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DIVERSITY AND SOLVING A THREE-DIMENSIONAL ALLOCATION PROBLEM

Abstract. *In mathematics and economics, transportation theory is a name given to the study of optimal transportation and allocation of resources. The problem was formalized by the French mathematician Gaspard Monge in 1781. Since, there are very many preoccupations and published results on the formulation and on the solving of diverse transportation problems. Using the weighted diversity measure, some new theoretical and practical results on the possibilities to formulate and to solve certain three-dimensional allocation problems are proposed in this paper.*
Key words: *weighted diversity index, optimal probability distributions, optimal diversification, Principle of Maximum Diversity* (*PMD*) *of Guiaşu and Guiaşu).*

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1. Preliminaries and the problem formulation.

Consider a three-dimensional probabilistic experiment characterized by a set of possible outcomes $[x,y,z]=[(x_i),(y_j),(z_k)]$ and by a joint probability distribution $\pi=(\pi_{ijk})$ denoted by

$$[X,Y,Z]=\begin{bmatrix}(x_i,y_j,z_k)\\(\pi_{ijk})\end{bmatrix};\ (\pi_{ijk})\geq 0,\ \sum_i\sum_j\sum_k \pi_{ijk}=\sum_{i,j,k}\pi_{ijk}=1 \qquad (1.1)$$

with the values $(i,j,k)\in\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\}$, where π_{ijk} is the probability that the three-dimensional outcome (x_i,y_j,z_k) will occur. We note that if we write the joint equality $[X,Y,Z]=[(x_i,y_j,z_j)]$, then we understand the equalities $X=(x_i)$, $Y=(y_j)$ and $Z=(z_k)$ for all values of (i,j,k) and if we write the joint equality $[X,Y,Z]=(x_i,y_j,z_k)$, then we understand the equalities $X=x_i$, $Y=y_j$ and $Z=z_k$ only for a certain three-dimensional value of (i,j,k). Using the three-dimensional probability distribution $\pi=(\pi_{ijk})$ given by (1.1) we can obtain the two-dimensional probability distributions $(\pi_{ij\bullet})$, $(\pi_{i\bullet k})$ and $(\pi_{\bullet jk})$ corresponding to the joint probabilistic experiments (X,Y), (X,Z) and (Y,Z), by the following relations:

$$\pi_{ij\bullet}=\sum_k \pi_{ijk}\ ;\ \pi_{i\bullet k}=\sum_j \pi_{ijk}\ ;\ \pi_{\bullet jk}=\sum_i \pi_{ijk} \tag{1.2}$$

and the one-dimensional probability distributions $(\pi_{i\bullet\bullet})$, $(\pi_{\bullet j\bullet})$ and $(\pi_{\bullet\bullet k})$ corresponding to the one-dimensional probabilistic experiments X, Y and Z, given by the following relations:

$$\pi_{i\bullet\bullet}=\sum_j\sum_k \pi_{ijk}\ ;\ \pi_{\bullet j\bullet}=\sum_i\sum_k \pi_{ijk}\ ;\ \pi_{\bullet\bullet k}=\sum_i\sum_j \pi_{ijk} \tag{1.3}$$

for all $(i,j,k)\in\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\}$ with the corresponding practical signification. Let $u=(u_{ijk})$ be a set of strict positive numbers, called ***weights,*** of the probabilistic distribution $\pi=(\pi_{ijk})$ of the experiment (1.1) or, generally, weights of the joint situation $[(x_i,y_j,z_k);(\pi_{ijk})]$. As a natural extension of the weighted diversity measure introduced by Guiaşu and Guiaşu (2003) for a two-dimensional probabilistic experiment (ecosystem) we can give the following definition.

Definition 1.1. *If we consider the probabilistic experiment (1.1), then the number:*

$$D(\pi;u)=\sum_i\sum_j\sum_k u_{ijk}\pi_{ijk}(1-\pi_{ijk}) \tag{1.4}$$

is called ***weighted diversity index*** *of the three-dimensional probability distribution* $\pi=(\pi_{ijk})$ *of the probabilistic experiment* (1.1) *with the weights* $u=(u_{ijk})$.

