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VECTOR SIMILARITY MEASURES OF HESITANT FUZZY SETS AND THEIR MULTIPLE ATTRIBUTE DECISION MAKING

***Abstract**: Three similarity measures between hesitant fuzzy sets are proposed based on the extension of the Jaccard, Dice, and cosine similarity measures in the vector space. Then multiple attribute decision making methods based on these three similarity measures are established to solve the hesitant fuzzy multiple attribute decision-making problem, in which the evaluated values of alternatives with respect to attributes are expressed by hesitant fuzzy elements. Through the weighted similarity measures between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best alternative can be easily identified as well. Finally, a practical example about investment alternatives is given to demonstrate the practicality and effectiveness of the developed approaches. The decision results demonstrate that the Jaccard and Dice similarity measures for the hesitant fuzzy multiple attribute decision-making problem are better than the cosine similarity measure in the similarity identification.*

***Keywords**: vector similarity measure, Jaccard similarity measure, Dice similarity measure, cosine similarity measure, hesitant fuzzy set, multiple attribute decision making.*

JEL Classification: C44, D8, D81

1. Introduction

Dealing with uncertainty is always a challenging problem, and different tools have been proposed to deal with it. In order to better understand the uncertainty of the objective world and thus being able to explain it, the fuzzy set theory (Zadeh, 1965) has been extended to many other forms, such as interval-valued fuzzy set (Zadeh, 1975) and intuitionistic fuzzy set (Atanassov, 1986). When defining the membership degree of an element to a set, however, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possible values, but because we have a set of possible values (Torra, 2010). The above generalization forms of fuzzy set cannot express this kind of uncertainty effectively. To deal with this situation, Torra and Narukawa (2009) and Torra (2010) firstly proposed a hesitant fuzzy set as another generalization form of fuzzy set. The characteristic of a hesitant fuzzy set is that it permits the membership degree of an element to a given set with a few different values, which can arise in a group decision making problem. For example, three experts discuss the degrees that an alternative should satisfy a criterion. The first expert provides 0.7, the second expert provides 0.8 and the third expert provides 0.9; however, these three experts cannot persuade each other, thus the degrees that the alternative satisfies the criterion can be represented by a hesitant fuzzy set {0.7, 0.8, 0.9}. Therefore, the hesitant fuzzy set can deal with all possible opinions in the group members and provide an intuitive description on the differences among the group members.

Recently, the hesitant fuzzy set has received more and more attention since its appearance. Xia and Xu (2011) proposed hesitant fuzzy information aggregation techniques and their application in decision making. Then, Xu and Xia (2011a) introduced a variety of distance measures for hesitant fuzzy sets and their corresponding similarity measures. Meantime, Xu and Xia (2011b) defined the distance and correlation measures for hesitant fuzzy information, and then discuss their properties in detail. Gu et al. (2011) investigated the evaluation model for risk investment with hesitant fuzzy information. They utilized the hesitant fuzzy weighted averaging operator to aggregate the hesitant fuzzy information corresponding to each alternative, and then rank the alternatives and select the most desirable one(s) according to the score function. Wei (2012) developed some prioritized aggregation operators for aggregating hesitant fuzzy information, and then applied them to develop some models for hesitant fuzzy multiple attribute decision making problems, in which the attributes are in different priority level. Rodriguez et al. (2012) introduced the concept of a hesitant fuzzy linguistic term set to provide a linguistic and computational basis to increase the richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars by using comparative terms, and then presented a multicriteria linguistic decision-making model in which experts provide their assessments by eliciting linguistic expressions. Chen et al (2013a) proposed some correlation coefficient formulas for hesitant fuzzy sets and apply them to clustering analysis under hesitant fuzzy environments. Xu et al. (2013) developed some hesitant fuzzy aggregation operators with the aid of quasi-arithmetic means and applied them to group decision making problems. Furthermore, Ye (2013a) introduced a trapezoidal hesitant fuzzy set as a generalization of hesitant fuzzy set and presented a multicriteria decision-making using expected values in trapezoidal hesitant fuzzy setting.

