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HEDGING OF DOWNSIDE RISK WITH PUT OPTIONS IN VARIOUS STOCHASTIC VOLATILITY ENVIRONMENT: HESTON MODEL APPROACH WITH ZERO CORRELATION AND ZERO MARKET PRICE OF VOLATILITY

Abstract. *The aim of this article is to find optimal strategy of proposed hedging strategy with put options in different stochastic volatility environment. Stochastic volatility and price movements follow processes described by Heston model. We simulate stock price paths with combinations of different stochastic volatility parameters and find optimal strategy for every environment. Market environment is defined with speed of reversion parameter, long run variance (volatility of volatility). Correlation between Brownian motions is set to zero. For evaluation of strategies and solution of optimization problem we use differences between proposed hedging strategy with active buying put options and basic hedging strategy with put option. Results show that optimal proposed strategy can bring additional positive expected return in strong bull market, but price for this additive return is smaller return of proposed hedging strategy in bear market in comparison with basic strategy. The results of the paper can be interpreted in practice as alternative to basic hedging strategy with minimal price evaluated with estimation of negative differences between proposed and basic strategy.*

Keywords: *hedging, stochastic volatility, Heston model, simulation, put options.*

AMS Subject Classification: 60H35, 91B25, 91G80, 97M30

1. Introduction

Derivatives such as options are very useful tool for hedging financial risks in wide range of financial and non financial markets. Dynamic growth of trading volumes in derivatives markets and also development of new, exotic types of derivatives such as weather derivatives, exotic options have been associated with growth and development in pricing approach area. The Black Scholes model [6] is fundamental and the most popular model for pricing European options. An elegant solution of pricing problem is based on assumption that stock price, or value of some other asset, can be described with geometric Brownian motion with constant volatility. Notoriously not constant volatility of such underlying assets as stocks, exchange rates and commodities causes systemic mispricing of out-of-the-money options and in-the-money options if the implied volatility of at-the-money options is used for pricing [7, 12]. Differences in implied volatility of options with same expiration day and different strikes are well known as implied volatility smile. This phenomenon of option market has been reason for development of new pricing approaches and techniques. Implementation of stochastic volatility process [9, 10, 16] and jump processes [5] in pricing can explain volatility smile. Among the family of stochastic volatility models, Heston model is one of the most popular. Under the Feller conditions the process nonnegativity is assured. The model assumes with relation between price movements and variance change. This relation is expressed by correlation between Brownian motions. With estimated implied parameters, one can explain market environment of underlying market participants' expectations and particular option market properties [4, 12]. With rising liquidity in option markets, options have become much more flexible tool for hedging and trading under same specific expectations [13]. With exotic options such as barrier options, Asian options, lookback ones he can hedge against special risk related to stock price process. Also it is possible to construct special option strategies which involve few option positions in vanilla or exotic options as described in Soltes [14, 15] and Rusnakova [11]. These strategies are passive and static, and performance of a portfolio depends only on asset price on expiration day. Our work is focused on hedging strategy with active buying of put options for eliminate downside risk. Strategy consists of long positions in put options with shorter maturity than investing horizon. In the first part of the work we describe fundamental properties of Heston model. In the second part basic option payoff, profit functions, are described. The second part also includes Heston model pricing formulas. The third section of the paper describes properties of simulations and hedging details.

2. Process behind stock price and variance

Let t be a time $t \geq 0$ and $S(t)$ value of a stochastic process, which represents stock price in time t. In our work $v(t)$ also represents stochastic process, specifically variance of particular stock price in time t.

Let $B(t) = (B_1(t), B_2(t))^T$ be a 2 dimensional independent Brownian motion on probability space (Ω, F, P), where P is real world or statistical measure. Correlation between relative changes of stock prices and changes of variance is reason of

multidimensional correlated Brownian motion creation. In our 2-dimensional case is ρ_{2x2} a correlation matrix and H_{2x2} is Cholesky decomposition of ρ_{2x2}.
Let W be 2 dimensional correlated Brownian motion with correlation defined in ρ_{2x2}. W is defined in the following equation:

$$\mathrm{W} = \mathrm{H}^T \mathrm{B} \tag{1}$$

Heston, see [9], described stock price $S(t)$ as geometric Brownian motion with stochastic volatility. The stochastic differential equation of this process is:

$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dW_1(t) \tag{2}$$

where $\mu(t)$ is drift in time t. If we assume $\mu(t) = \mu$, then solution of $S(\mathrm{T})$ is described by following equation:

$$S(t) = S(0)exp\left(\mu T - \frac{1}{2}\int_0^T v(t)\,dt + \int_0^T \sqrt{v(t)}dW_1(t)\right) \tag{3}$$

Process $v(t)$ is defined as square root process or CIR process with stochastic differential equation:

$$dv(t) = \theta(\overline{v} - v(t))dt + \xi\sqrt{(v(t))}dW_2(t) \tag{4}$$

In (4) θ is speed reversion parameter, $\overline{v}$ is long run variance and ξ is defined as volatility of volatility. The parameters described above are constant and defined on R_+.

