et al. (2011), the objective values obtained from solving the reformulated model are less than those in Sadjadi et al. (2011). For, the fact that the reformulated model contains some modifications compared to that in Sadjadi et al. (2011); as adding transaction costs, adding and removing some variables and constraints. Regarding all these modifications, of course, the main reason for this reduction is adding the realistic assumption that net terminal liabilities must be repaid to the creditors. It is obvious that this assumption solely entails a decrease in the investor's utility.

Table 2 α-level confidence intervals of objective function and variables for different values of α

confidence level		$\alpha = 0$	$\alpha = 0.7$	$\alpha = 1$
U		[19739.762,24077.12]	[21061.058,22403.498]	[21701.495,21701.495]
***t* = 0**	X_0^0	[1000,1000]	[1000,1000]	[1000,1000]
	X_0^1	[2000,2000]	[2000,2000]	[2000,2000]
	X_0^2	[3000,3000]	[3000,3000]	[3000,3000]
	X_0^3	[4000,4000]	[4000,4000]	[4000,4000]
	X_0^4	[5000,5000]	[5000,5000]	[5000,5000]
	$X_0'^1$	[2000,2000]	[2000,2000]	[2000,2000]
	$X_0'^2$	[3000,3000]	[3000,3000]	[3000,3000]
	$X_0'^3$	[4000,4000]	[4000,4000]	[4000,4000]
	$X_0'^4$	[5000,5000]	[5000,5000]	[5000,5000]
***t* = 1**	X_1^0	[0,0]	[0,0]	[0,0]
	X_1^1	[3208.544,3267.961]	[2174,3247.165]	[3238.252,3238.252]
	X_1^2	[3240,3300]	[4316.34,3279]	[3270,3270]
	X_1^3	[4280,4360]	[4308,4332]	[4320,4320]
	X_1^4	[5450,5550]	[5485,5515]	[5500,5500]
	$X_1'^1$	[1980,2060]	[2008,2032]	[2020,2020]
	$X_1'^2$	[29.035,5137.961]	[3012,5187.165]	[5208.252,5208.252]
	$X_1'^3$	[38.854,4080]	[3976,4024]	[4000,4000]
	$X_1'^4$	[5000,5200]	[5070,5130]	[5100,5100]
***t* = 2**	X_2^0	[0,0]	[0,0]	[0,0]

	X_2^1	[3497.313,3627.437]	[2384.878,3581.623]	[3562.078,3562.078]
	X_2^2	[3499.2,3630]	[4691.861,3583.947]	[3564.3,3564.3]
	X_2^3	[4622.4,4796]	[4682.796,4734.876]	[4708.8,4708.8]
	X_2^4	[5831.5,6049.5]	[5907.345,5972.745]	[5940,5940]
	$X_2'^1$	[1998.2,2166.6]	[2056.776,2107.296]	[2082,2082]
	$X_2'^2$	[21.255,5348.629]	[3055.044,5322.941]	[5311.128,5311.128]
	$X_2'^3$	[28.442,5219.708]	[4029.912,5222.164]	[5224.05,5224.05]
	$X_2'^4$	[32.942,5368]	[5095.39,5220.79]	[5158,5158]
***t* = 3**	X_3^0	[0,0]	[0,0]	[0,0]
	X_3^1	[3742.125,3953.906]	[2568.514,3878.898]	[3847.044,3847.044]
	X_3^2	[3814.128,4029.3]	[5146.972,3953.094]	[3920.73,3920.73]
	X_3^3	[4992.192,5275.6]	[5090.199,5175.219]	[5132.592,5132.592]
	X_3^4	[6298.02,6654.45]	[6421.284,6528.21]	[6474.6,6474.6]
	$X_3'^1$	[6.851,2221.594]	[2049.148,2128.202]	[2088.56,2088.56]
	$X_3'^2$	[11.802,5583.328]	[3102.383,5472.838]	[5424.106,5424.106]
	$X_3'^3$	[15.506,6553.535]	[4048.514,5315.253]	[5281.602,5281.602]
	$X_3'^4$	[18.124,5554.8]	[5123.689,5321.323]	[5222.22,5222.22]
***t* = 4**	X_4^0	[0,0]	[0,0]	[0,0]
	X_4^1	[4041.494,4349.297]	[2791.974,4239.635]	[4193.278,4193.278]
	X_4^2	[4081.117,4391.937]	[5543.289,4281.2]	[4234.388,4234.388]
	X_4^3	[5441.489,5855.916]	[5583.949,5708.267]	[5645.851,5645.851]
	X_4^4	[6738.881,7253.351]	[6915.723,7070.052]	[6992.568,6992.568]
	$X_4'^1$	[0,2283.753]	[2041.424,2152.124]	[2096.53,2096.53]

	$X_4'^2$	[0,5681.655]	[3062.267,5476.982]	[5387.634,5387.634]
	$X_4'^3$	[0,6784.364]	[4069.22,5419.168]	[5345.574,5345.574]
	$X_4'^4$	[0,5654.732]	[5053.213,5327.993]	[5189.998,5189.998]

Source: Authors' computations using GAMS 22.2

6 Concluding Remarks

This paper makes a closer look to the multiperiod portfolio selection with different borrowing and lending rates. First of all, some conceptual and mathematical points about related studies in the literature have been addressed. Afterward, notations have been introduced and the multi-period portfolio selection problem has been formulated. Furthermore, we considered the rates of return for risky assets as well as borrowing and lending rates to be triangular fuzzy numbers, and used α-cut on membership functions to yield α-level confidence intervals for these rates. Finally, the numerical example used in Sadjadi et al. (2011) was utilized to implement the formulated model and the computational results were presented. The computational results confirmed that when the confidence level increases, the interval pertaining to the investor's utility becomes tighter. When the investor seeks a confidence level equal to 1, his/her optimal utility has a single value.

Taking advantage of stochastic programming models, to deal with underlying problem, is a promising research direction in this area. In addition, transaction costs can be assumed to change between distinct periods. Furthermore, to deal with rates of return for risky assets and borrowing and lending rates, other definitions of fuzzy numbers such as trapezoidal fuzzy numbers can be used.

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