From above review, we can see that the hesitant fuzzy set is a very useful tool to deal with uncertainty and decision making problems. More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed in recent years [(Qian et al. (2013), Chen et al. (2013a), (2013b), Ye (2013a)]. Then, similarity measure is one of the most broadly applied indices in many fields [Wu and Mendel (2008), (2009), Ye (2011), (2012), (2013b)] and also an important measure in data analysis and classification, pattern recognition, decision making and so on. Under the hesitant fuzzy environment, current methods only study the similarity measures based on the distances of hesitant fuzzy sets. However, the vector similarity measures play an important role in pattern recognition (Ye, 2011) and fuzzy multiple attribute decision making problems (Ye, 2012, 2013b). Therefore, in this paper, three vector similarity measures between hesitant fuzzy sets are proposed by the extension of the Jaccard, Dice, and cosine similarity measures. Then, we utilize these vector similarity measures to solve hesitant fuzzy multiple attribute decision making problems in which attribute values take the form of hesitant fuzzy elements. To do so, the remainder of this paper is set out as follows. In Section 2, we introduce some basic concepts related to hesitant fuzzy sets and the Jaccard, Dice, and cosine similarity measures in the vector space. In Section 3, we propose three vector similarity measures: the Jaccard, Dice, and cosine similarity measures of hesitant fuzzy sets. In Section 4, we establish decision-making methods based on these vector similarity measures to deal with hesitant fuzzy multiple attribute decision making problems in which the evaluated values of alternatives under attributes take the form of hesitant fuzzy elements. In Section 5, a practical example about investment alternatives is given to demonstrate the practicality and effectiveness of the developed approaches. We conclude the paper and give some remarks and future work in Section 6.

2. Preliminaries

2.1. Hesitant fuzzy sets

Torra and Narukawa (2009) and Torra (2010) recently put forward the concept of hesitant fuzzy set which is defined as follows:

Definition 1 [Torra and Narukawa (2009), Torra (2010)]. Let X be a fixed set, a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\},$$

where $h_A(x)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to A. For convenience, we call $h_A(x)$ a hesitant fuzzy element (Xia and Xu, 2011).

Definition 2 [Torra and Narukawa (2009), Torra (2010)]. Given a hesitant fuzzy element h, its lower and upper bounds are defined as $h^-(x) = \min h(x)$ and $h^+(x) = \max h(x)$, respectively.

Definition 3 [Torra and Narukawa (2009), Torra (2010)]. Given a hesitant fuzzy element h, $A_{env}(h)$ is called the envelope of h which is represented by $(h^-, 1 - h^+)$, with the lower bound h^- and upper bound h^+.

From this definition, Torra and Narukawa (2009) and Torra (2010) gave the relation between a hesitant fuzzy set and an intuitionistic fuzzy set, i.e., $A_{env}(h)$

is defined as $\{< x, \mu(x), v(x) >\}$ with μ and v defined by $\mu(x) = h^{-}(x)$, $v(x) = 1 - h^{+}(x)$, $x \in X$.

Then, the values of a hesitant fuzzy element are usually given a disorder, so we arrange them in a decreasing order. For a hesitant fuzzy element h, let σ: (1, 2, …, n) → (1, 2, …, n) be a permutation satisfying $h_{\sigma(j)} \geq h_{\sigma(j+1)}$ for j = 1, 2, …, n − 1 and $h_{\sigma(j)} \in h$ be the jth largest value in h.