3. Options and options pricing techniques

Payoff of one European style vanilla option with underlying asset with price $S(T)$, time of maturity T and strike K is described by following equation:

$$H_{\omega_1,\omega_2,K,T} = \omega_2 \max(\omega_1(S(T) - K), 0). \tag{5}$$

ω_1 represents type of options ($\omega_1 = 1$ means call option, $\omega_1 = -1$ means put option) and ω_2 is defined as position of the option ($\omega_2 = 1$ means long position, $\omega_2 = -1$ means short position). Payoff determines only cash flow of financial instrument in the time point T, not profit or portfolio return. Let t be the time point of buying or selling one option type ω_1 with maturity at T, strike K and option premium $p(t)_{T,K,\omega_1}$. The profit of single option position is defined as:

$$z_{\omega_1,\omega_2,K,T,t} = H_{\omega_1,\omega_2,K,T} - \omega_2 p(t)_{T,K,\omega_1} \tag{6}$$

Return of portfolio with single asset, cash amount $L, L \geq hp(t)_{T,K,\omega_1}$ (cash is source for option buying ($h = 1$) or for short position collateral $-h$ depends on trading platform or broker) and one option contract can be defined by following equation:

$$r = \ln\left(\frac{S(T) + z_{\omega_1,\omega_2,K,T,t} + \exp(u(T-t))(L - hp(t)_{T,K,\omega_1})}{S(t) + L}\right) \tag{7}$$

In case of short selling ($\omega_A = -1$) or leverage buying of assets and opening positions in futures one can use equation (8), where C is maximal collateral hold by broker for asset or futures position.

$$r = \ln\left(\frac{C\ exp(u_A(T-t)+\omega_A(S(T)-S(t))+z_{\omega_1,\omega_2,K,T,t}+\exp(u(T-t))(L-hp(t)_{T,K,\omega_1})}{C+L}\right) \quad (8)$$

In general, methods of option pricing consist of the change of measure from the measure P to the measure Q. P represents real world or statistical measure and Q represents risk neutral measure. Value of option premium is then equal to:

$$p(t)_{T,K,\omega_1} = E^Q[H_{\omega_1,\omega_2=1,K,T}|S(t),v(t),\ldots] \quad (9)$$

2 dimensional Brownian motion W in Heston model is changed to W^Q by implementation of risk premiums:

$$W^Q(t) = W(t) - \frac{1}{\sqrt{v(t)}}\begin{pmatrix}\mu(t)-r\\ \lambda v(t)\end{pmatrix} \quad (10)$$

$\frac{\mu(t)-r}{\sqrt{v(t)}}$ is the market price of risk with risk free rate r and λ is risk premium in market price of volatility. Pricing of options is then based on following stochastic differential equations:

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1^Q \quad (11)$$

$$dv(t) = \left(\underbrace{\theta+\lambda}_{\theta^*}\right) + \left(\frac{\bar{v}\,\theta}{\underbrace{\theta+\lambda}_{\theta^*}} - v(t)\right)dt + \xi\sqrt{v(t)}S(t)dW_{12}^Q \quad (12)$$

The solution of call option price has form:

$$p = S(t)P_1 + \exp\big(-r(T-t)\big)\,KP_2 \quad (13)$$

where:

$$P_j = \frac{1}{2} + 1/\pi \int_0^\infty Re\frac{f_j(S(t),v(t),\phi)\exp(-i\phi\ln(K))}{i\phi}d\phi \quad (14)$$

$$f_{j(S(t),v(t),\phi)} = \exp(C_{j(T-t,\phi)} + D_j(T-t,\phi)v(t) + i\phi S(t))$$

$$C_j(T-t,\phi) = rT\phi i + \frac{a}{\xi^2}\left[\big(b_j - \rho\xi\phi i + d\big)(T-t) - 2\ln\left(\frac{1-ge^{d(T-t)}}{1-q}\right)\right]$$

$$D_j(T-t,\phi) = \frac{b_j-\rho\xi\phi i+d}{\xi^2}\left[\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\right],\quad g = \frac{b_j-\rho\xi\phi i+d}{b_j-\rho\xi\phi i-d}$$

$$d = \sqrt{\big(-b_j+\rho\xi\phi i\big)^2 - \xi^2(2y_j\phi i\,\phi^2)},\quad u_1 = -u_2 = \frac{1}{2},\quad a = \theta\,\bar{v}$$

$$b_1 = \theta+\lambda-\rho\xi,\quad b_2 = \theta+\lambda$$

The details of pricing formula and its derivation can be found in Heston work [9]. To calculate price of options with Heston model one needs to input more parameters than in Black Scholes formula. Black Scholes formula includes 3 parameters from market: risk free rate, price or value of underlying asset and estimation or expected value of volatility. In Heston model, dynamics of variance is defined by 3 parameters $(\theta,\bar{v},\xi)$, relationship between returns and variance is defined by correlation ρ. Heston model also includes risk free rate, value of underlying asset and initial variance (the same parameters as in Black Scholes). Although Heston model can evaluate option premium in way which captures volatility smile, parameters to input the pricing formulae are not observable from market data, see [3, 12]. In practice, variance $v(t)$ is not directly observed. To estimate $v(t)$ one can focus on local volatility or implied volatility techniques based on cross-sectional option data, see [1, 3, 7]. Similar to estimation of $v(t)$, many studies focused (see [4]) on cross-sectional option data to calibrate parameters of stochastic volatility model. Results of this kind of calibration don't refer information on volatility parameters for data directly observed from market asset price or value of some non tradeable underlying asset. Another way to estimate parameters $(\theta, \bar{v}, \xi)$ is usage of indirect inference methods. Convergence criteria of this method are described in Gourieroux et al., see [8]. Shu & Zhang developed a method for Heston model calibration based on GARCH(1,1) estimation and applied this method on S&P 500 data [12]. Estimation of the remaining parameters is based again on cross sectional option data. Another paper AitSahlia et al., see [1], applies same method like Shu & Zhang on S&P 100 data but with another time interval. Estimation of $(\theta, \bar{v}, \xi)$ in Shu & Zhang differs from the estimation based only on cross-sectional option data. They fit $(\theta, \bar{v}, \xi)$ to (2.75,0.035,0.425) in time interval 1/1995-12/1999. These parameters are constant for all cross sectional models and the remaining parameters $(v(t), \rho, \lambda)$ differ over the time. AitSahlia et al. fit $(\theta, \bar{v}, \xi)$ to (2.65,0.029,0.154). Both studies show that properties of implied volatility surface are very unstable because of correlation variability, which changes skew of volatility smile. Another reason of instability of implied volatility surface is volatility of risk premium λ which differs speed reversion parameter to θ^* and long run variance to $\bar{v}^*$.