Proposition 1.1. *For the weighted diversity index (1.4), we have the inequalities:*

$$0 \le D(\pi;u) \le D_{\max}(\pi;u) = \frac{1}{4}\left[U - \frac{(mnp-2)^2}{V}\right] \tag{1.5}$$

with $D(\pi;u)=0$ *if and only if there is the situation* $\pi_{i_0 j_0 k_0}=1$ *and* $\pi_{ijk}=0$ *for all* (i,j,k), $(i,j,k)\neq(i_0,j_0,k_0)$ *and with the equality* $D(\pi;u)=D_{\max}(\pi;u)$ *if and only if we have the three-dimensional probability distribution* $\pi^0=(\pi^0_{ijk})$ *given by the following numbers* [where $(i,j,k)\in\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\}$]*:*

$$\pi^0_{ijk} = \frac{1}{2}\left[1-\frac{(mnp-2)}{V}v_{ijk}\right];\ v_{ijk}=\frac{1}{u_{ijk}} \tag{1.6}$$

$$U=\sum_i\sum_j\sum_k u_{ijk}\ ;\ V=\sum_i\sum_j\sum_k v_{ijk} \tag{1.7}$$

only for those numbers $u=(u_{ijk})$ *for which* $(\pi_{ijk})\ge 0$ *or the equivalent inequality*:

$$V \ge (mnp-2)\times\max\{v_{ijk}\} \tag{1.8}$$

Consequence. *If* $u_{ijk}=1$ *for all values of* (i,j,k) *then the optimal probability distribution (1.6) is just the uniform three-dimensional probability distribution:*

$$\pi_{ijk}=1/mnp\ ;\ (i,j,k)\in\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\} \tag{1.9}$$

and the maximum value of the diversity index (1.4) is given by the number:

$$D_{\max}=\frac{mnp-1}{mnp} \tag{1.10}$$

Remarks. Generally, the index of diversity (1.4) is directly calculated for certain decisions but, sometimes, it is fixed (pre-established) as a constraint by the type:

$$D(\pi;u)=D_0\,(known)\ ;\ 0<D_0\le D_{\max} \tag{1.11}$$

From the relations (1.5) and (1.11), we can write the following important inequality:

$$UV-(mnp-2)^2-4VD_0\geq 0 \qquad (1.12)$$

Let $q=(q_{ijk})$ be a set of real numbers and an economical indicator $E(\pi;q)$ associated of the experiment (1.1) defined as a mean value of the numbers $q=(q_{ijk})$:

$$E(\pi;q)=\sum_i\sum_j\sum_k \pi_{ijk}q_{ijk} \qquad (1.13)$$

Remark. Generally, for some probabilistic experiments, the indicator $E(\pi;q)$ is calculated for certain conclusions and decisions but, sometimes, this indicator can be fixed or pre-established as a theoretical or practical constraint by the following type:

$$E(\pi;q)=E_0(known)\;;\;\; \min\{q_{ijk}\}\leq E_0\leq \max\{q_{ijk}\} \qquad (1.14)$$

Let $\{\,w_{ijk}\geq 0\,\}$ be a set of positive real numbers with certain economical significations and with the following properties (where: $kno=known$):

$$\sum_k w_{ijk}=w_{ij\bullet}\ (kno)\;;\;\sum_j w_{ijk}=w_{i\bullet k}\ (kno)\;;\;\sum_i w_{ijk}=w_{\bullet jk}\ (kno) \qquad (1.15)$$

$$\sum_j\sum_k w_{ijk}=w_{i\bullet\bullet}\ (kno);\;\;\sum_i\sum_k w_{ijk}=w_{\bullet j\bullet}\ (kno);\;\;\sum_i\sum_j w_{ijk}=w_{\bullet\bullet k}\ (kno) \qquad (1.16)$$

$$\sum_i w_{i\bullet\bullet}=\sum_j w_{\bullet j\bullet}=\sum_k w_{\bullet\bullet k}=W\ (kno) \qquad (1.17)$$

By a general analogy, for example, we can consider the following three-dimensional problem by type transportation (or allocation) in which ***we transport the quantity $\{w_{ijk}\}$ from the source $\{i\}$, to the destination $\{j\}$, at the moment $\{k\}$.*** In this practical situation we have the following significations:

$\{w_{ij\bullet}\}=$ *total received quantity, at all moments, from the source* $\{i\}$ *to the destination* $\{j\}$ *;*

$\{w_{i\bullet k}\}=$ *total demanded quantity, to all destinations, from the source* $\{i\}$ *at the moment* $\{k\}$ *;*

$\{w_{\bullet jk}\}$ = *total offered quantity, from all sources, to the destination* $\{j\}$ *at the moment* $\{k\}$ *;*

$\{w_{i\bullet\bullet}\}$ = *total offered quantity by the source* $\{i\}$ *;*

$\{w_{\bullet j\bullet}\}$ = *total demanded quantity by the destination* $\{j\}$ *;*

$\{w_{\bullet\bullet k}\}$ = *total received quantity at the moment* $\{k\}$ *;*

$\{T\}$ = *total transported quantity from all sources to all destinations at all moments.*