Definition 4 (Xia and Xu, 2011). For a hesitant fuzzy element h, $s(h) = \frac{1}{l}\sum_{j=1}^{l} h_{\sigma(j)}$ is called the score function of h, where l is the number of the elements in h. For two hesitant elements h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

2.2. Vector similarity measures

Let $\boldsymbol{X} = (x_1, x_2,..., x_n)$ and $\boldsymbol{Y} = (y_1, y_2,..., y_n)$ be the two vectors of length n where all the coordinates are positive. The Jaccard index of these two vectors (measuring the “similarity” of these vectors) (Jaccard, 1901) is defined as

$$J(\boldsymbol{X},\boldsymbol{Y}) = \frac{\boldsymbol{X}\cdot\boldsymbol{Y}}{\|\boldsymbol{X}\|_2^2 + \|\boldsymbol{Y}\|_2^2 - \boldsymbol{X}\cdot\boldsymbol{Y}} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i y_i}, \tag{1}$$

where $\boldsymbol{X}\cdot\boldsymbol{Y} = \sum_{i=1}^{n} x_i y_i$ is the inner product of the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ and $\|\boldsymbol{X}\|_2 = \sqrt{\sum_{i=1}^{n} x^2}$ and $\|\boldsymbol{Y}\|_2 = \sqrt{\sum_{i=1}^{n} y^2}$ are the Euclidean norms of $\boldsymbol{X}$ and $\boldsymbol{Y}$ (also called the L_2 norms).

Then, the Dice similarity measure (Dice, 1945) is defined as follows:

$$D(\boldsymbol{X},\boldsymbol{Y}) = \frac{2\boldsymbol{X}\cdot\boldsymbol{Y}}{\|\boldsymbol{X}\|_2^2 + \|\boldsymbol{Y}\|_2^2} = \frac{2\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}. \tag{2}$$

Moreover, a cosine similarity measure (Salton and McGill, 1987) is defined as the inner product of two vectors divided by the product of their lengths. This is nothing but the cosine of the angle between two vectors. The cosine similarity measure (angular coefficient) can be defined as follows (Salton and McGill, 1987):

$$C(\boldsymbol{X},\boldsymbol{Y}) = \frac{\boldsymbol{X}\cdot\boldsymbol{Y}}{\|\boldsymbol{X}\|_2 \|\boldsymbol{Y}\|_2} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}}. \tag{3}$$

These three formulas are similar in the sense that they take values in the interval [0, l]. The Jaccard and Dice formulas are undefined if $x_i = y_i = 0$ for i = 1, 2, …, n. However, the cosine formula is undefined if $x_i = 0$ and/or $y_i = 0$ for $i = 1, 2, \ldots, n$, and then we let the cosine similarity measure value be zero when $x_i = 0$ and/or $y_i = 0$ for $i = 1, 2, \ldots, n$.

These vector similarity measures for the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy the following properties:

(1) $J(\boldsymbol{X}, \boldsymbol{Y}) = J(\boldsymbol{Y}, \boldsymbol{X})$, $D(\boldsymbol{X}, \boldsymbol{Y}) = D(\boldsymbol{Y}, \boldsymbol{X})$, and $C(\boldsymbol{X}, \boldsymbol{Y}) = C(\boldsymbol{Y}, \boldsymbol{X})$;

(2) $0 \le J(\boldsymbol{X}, \boldsymbol{Y}), D(\boldsymbol{X}, \boldsymbol{Y}), C(\boldsymbol{X}, \boldsymbol{Y}) \le 1$;

(3) $J(\boldsymbol{X}, \boldsymbol{Y}) = D(\boldsymbol{X}, \boldsymbol{Y}) = C(\boldsymbol{X}, \boldsymbol{Y}) = 1$, if $\boldsymbol{X} = \boldsymbol{Y}$.

3. Vector similarity measures of hesitant fuzzy sets

Let A and B be two hesitant fuzzy sets on a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ denoted as $A = \{\langle x_i, h_A(x_i)\rangle \mid x_i \in X\}$ and $B = \{\langle x_i, h_B(x_i)\rangle \mid x_i \in X\}$, respectively. Then, we can consider any two hesitant fuzzy elements $h_A(x_i)$ and $h_B(x_i)$ in A and B as two vectors. According to the aforementioned similarity measures in the vector space, we can define the Jaccard similarity measure, Dice similarity measure, and cosine similarity measure between hesitant fuzzy sets A and B as follows:

Definition 5. Let A and B be two hesitant fuzzy sets on a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ denoted as $A = \{\langle x_i, h_A(x_i)\rangle \mid x_i \in X\}$ and $B = \{\langle x_i, h_B(x_i)\rangle \mid x_i \in X\}$, respectively. Then, the Jaccard similarity measure, Dice similarity measure, and cosine similarity measure between A and B are defined, respectively, as follows:

$$J_{HFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) h_{B\sigma(j)}(x_i)}{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2 + \sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2 - \sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) h_{B\sigma(j)}(x_i)}, \tag{4}$$

$$D_{HFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) h_{B\sigma(j)}(x_i)}{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2 + \sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2}, \tag{5}$$

$$C_{HFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) h_{B\sigma(j)}(x_i)}{\sqrt{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2}\sqrt{\sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2}}. \tag{6}$$

Here, $l_i = \max\{l(h_A(x_i)), l(h_B(x_i))\}$ for each x_i in X, where $l(h_A(x_i))$ and $l(h_B(x_i))$ represent the number of values in $h_A(x_i)$ and $h_B(x_i)$, respectively. When $l(h_A(x_i)) \neq l(h_B(x_i))$, one can make them have the same number of elements through adding some elements to the hesitant fuzzy element which has less number of elements. According to the pessimistic principle, the smallest element will be added. Therefore, if $l(h_A(x_i)) < l(h_B(x_i))$, $h_A(x_i)$ should be extended by adding the minimum value in it until it has the same length as $h_B(x_i)$. This idea has been successfully applied to distance and similarity measures for hesitant fuzzy sets (Xu and Xia, 2011a).

According to the aforementioned properties of three vector similarity measures, it is obvious that these vector similarity measures for the hesitant fuzzy sets A and B also satisfy the following properties (Ye, 2011, 2012, 2013b):

(P1) $J_{HFS}(A, B) = J_{HFS}(B, A)$, $D_{HFS}(A, B) = D_{HFS}(B, A)$, and $C_{HFS}(A, B) = C_{HFS}(B, A)$;

(P2) $0 \leq J_{HFS}(A, B), D_{HFS}(A, B), C_{HFS}(A, B) \leq 1$;

(P3) $J_{HFS}(A, B) = D_{HFS}(A, B) = C_{HFS}(A, B) = 1$, if and only if $A = B$.

Proof.

(P1) It is obvious that the property is true.

(P2) It is obvious that the property is true according to the inequality $a^2 + b^2 \geq 2ab$ for Eqs. (4) and (5), and cosine value for Eq. (6).

(P3) When $A = B$, there is $h_A(x_i) = h_B(x_i)$ for $i = 1, 2, \ldots, n$. So there are $J_{HFS}(A, B) = 1$, $D_{HFS}(A, B) = 1$, and $C_{HFS}(A, B) = 1$. When $J_{HFS}(A, B) = 1$, $D_{HFS}(A, B) = 1$, and $C_{HFS}(A, B) = 1$, there is $h_A(x_i) = h_B(x_i)$ for $i = 1, 2, \ldots, n$. So there is $A = B$.

For example, let A and B be two hesitant fuzzy sets in $X = \{x_1, x_2, x_3\}$, and then

$$A = \{\langle x_1, \{0.6, 0.4\}\rangle, \langle x_2, \{0.8, 0.7, 0.5\}\rangle, \langle x_3, \{0.5, 0.4, 0.3\}\rangle\},$$

$$B = \{\langle x_1, \{0.5, 0.3\}\rangle, \langle x_2, \{0.7, 0.6, 0.5\}\rangle, \langle x_3, \{0.7, 0.5, 0.4\}\rangle\}.$$