4. Hedging strategies and simulations

We simulate hedging strategies in different market conditions. Market conditions are represented by $\Theta = (\theta, \bar{\mathbf{v}}, \xi)$. Concrete market condition of k-*th* market is described:

$$\Theta_{ijl} = ((\theta)_i, (\bar{\mathbf{v}})_j, (\xi)_l) \qquad (15)$$

$(\ldots)_i$ is i-*th* component of particular parameter vector. In our numerical results, for simulation we use:

$$\begin{aligned} \theta &= (1.50, 3.00, 4.50, 6.00) \\ \bar{\mathbf{v}} &= (0.15, 0.25, 0.35, 0.45)^2 \\ \xi &= (0.15, 0.25, 0.35, 0.45)^2 \end{aligned} \qquad (16)$$

θ represents different speed of reversion parameters. Speed of reversion parameter can be interpreted as a time which mean of variance needs to revert to half difference between long run variance and initial variance. Small θ revert variance to long run variance with less power and stochastic process is much more driven by stochastic part. High θ indicates that a mean of variance has a tendency for moving around the long run value. θ refers that mean of variance needs 116 trading days to revert to half difference between long run variance and initial variance. $\theta = 6$ shows market with very strong power of reverting process. Variance needs only less then 30 trading days to the half difference.

$\bar{v}$ points various market conditions with different deterministic (long run) variance. Higher level of variance (volatility) indicates asset with higher risk and also more expensive option premiums. Volatility of 15% is characteristic for some forex exchange rates, volatility between 15% – 25% is characteristic for stock market indices and some commodities. Higher volatilities 35% – 45% characterize higher risk stock and commodities. Volatility of volatility, which we set by ξ, represents also stochastic volatility dynamics and shows the importance of stochastic part of stochastic differential equation.

For elimination effects of changes in implied volatility surface properties, we set theoretical values of ρ and λ as:

$$\rho = 0, \ \lambda = 0 \tag{17}$$

Correlation $\rho = 0$ is typical in situation, when market participants are not sensitive to any direction of market price movements. This is typical for exchange rate of 2 strong currencies in case when expectations or risks between these currencies are not too much different [7]. Correlation $\rho = 0$ can be found in some time intervals in various markets such as commodities or stocks and stock market indices. Risk premium $\lambda = 0$ means also no market price of volatility. Implied risk premium of volatility is option trader's preference from observed option data. If this premium is set to zero, option trader's properties of variance expectations dynamics are same as properties of variance dynamics associated with stochastic model parameters in Θ_k [12].

The aim of our hedging strategy is to reduce exposure to downside risk involving long position in put options in time horizon between $\langle 0, T \rangle$. We set T as 1 year (252 trading days).The basic strategy is long in put option with strike equal to underlying price in time t and expiration day in time T. Despite our strategy involves buying options with shorter maturity then time sequence $\langle 0, T \rangle$. Its conditioned step by step hedging strategy with long positions in put options with maturity $t^* = (t_1, t_2)$. If underlying price in time of last bought put option expiration is higher than minimum of all strike prices of previous bought option, then we buy put option with strike equal to minimum strike of previous bought options and with maturity time t_1. If the price is below minimum of all strike prices of previous bought options, then we buy put option with strike equal to price of underlying with maturity time t_2.

We simulate different hedging strategies defined by $t^* = (t_1, t_2)$ in different market conditions defined by Θ_k. With parameters of stochastic volatility

$(\theta, \overline{v}, \xi)$ we have 64 market conditions. In these conditions we simulate hedging strategies defined in following description:

$$t_1 = \left(\frac{25}{252}, \frac{50}{252}, \frac{50}{252}, \frac{75}{252}, \frac{100}{252}, \frac{125}{252}, \frac{150}{252}\right) \quad (18)$$

$$t_2 = \left(\frac{25}{252}, \frac{50}{252}, \frac{50}{252}, \frac{75}{252}, \frac{100}{252}, \frac{125}{252}, \frac{150}{252}\right) \quad (19)$$

Strategy is a combination of one component of t_1 and one component of t_2. So we simulate 36 strategies on 64 market conditions. Every simulation of the strategy is divided into these 4 steps:

1. The first step consists of independent Brownian motion simulation defined in $\boldsymbol{B}(t)$. Because of $\rho = 0$ correlated Brownian motion $\boldsymbol{W}(t)$ is equal to the independent.
2. The second step is calculation of $S(t)$ and $v(t)$ with simulated Brownian motion with Θ_k. We set $S(0) = 100$ and $u = 0$. For $v(t)$ we use Milstein scheme of discretization and for price in time t we use equation (3) and $v(0)$ values were generated with uniform distribution with parameters $min = 0.001$; $max = 0.275$.
3. The object of the next step is finding the time points of buying option and finding option properties. At the beginning of the simulation we buy put option with strike equal to $S(0)$ and maturity t_1. After expiration of the first option we buy another option. Let t_x be the time of expiration of last bought option and **K** set with strike prices of previously bought options. Maturity of next put option τ in our strategy is described by following condition:

$$\tau(t_x) = \begin{cases} t_1 & \text{if} \quad \min(\mathbf{K}) < S(t_x) \\ t_2 & \text{if} \quad \min(\mathbf{K}) \geq S(t_x) \end{cases} \quad (20)$$

   Next put option strike $K(t\_x)$ is also described by condition connected in relationship between minimum of **K** and underlying price in t_x:

$$K(t_x) = \begin{cases} \min(\mathbf{K}) & \text{if} \quad \min(\mathbf{K}) < S(t_x) \\ S(t_x) & \text{if} \quad \min(\mathbf{K}) \geq S(t_x) \end{cases} \quad (21)$$

   The next time of another buy position in option is $t_{x+1} = t_x + \tau(t_x)$, if $t_x + 1 < T$.
4. The fourth step involves pricing options with Heston model (see equation (14)) with parameters Θ_k. For pricing put options one can use put-call parity:

$$p_{\omega=-1,K,T}(t) = p_{\omega=1,S,T}(t) + K \cdot \exp(-r(T-t)) - S(t) \quad (22)$$

In case of stock options, change of measure from \mathbb{P} to \mathbb{Q} is characterized with usage of risk free rate. For simulations we set 4 risk free rate:

$$r = (2\%, 4\%, 6\%\ 8\%) \tag{23}$$

This step also includes summation of option payoff, total profit/loss of all option positions (latest bought option expires in time after T, so we use difference between price of option in T and price of option in time of buying).

5. Results

To evaluate strategies we focus on differences between strategy defined in t^* and the basic strategy. For comparison of strategies the same initial value of portfolio must be set. We set that $S(0) = 100$, but this is not the value of entire portfolio. For long positions in put option we have to dispose with cash. In basic strategy we use cash for one long position in put option with expiration day in time T. Rest of cash is invested in risk free asset with risk free rate defined in **r**. In our simulated strategies we also use cash for option buying and rest of cash and also cash from option payoff is invested in risk free asset with maturity in next time of option buying. Differences can be described by following equation:

$$d = \ln\left(\frac{S(T)+\sum_i H_i - \sum_i p_i + \sum_i (L_i) e^{rt_i}}{S(0)+L}\right) - \ln\left(\frac{S(T)+z_b+(L-p_b)e^{rT}}{S(0)+L}\right) \tag{24}$$

where z_b is profit from the basic strategy with put option with price p_b. H_i is i-th payoff of option in our active strategy with option of value in time of buying p_i and cash amount L_i. For liquidity purposes we set $L = 50$. We found that within different interval of $S(T)$, can d be described with following relation:

$$E[d(\Theta,t^*)] = \begin{cases} a(\Theta,t^*) + b(\Theta,t^*)S(T,\Theta) & \text{if} \quad 0.6\,S(0) < S(T,\Theta) < S(0) \\ c(\Theta,t^*) & \text{if} \quad S(0) \end{cases} \tag{25}$$

where $a(\Theta,t^*), b(\Theta,t^*)$ are regression coefficients of linear model defined as: $d = a(\Theta,t^*) + b(\Theta,t^*) \cdot S(T,\Theta)$ and $c(\Theta,t^*)$ is regression coeficient of linear model defined as: $d = c(\Theta,t^*)$. Regression results are: $a(\Theta,t^*) < 0$, $b(\Theta,t^*) > 0$ and $c(\Theta,t^*) > 0$. These results show that strategy is in general more profitable in market with growing price. In market situation where price declines, the basic strategy is much more effective. Our proposed strategy with active buying of put options is then focused on some predictions. If investors or portfolio managers can predict price growth and also want to be protected against situation on market which is opposite to their prediction, active strategy can be good way of hedging. Negative $E[d(\Theta,t^*)]$ for $0.6S(0) < S(T) < S(0)$ can be interpreted as price for greater average return for $S(T) > S(0)$. Optimization in these conditions can be focused on finding strategy, which has the smallest price for estimated $c(\Theta, t^*)$. We set this price to $\int_{0.6S(0)}^{S(0)} \left(a(\Theta, t^*)\right) + b(\Theta, t^*) S(T, \Theta) dS(T, \Theta)$, which is in general negative number. The best strategy can be found with minimization of the price and $c(\Theta, t^*)$ ratio:

$$\bar{t}^*(\Theta) = \arg\min_{t_1, t_2} \left(-\frac{\int_{0.6S(0)}^{S(0)} \left(a(\Theta, t^*)\right) + b(\Theta, t^*) S(T, \Theta) dS(T, \Theta)}{c(\Theta, t^*)} \right) \tag{26}$$

The results are in tables, for every combination Θ as set of 4 numbers. This set represents result under risk free rates in **r**. We found that in market environment with very low long run variance $\bar{v} = 0.0225$ (see Table 1 and Table 2) the best strategies are with high t_1 and also with high t_2 (125-150) for all simulated ξ in case that variance has no such a power to return to long run variance $(\theta = 1.5; 3.0)$. In case of stronger speed reversion parameter $(\theta = 4.5; 6.0)$ and low volatility of volatility we found that strategies with short t_1 is better than other. Volatility is not so dynamic and prices of options don't have tendency to vary much because of stability of implied volatility. Hedging costs in this type of market in situation of price growth are steady and low and price of average additional return $c(\Theta, t^*)$ is not so high. t_2 stays high in case of stronger speed reversion parameter and low volatility of volatility. Long maturity represented with t_2 can save hedging costs in situation of negative trend in asset price movements. Additional return is described in Table 2. The table shows, that market environment with low risk free rate brings greater average additional return. With strong speed reversion, lower volatility of volatility and risk free rate on 2% level average additional return is over 1%. With rising dynamics of variance (greater ξ) $c(\Theta, t^*)$ declines.