By using Eq. (4), we calculate the Jaccard similarity measure:

$$\begin{aligned} J_{HFS}(A, B) = \frac{1}{3}\Bigg(& \frac{(0.6\times 0.5 + 0.4\times 0.3)}{0.6^2 + 0.4^2 + 0.5^2 + 0.3^2 - (0.6\times 0.5 + 0.4\times 0.3)} \\ & + \frac{(0.8\times 0.7 + 0.7\times 0.6 + 0.5\times 0.5)}{0.8^2 + 0.7^2 + 0.5^2 + 0.7^2 + 0.6^2 + 0.5^2 - (0.8\times 0.7 + 0.7\times 0.6 + 0.5\times 0.5)} \\ & + \frac{(0.5\times 0.7 + 0.4\times 0.5 + 0.3\times 0.4)}{0.5^2 + 0.4^2 + 0.3^2 + 0.7^2 + 0.5^2 + 0.4^2 - (0.5\times 0.7 + 0.4\times 0.5 + 0.3\times 0.4)}\Bigg) \\ = {} & 0.9521 \end{aligned}$$

With Eq. (5), we obtain the Dice similarity measure:

$$D_{HFS}(A,B)=\frac{1}{3}\left(\frac{2\times(0.6\times 0.5+0.4\times 0.3)}{0.6^2+0.4^2+0.5^2+0.3^2}+\frac{2\times(0.8\times 0.7+0.7\times 0.6+0.5\times 0.5)}{0.8^2+0.7^2+0.5^2+0.7^2+0.6^2+0.5^2}\right.$$
$$\left.+\frac{2\times(0.5\times 0.7+0.4\times 0.5+0.3\times 0.4)}{0.5^2+0.4^2+0.3^2+0.7^2+0.5^2+0.4^2}\right)=0.9753$$

Then, we use Eq. (6) to obtain the cosine similarity measure:

$$C_{HFS}(A,B)=\frac{1}{3}\left(\frac{(0.6\times 0.5+0.4\times 0.3)}{\sqrt{0.6^2+0.4^2}\times\sqrt{0.5^2+0.3^2}}+\frac{(0.8\times 0.7+0.7\times 0.6+0.5\times 0.5)}{\sqrt{0.8^2+0.7^2+0.5^2}\times\sqrt{0.7^2+0.6^2+0.5^2}}\right.$$
$$\left.+\frac{(0.5\times 0.7+0.4\times 0.5+0.3\times 0.4)}{\sqrt{0.5^2+0.4^2+0.3^2}\times\sqrt{0.7^2+0.5^2+0.4^2}}\right)=0.9987$$

In practical applications, the elements x_i (i = 1, 2, . . . , n) in an universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ have different weights. Let $w = (w_1, w_2, . . . ,w_n)^T$ be the weight vector of x_i (i = 1, 2, . . . , n) with $w_i \geq 0$, i = 1,2,. . . , n, and $\sum_{i=1}^{n} w_i = 1$. Then, we further extend the vector measure formulas given in Eqs. (4), (5) and (6) as follows:

$$WJ_{HFS}(A,B)=\sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)h_{B\sigma(j)}(x_i)}{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2+\sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2-\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)h_{B\sigma(j)}(x_i)}, \quad (7)$$

$$WD_{HFS}(A,B)=\sum_{i=1}^{n} w_i \frac{2\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)h_{B\sigma(j)}(x_i)}{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2+\sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2}, \quad (8)$$

$$WC_{HFS}(A,B)=\sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)h_{B\sigma(j)}(x_i)}{\sqrt{\sum_{j=1}^{l_i}\left[h_{A\sigma(j)}(x_i)\right]^2}\sqrt{\sum_{j=1}^{l_i}\left[h_{B\sigma(j)}(x_i)\right]^2}}. \quad (9)$$

It can be seen that if $w = (1/n, 1/n, . . ., 1/n)^T$, then Eqs. (7), (8) and (9) reduced to Eqs. (4), (5) and (6), respectively. Similarly, the three weighted similarity measures also satisfy the following three properties:

(P1) $WJ_{HFS}(A, B) = WJ_{HFS}(B, A)$, $WD_{HFS}(A, B) = WD_{HFS}(B, A)$, and $WC_{HFS}(A, B) = WC_{HFS}(B, A)$;

(P2) $0 \le WJ_{HFS}(A, B)$, $WD_{HFS}(A, B)$, $WC_{HFS}(A, B) \le 1$;

(P3) $WJ_{HFS}(A, B) = WD_{HFS}(A, B) = WC_{HFS}(A, B) = 1$, if and only if $A = B$.