Table 1. $\bar{t}^*$ for $\bar{v} = 0.0225$

t_1				
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	125, 125, 150, 125	125, 125, 150, 125	125, 125, 150, 125	125, 125, 150, 125
3.0	125, 125, 150, 125	125, 125, 150, 125	125, 125, 150, 125	125, 125, 150, 125
4.5	25, 25, 150, 25	25, 150, 150, 25	150, 150, 150, 150	150, 25, 150, 150
6.0	25, 25, 150, 25	25, 25, 150, 25	150, 25, 150, 150	150, 25, 150, 150
t_2				
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	125, 125, 25,	125, 125, 25,	125, 125, 25,	125, 125, 25,

	125	125	125	125
3.0	125, 125, 25, 125	125, 125, 25, 125	125, 125, 25, 125	125, 125, 25, 125
4.5	150, 150, 25, 150	150, 25, 25, 150	150, 25, 25, 150	150, 150, 25, 150
6.0	150, 150, 25, 150	150, 150, 25, 150	150, 150, 25, 150	150, 150, 25, 150

Table 2. $c(\Theta, t^*)$ in % for $\bar{v} = 0.0225$

	c			
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	10.68; 0.63; 0.58; 0.60	0.66; 0.61; 0.55; 0.58	0.62; 0.56; 0.51; 0.53	0.57; 0.50; 0.40; 0.40
3.0	0.55; 0.48; 0.43; 0.43	0.57; 0.51; 0.46; 0.46	0.54; 0.47; 0.41; 0.41	0.47; 0.40; 0.36; 0.35
4.5	1.06; 0.89; 0.38; 0.37	1.06; 0.43; 0.39; 0.39	0.46; 0.43; 0.39; 0.39	0.46; 0.84; 0.34; 0.33
6.0	1.02; 0.85; 0.37; 0.36	1.02; 0.85; 0.37; 0.37	0.46; 0.83; 0.37; 0.37	0.47; 0.81; 0.35; 0.34

With $\overline{v} = 0.06225$ (see Table 3 and Table 4) results of optimal strategies differ with changing risk free rate. For risk free rate on levels 2% and 8% we found that t_1 is small (short maturity options). Greater ξ changed some values of t_1 to smaller (from 150 days to 125) or to the smallest value (25 days). Second maturity t_2 is between 100 and 150 except cases when $t_1 = 150$. In such a case, t_2 is equal to 25 days. So in general, if one maturity is short (25 days), then another is long - mostly 125 -150 days. More dynamic variance in this case shows, that traders or investors can catch additional return with our hedging strategy with short maturity options, but in case of bear market is better use longer maturity options. In comparison with $\overline{v} = 0.0225$ average addition return c with our strategy is greater in market environment with $\overline{v} = 0.0625$. Again, here we can observe same negative relation between risk free rate and c. In case of small risk free interest rate can portfolio manager with proposed hedging strategy expect the gain of more than 1,6%. Other risk free rates show expected additional return on levels between 0.6% and 0.8%.

Table 3. $\bar{t}^*$ for $\bar{v} = 0.0625$

	t_1			
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	25, 125, 150, 25	25, 125, 150, 25	125, 125, 150, 125	125, 125, 125, 125
3.0	25, 125, 150, 25	25, 125, 150, 25	25, 125, 125, 25	25, 125, 125, 25

4.5	25, 125, 150, 25	25, 125, 150, 25	25, 25, 125, 25	25, 25, 125, 25
6.0	25, 125, 150, 25	25, 25, 150, 25	25, 25, 125, 25	25, 125, 125, 25
		t_2		
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	150, 125, 25, 150	150, 125, 25, 150	125, 125, 25, 125	125, 125, 125, 125
3.0	150, 125, 25, 150	150, 125, 25, 150	150, 125, 125, 150	150, 125, 125, 150
4.5	150, 125, 25, 150	150, 125, 25, 150	150, 100, 125, 150	150, 150, 125, 150
6.0	150, 125, 25, 150	150, 100, 25, 150	150, 100, 125, 150	150, 125, 125, 150

Table 4. $c(\theta, t^*)$ in % for $\bar{v} = 0.0225$

		c		
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	1.68; 0.79; 0.74; 0.79	1.61; 0.74; 0.67; 0.72	0.77; 0.76; 0.,65; 0.70	0.80; 0.77; 0.74; 0.66
3.0	1.61; 0.75; 0.69; 0.74	1.60; 0.74; 0.66; 0.71	1.56; 0.75; 0.,73; 0,68	1,52; 0,71; 0,69; 0,62
4.5	1.60; 0.74; 0.69; 0.74	1.60; 0.75; 0.69; 0.74	1.58; 1.63; 0.70; 0,67	1,55; 1,43; 0,63; 0,61
6.0	1.60; 0.75; 0.71; 0.76	1.59; 1.68; 0.68; 0.73	1.61; 1.67; 0.70; 0.69	1.56; 0.70; 0.68; 0.61

In markets with $\bar{v} = 0.1225$, optimal strategies consist of various different combinations (see Table 5 and Table 6) . Relations to parameters of stochastic volatility (variance) are not so clear. However, expected additional returns in market with $S(T) > S(0)$ are greater then in markets with $\bar{v} = 0.0225$ or $\bar{v} = 0.0625$. The best expected additional returns are in interval from 2.07 to 3.1%. The last level of long run variance, $\bar{v} = 0.2025$ (see Table 7 and Table 8) has very significant influence of strategy selection. Strategy for markets with strong speed of reversion parameter and low volatility of volatility is characterized with longer maturities of t_1 and short maturities of t_2. This strategy consists of insurance for longer time in bull markets and more active hedging in bear market. We have pointed out, that the best returns are in market environment with low risk free rate and other expected returns are much more smaller. In the case

$\overline{v} = 0.2025$ we found, that differences in $c(\Theta, t^*)$ are not so big with change of risk free rate.

Table 5. $\bar{t}^*$ for $\bar{v} = 0.1225$

t_1				
$\theta \backslash \xi$	$\xi = 0,15$	$\xi = 0,25$	$\xi = 0,35$	$\xi = 0,45$
1.5	25, 25, 150, 25	25, 25, 150, 25	25, 125, 125, 25	125, 125, 125, 125
3.0	25, 25, 150, 25	25, 25, 150, 25	25, 25, 125, 25	25, 125, 125, 25
4.5	25, 125, 150, 25	25, 125, 150, 25	25, 25, 125, 25	25, 125, 125, 25
6.0	25, 125, 150, 25	125, 125, 150, 125	75, 125, 125, 75	25, 125, 125, 25
t_2				
$\theta \backslash \xi$	$\xi = 0,15$	$\xi = 0,25$	$\xi = 0,35$	$\xi = 0,45$
1.5	100, 100, 25, 100	75, 100, 25, 75	100, 125, 125, 100	125, 125, 125, 125
3.0	100, 100, 25, 100	100, 100, 25, 100	75, 75, 125, 75	100, 125, 125, 100
4.5	100, 125, 25, 100	100, 125, 25, 100	75, 75, 125, 75	50, 125, 125, 50
6.0	100, 125, 25, 100	50, 125, 25, 50	25, 125, 125, 25	50, 125, 125, 50

Table 6. $c(\Theta, t^*)$ in % for $\bar{v} = 0.1225$

c				
$\theta \backslash \xi$	$\xi = 0,15$	$\xi = 0,25$	$\xi = 0,35$	$\xi = 0,45$
1.5	2.35; 2.20; 0.81; 0.90	2.56; 2.14; 0.78; 0.86	2.33; 0.95; 0.96; 0.82	0.85; 0.87; 0.88; 0.87
3.0	2.42; 2.28; 0.84; 0.93	2.41; 2.27; 0.85; 0.95	2.65; 2.48; 0.97; 0.97	2.38; 0.96; 0.97; 0.98
4.5	2.48; 1.01; 0.89; 0.98	2.49; 1.05; 0.91; 1.00	2.71; 2.54; 1.02; 1.03	3.05; 0.98; 1.00; 1.01
6.0	2.50; 1.01; 0.87; 0.97	1.07; 1.05; 0.90; 0.99	2.07; 1.01; 1.03; 1.03	3.10; 1.03; 1.04; 1.05

Table 7. $\bar{t}^*$ for $\bar{v} = 0.2025$

t_1				
$\theta \backslash \xi$	$\xi = 0,15$	$\xi = 0,25$	$\xi = 0,35$	$\xi = 0,45$
1.5	75, 75, 150, 75	75, 75, 150, 75	50, 50, 25, 50	25, 125, 125, 25

3.0	125, 125, 150, 125	125, 125, 150, 125	50, 50, 125, 50	75, 125, 125, 75
4.5	125, 125, 150, 125	125, 125, 150, 125	50, 125, 125, 50	125, 125, 125, 125
6.0	125, 125, 150, 125	125, 125, 150, 125	125, 125, 125, 125	125, 125, 125, 125
t_2				
$\theta\backslash\xi$	$\xi = 0.15$	$\xi = 0.25$	$\xi = 0.35$	$\xi = 0.45$
1.5	25, 25, 25, 25	25, 25, 25, 25	25, 25, 75, 25	100, 125, 125, 100
3.0	50, 50, 25, 50	50, 50, 25, 50	25, 25, 25, 25	25, 25, 125, 25
4.5	50, 50, 25, 50	50, 50, 25, 50	25, 25, 25, 25	25, 25, 125, 25
6.0	50, 50, 25, 50	50, 50, 25, 50	25, 25, 25, 25	25, 25, 125, 25