Similar to the previous proof method, we can prove the properties (P1)–(P3).

4. Multiple attribute decision making with hesitant fuzzy information

In this section, we shall utilize the vector similarity measures of hesitant fuzzy sets to multiple attribute decision making with hesitant fuzzy information.

For a multiple attribute decision making problem with hesitant fuzzy information, let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a discrete set of attributes. If the decision makers provide several values for the alternative A_i ($i = 1, 2, \ldots, m$) under the attribute C_j ($j = 1, 2,\ldots, n$), these values can be considered as a hesitant fuzzy element h_{ij} ($j = 1, 2,\ldots, n$; $i = 1, 2,\ldots, m$). Therefore, we can elicit a hesitant fuzzy decision matrix $D = (h_{ij})_{m\times n}$, where h_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) is in the form of hesitant fuzzy elements.

In multiple attribute decision making environments, the concept of ideal point has been used to help the identification of the best alternative in the decision set. Although the ideal alternative does not exist in real world, it does provide a useful theoretical construct to evaluate alternatives. Therefore, we define each value in each ideal hesitant fuzzy element h_j^* for the ideal alternative $A^* = \{\langle C_j, h_j^*\rangle \mid C_j \in C\}$ as $h^*_{j\sigma(k)} = 1$ for $k = 1, 2, \ldots, l_j$, where l_j is the number of values in h_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).

The weighting vector of attributes for the different importance of each attribute is given as $w = (w_1, w_2,\ldots, w_n)^{\mathrm{T}}$, where $w_j \ge 0$, $i = 1, 2, \ldots, m$, and $\sum_{j=1}^{n} w_j = 1$.

Then, we utilize the three weighted vector similarity measures for multiple attribute decision making problems with hesitant fuzzy information, which can be described as follows:

Step 1. Calculate the one of three weighted vector similarity measures between an alternative A_i ($i = 1, 2, \ldots, m$) and the ideal alternative A^* by using one of the following three formulas:

$$MJ_{HFS}(A_i, A^*) = \sum_{j=1}^{n} w_j \frac{\sum_{k=1}^{l_i} h_{ij\sigma(k)} h^*_{j\sigma(k)}}{\sum_{k=1}^{l_i}\left[h_{ij\sigma(k)}\right]^2 + \sum_{k=1}^{l_i}\left[h^*_{j\sigma(k)}\right]^2 - \sum_{k=1}^{l_i} h_{ij\sigma(k)} h^*_{j\sigma(k)}}, \tag{10}$$

$$WD_{HFS}(A_i, A^*) = \sum_{j=1}^{n} w_j \frac{2\sum_{k=1}^{l_i} h_{ij\sigma(k)} h^*_{j\sigma(k)}}{\sum_{k=1}^{l_i}\left[h_{ij\sigma(k)}\right]^2 + \sum_{k=1}^{l_i}\left[h^*_{j\sigma(k)}\right]^2}, \quad (11)$$

$$WC_{HFS}(A_i, A^*) = \sum_{j=1}^{n} w_j \frac{\sum_{k=1}^{l_i} h_{ij\sigma(k)} h^*_{j\sigma(k)}}{\sqrt{\sum_{k=1}^{l_i}\left[h_{ij\sigma(j)}\right]^2}\sqrt{\sum_{k=1}^{l_i}\left[h^*_{j\sigma(k)}\right]^2}}. \quad (12)$$

Step 2. Rank the alternatives and select the best one(s) in accordance with the weighted similarity measures.

Step 3. End.