Table 8. $c(\Theta, t^*)$ in % for $\bar{v} = 0.2025$

c				
$\theta\backslash\xi$	$\xi = 0.15$	$\xi = 0.25$	$\xi = 0.35$	$\xi = 0.45$
1.5	2.37; 2.28; 0.91; 1.03	2.33; 2.25; 0.93; 1.05	2.78; 2.64; 2.72; 1.08	2.67; 1.04; 1.08; 1.11
3.0	1.16; 1.24; 0.98; 1.11	1.19; 1.26; 0.97; 1.10	2.98; 2.85; 1.30; 1.17	2.34; 1.25; 1.18; 1.22
4.5	1.23; 1.31; 1.02; 1.15	1.26; 1.33; 1.01; 1.14	3.12; 1.37; 1.42; 1.29	1.29; 1.35; 1.28; 1.05
6.0	1.25; 1.33; 1.01; 1.14	1.29; 1.37; 1.02; 1.15	1.34; 1.41; 1.46; 1.33	1.35; 1.41; 1.35; 1.39

It is also important to focus on $E[d(\Theta, t^*)]$ in conditions $0.6S(0) < S(T(\Theta, t^*)) < S(0)$. We found, that in markets with very low long run variance is $b(\Theta, t^*)$ not differ much in different Θ.Estimated $b(\Theta, t^*)$ are between $9.0 \cdot 10^{-4}$ and $12.0 \cdot 10^{-4}$. Values on this levels mean, that $S(T)$ decline of 1.00 in interval between $0.6S(0)$ and $S(0)$ causes decline of $9.0 \cdot 10^{-4} - 12.0 \cdot 10^{-4}$ in $E[d(\Theta, t^*)]$. With bigger differences in $c(\Theta, t^*)$ estimation of $b(\Theta, t^*)$ also rises. Differences among $b(\Theta, t^*)$ for market characterize with $\overline{v} = 0.06225$; 0.1225, 0.2025 are much more variable then in market $\overline{v} = 0.0225$. For $\overline{v} = 0.0225$ is estimation of $b(\Theta, t^*)$ mostly between $11.0 \cdot 10^{-4}$ and $17.0 \cdot 10^{-4}$. For markets

with environment of high volatilities we found, that $b(\Theta, t^*)$ values are mostly between $12.0 \cdot 10^{-4}$ and $26.0 \cdot 10^{-4}$. Comparison of market with $\overline{v} = 0.0225$ shows that for catching optimal strategy one have to risk more in environment characterized with higher deterministic volatility.
Estimation of intercept $a(\Theta, t^*)$ characterizes some average terms of $d(\Theta, t^*)$ in bear market. These values are negative and for optimal strategies decline to deeper negative values in market environment with higher volatilities (long run variances). Intervals of $a(\Theta, t^*)$ and $b(\Theta, t^*)$ are in Table 9.

Table 9. $a(\Theta, t^*)$ and $b(\Theta, t^*)$ in different $\overline{v}$

$\overline{v}$	$b(\Theta, t^*)$	$a(\Theta, t^*)$
0.0225	$6{,}78 \cdot 10^{-4} - 14{,}01 \cdot 10^{-4}$	$-13{,}31 \cdot 10^{-2} - 6{,}05 \cdot 10^{-2}$
0.0625	$10{,}60 \cdot 10^{-4} - 18{,}33 \cdot 10^{-4}$	$-16{,}67 \cdot 10^{-2} - 8{,}97 \cdot 10^{-2}$
0.1225	$11{,}01 \cdot 10^{-4} - 20{,}68 \cdot 10^{-4}$	$-25{,}50 \cdot 10^{-2} - 9{,}20 \cdot 10^{-2}$
0.2025	$11{,}91 \cdot 10^{-4} - 32{,}44 \cdot 10^{-4}$	$-28{,}03 \cdot 10^{-2} - 11{,}01 \cdot 10^{-2}$

Another approach for the strategies evaluation is comparison of $d(\Theta, t^*)$ in different average price in interval $0, T$. We found that if average of price is under $S(T)$, then $d(\Theta, t^*)$ is positive in situations defined as $S(T) < S(0)$ or $S(T) < 1.1S(0)$. For optimization purposes ratio m have been constructed as:

$$\bar{t}^*(\Theta) = \mathrm{m} = \frac{\mathrm{E}[\mathrm{d}(\Theta, t^*, \Phi)]}{\sigma(\mathrm{d}(\Theta, t^*, \backslash\mathrm{Phi}))} \quad (27)$$

where $E[d(\Theta, t_1, t_2, \Phi)]$ is mean of differences between the active and basic strategy, $\sigma(d(\Theta, t_1, t_2, \Phi))$ is standard deviation of these differences and $\Phi = \ln\left(\frac{\bar{S}}{S(T)}\right)$ where $\bar{S}$ is average price. The best strategy $\bar{t}^*$ can be found by solving following optimization problem:

$$\bar{t}^*(\Theta, \mathrm{t}_1, \mathrm{t}_2, \Phi) = \arg\max_{t_1, t_2} m(\Theta, t_1, t_2, \Phi) \quad (28)$$

In market $\overline{v} = 0{,}0625$ (see Table 10) and market situation, when $\Phi < -0{,}15$ we found that in case of $S(T) < 1{,}1S(0)$ is proposed strategy with active put option buying much more effective, because of positive expected $d(\Theta, t^*)$. In market environment with small ξ and θ the best returns are observed. Again in this comparison we can see negative relation between differences of strategies and risk free rate. Proposed strategy can profit not only from price $S(T)$ but also source of profit is path of stock price. If managers can predict behaviour of stock price movements in average terms, then active strategy can be source of additional return. For comparison, we show results in the same market $\overline{v} = 0.0625$ when $-0.075 < \Phi < -0.15$. $E[d(\Theta, t^*)]$ and $m(\Theta \ldots)$ are smaller (see Table 11). Trading with average is also connected to Asian options. One can construct strategy with Asian option and change exposure to average in active strategy. Interesting are also optimal strategies in these cases. Of course, investor or manager can only predict average and path of the stock. It is important to know, that active strategy works with expectations and a success of these expectation depends on manager skills. In case of $-0.075 < \Phi < -0.15$ optimal strategy is in major environment defined with long $(125-150)$ or middle $(75-100)$ maturities for both t_1 and t_2. That means only little activity, which is not possible to capture average of asset price. This strategies are also evidence, that only few positions in options during period of temporary bear market can cause profit related to average prices.

Table 10. $E[d(\Theta, t^*, \Phi)]$ and $\sigma(d)$ for $\bar{v} = 0.0625$, $\Phi < -0{,}15$

$E[d(\Theta, t_1, t_2, \Phi)$				
$\theta\backslash\xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	8.84; 8.70; 8.57; 8.46	9.82; 7.52; 7.43; 7.35	9.04; 8.93; 8.83; 8.75	8.91; 8.81; 8.70; 8.60
3.0	9.11; 9.03; 8.93; 8.83	8.42; 8.92; 8.82; 8.74	7.30; 7.22; 7.14; 7.05	9.28; 9.13; 9.01; 8.90
4.5	10.03; 9.93; 9.81; 9.69	8.12; 8.02; 7.92; 7.81	7.83; 7.77; 7.72; 7.66	6.12; 6.03; 5.94; 5.88
6.0	9.18; 9.04; 8.92; 8.79	10.30; 10.18; 10.07; 9.98	8.36; 8.25; 8.15; 8.04	6.49; 6.43; 6.37; 6.32
$\sigma[d(\Theta, t_1, t_2, \Phi)$				
$\theta\backslash\xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	6.35; 6.21; 6.07; 5.92	6.59; 4.98; 4.81; 4.65	5.12; 5.03; 4.93; 4.79	6.07; 5.88; 5.70; 5.52
3.0	5.98; 5.87; 5.76; 5.65	4.67; 4.92; 4.83; 4.72	5.31; 5.14; 4.98; 4.82	6.25; 6.08; 5.89; 5.70
4.5	5.16; 5.06; 4.97; 4.86	4.34; 4.20; 4.07; 3.94	4.83; 4.74; 4.63; 4.53	4.91; 4.79; 4.67; 4.55
6.0	4.85; 4.71; 4.56; 4.41	4.64; 4.60; 4.53; 4.41	4.28; 4.14; 4.01; 3.88	4.99; 4.83; 4.69; 4.54

Table 11. $E[d(\Theta, t^*, \Phi)]$ and $\sigma(d)$ for $\bar{v} = 0.0625$, $-0{,}075 < \Phi < -0{,}15$

$E[d(\Theta, t_1, t_2, \Phi)$				
$\theta\backslash\xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	5.21; 4.42; 4.43; 4.46	3.89; 4.28; 4.29; 4.33	3.69; 3.56; 3.56; 3.59	3.43; 4.60; 4.49; 4.43
3.0	4.97; 4.90; 5.24; 5.18	4.47; 4.24; 4.25; 4.29	3.91; 3.88; 3.88; 3.89	5.11; 4.95; 4.86; 4.80
4.5	5.29; 4.53; 4.53; 4.55	4.80; 4.58; 4.58; 4.60	5.07; 4.61; 4.57; 4,59	4,55; 4,47; 4,43; 4,40
6.0	4.92; 4.86; 4,50; 4.52	4,33; 4.27; 4.25; 5.15	4.40; 4.34; 4.30; 4.28	4.58; 4.52; 4.92; 4.83

$\sigma[d(\Theta, t_1, t_2, \Phi)$				
$\theta \backslash \xi$	$\xi = 0{,}15$	$\xi = 0{,}25$	$\xi = 0{,}35$	$\xi = 0{,}45$
1.5	4.86; 4.12; 4.09; 4.02	3.88; 4.22; 4.16; 4.09	4.24; 4.10; 4.06; 3.99	4.04; 5.38; 5.28; 5.17
3.0	4.27; 4.18; 4.42; 4.35	4.24; 4.02; 3.98; 3.90	3.88; 3.78; 3.69; 3.59	5.05; 4.94; 4.85; 4.75
4.5	4,60; 3.87; 3.84; 3.77	4.17; 3.91; 3.88; 3.82	4.47; 3.99; 3.90; 3.83	4.38; 4.30; 4.17; 4.04
6.0	4.38; 4.27; 3.92; 3.85	3.79; 3.73; 3.65; 4.35	3.74; 3.64; 3.54; 3.44	4.29; 4.20; 4.50; 4.43

6. Conclusion

Proposed strategies with active buying of put options with 2 types of option maturity represent alternative to the basic hedging strategy with put option. These kinds of strategies are in average better than basic strategy in strong bull market. On the other hand, price for these additional expected returns are negative differences between active and the basic strategy in bear market situation. We found best strategies with minimizing ratio of expected additional return and price for this return, which is defined in the work. Although there is no general rule for description of relation between market environment and choice of strategy, most strategies include options buying with longer maturity in one or both types. In optimal strategies, expected optimal addition return can rise with risk free rate decline and in environment with greater long run variance. Proposed strategy with active buying of put options can be also better in bear or neutral direction moving market but only if average of prices is below the price in the end of investing time interval. In case of big negative difference between average and price, proposed strategies returns are 6 to 10 % above returns of basic the strategy.

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