5. Practical example

A practical example about investment alternatives for a multiple attribute decision-making problem adapted from (Herrera and Herrera-Viedma, 2000; Ye, 2013b) is used as the demonstration of the applications of the proposed decision-making method in a realistic scenario. There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the following three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environmental impact. The attribute weight vector is given as w = (0.35, 0.25, 0.40)^T. The four possible alternatives A_i (i = 1, 2, 3, 4) are to be evaluated using the hesitant fuzzy information by three decision makers under the three attributes C_j (j = 1, 2, 3), as listed in the following hesitant fuzzy decision matrix D:

$$D = \begin{bmatrix} \{0.5,0.4,0.3\} & \{0.6,0.4\} & \{0.3,0.2,0.1\} \\ \{0.7,0.6,0.4\} & \{0.7,0.6\} & \{0.7,0.6,0.4\} \\ \{0.6,0.4,0.3\} & \{0.6,0.5\} & \{0.6,0.5\} \\ \{0.8,0.7,0.6\} & \{0.7,0.6\} & \{0.4,0.3\} \end{bmatrix}.$$

Then, we utilize the developed approaches to obtain the ranking order of the alternatives and the most desirable one(s).

Step 1. By applying Eq. (10) or Eq. (11) or Eq. (12), we can obtain the computing results of all the alternatives as shown in Table 1.

Step 2. We can rank the alternatives in accordance with the weighted similarity measures, which are shown in Table 1.

Therefore, from Table 1 we can see that the two ranking orders of them are the same and the alternative A_2 is the best choice by using the Jaccard similarity measure and Dice similarity measure, and then these results are in agreement with the ones obtained in (Ye, 2013b). By using cosine similarity measure, however, we have another ranking order and the alternative A_4 is the best choice. The ranking orders may be different according to different measures because each algorithm focuses on different point of view.

Table 1. Decision results of different similarity measures

	$WJ_{HFS}(A_i, A^*)$	$WD_{HFS}(A_i, A^*)$	$WC_{HFS}(A_i, A^*)$
A_1	0.4416	0.5913	0.9584
A_2	0.7616	0.8640	0.9817
A_3	0.6623	0.7944	0.9838
A_4	0.6881	0.7950	0.9930
Ranking order	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_4 \succ A_3 \succ A_2 \succ A_1$

Furthermore, the results of Table 1 demonstrated that the Jaccard and Dice similarity measures for the hesitant fuzzy multiple attribute decision-making problem are better than the cosine similarity measure in the similarity identification and have a strong influence on the change of values in hesitant fuzzy elements.

The example clearly indicates that the proposed decision-making methods are simple and effective under hesitant fuzzy environments and the true need of new types of models based on the vector similarity measures of hesitant fuzzy sets for dealing with hesitant fuzzy multiple attribute decision making problems.

The hesitant fuzzy set is a comprehensive set encompassing its membership degree represented by a set of possible values. Therefore, it has the desirable characteristics and advantages of its own and appears to be a more flexible method to be evaluated in hesitant ways according to the practical demands than existing regular type-1 and type-2 fuzzy sets. Then, the proposed decision-making methods can automatically take into account much more information than existing regular fuzzy decision-making methods and show the differences of the evaluation data given by different experts or decision makers, which allow the proposed methods to have more wide practical application potentials.

6. Conclusions

In this paper, we proposed the Jaccard, Dice, and cosine similarity measures between two hesitant fuzzy sets. Based on the three similarity measures of hesitant fuzzy sets, we established the multiple attribute decision-making methods, in which the evaluated values of alternatives under attributes take the form of hesitant fuzzy elements. Then, one of three weighted similarity measures between each alternative and the ideal alternative was utilized for ranking alternatives and choosing the best one(s). Finally, a practical example for the developed methods was given to select the investment alternatives. The numerical example showed that the proposed methods in this paper are applicable and effective. The decision results demonstrated that the Jaccard and Dice similarity measures are better than the cosine similarity measure in the similarity identification.

As future work, we shall seek for the potential applications of the vector similarity measures between hesitant fuzzy sets, such as pattern recognition and medical diagnosis.

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