Generally, the relations (1.15) or (1.16) represent the classical constraints for a three-dimensional transportation or allocation problem. If we have the equality (1.17), then we say that the problem is in equilibrium (or balanced). In these conditions, we can easy introduce the following ratios:

$$\pi_{ijk} = \frac{w_{ijk}}{W}; \quad (i, j, k) \in \{1,2,...,m\} \times \{1,2...,n\} \times \{1,2...,p\} \tag{1.18}$$

which represent just a probability distribution by type (1.1) and, using the relations (1.1)-(1.3) and (1.15)-(1.17), we can determine the partial probability distributions (1.2) and (1.3) too. As a result of these analogy, the determination of the numbers (w_{ijk}) represent the determination of the numbers (π_{ijk}) and reciprocally. If the distribution $\pi = (\pi_{ijk})$ is obtained, then the solution of the transportation problem is immediately given by the numbers $(w_{ijk}) = (\pi_{ijk} \times W)$. Also, in the case of a practical problem of transportation, if the numbers $q = (q_{ijk})$ are the costs (profits) per unity of transportation, then the indicator $E(\pi;q)$ given by (1.13) represent the total mean cost (profit) per unity implying a total mean cost (profit) $C_{total} = E(\pi;q) \times W$ of transportation. As a result of these considerations, we can easy formulate the following two non-linear three-dimensional problems, as a concrete situation of the optimal diversification in the case of general allocation problem.

Problem 1. ***Optimal diversification decisions with maximum diversity degree [Optimal probability distributions with maximum diversity degree]***.

For solving the **Problem 1**, we can use the ***Principle of Maximum Diversity*** (***PMD***) of Guiaşu and Guiaşu (2003), according to which: *"from the set of all probability distributions compatible with one or several mean values of one or several random variables, choose the one that maximize the diversity index".* As a result, we search the numbers (π_{ijk}) compatible with certain constraints, one of constraints being (1.14) signifying a given or pre-established mean value of the indicator $E(\pi;q)$ defined by (1.13), which maximize the diversity $D(\pi;u)$ defined by (1.4) and the optimal decisions are given by the numbers (π_{ijk}).

Problem 2. ***Optimal diversification decisions with pre-established diversity degree*** [***Optimal probability distributions with pre-established diversity degree***].

For solving the **Problem 2,** we can reformulate the ***PMD*** of Guiaşu and Guiaşu (2003) under the name ***Principle of Optimum Diversification*** (**POD**), according to which: *"from the set of all probability distributions compatible with one or several mean values of one or several random variables and with a pre-established value of the diversity degree, choose the one that optimizes (maximizes or minimizes) a certain economical indicator".* As a result, we search the numbers (π_{ijk}) compatible with certain constraints, one of constraints being (1.11) signifying a pre-established diversity degree $D(\pi;u)$ defined by (1.4), which optimizes the economic indicator $E(\pi;q)$ defined by (1.13) and the optimal decisions are given by the numbers (π_{ijk}).

Note. By analogy with the proof of Proposition 1.1, using the method of the Lagrange multipliers, we present, without mathematical demonstrations, some practical results for solving a three-dimensional problem of transportation or allocation of resources.

2. Optimal diversification decisions with maximum diversity degree.

Note. *As an optimal solution for a three-dimensional transportation problem, in this section, we search the three-dimensional probability distributions compatible with certain constraints equalities, one constraint can be even (1.14) signifying a fixed mean value, and which maximizes the weighted diversity degree (1.4).*

Proposition 2.1. *The three-dimensional probability distribution which is compatible with the constraints:*

$$1)\ \sum_i \sum_j \sum_k \pi_{ijk} = 1;\quad 2)\ \sum_i \sum_j \sum_k \pi_{ijk} q_{ijk} = E_0 \tag{2.1}$$

and which maximizes the weighted diversity (1.4) is given by the numbers $\pi^0 = (\pi^0_{ijk})$ *:*

$$\pi^0_{ijk} = \frac{1}{2}\left[1 + av_{ijk} + bv_{ijk} q_{ijk}\right] \tag{2.2}$$

$$a = \frac{(2-mnp)B - (2E_0 - Q)A}{VB - A^2};\quad b = \frac{(2E_0 - Q)V - (2-mnp)A}{VB - A^2} \tag{2.3}$$

$$A = \sum_i \sum_j \sum_k v_{ijk} q_{ijk} \;;\; B = \sum_i \sum_j \sum_k v_{ijk} q_{ijk}^2 \;;\; Q = \sum_i \sum_j \sum_k q_{ijk} \;;\; VB - A^2 > 0 \qquad (2.4)$$

$$D_{\max} = \frac{1}{4}\left[U - \frac{(mnp-2)^2}{V} - \frac{[(mnp-2)A + (2E_0 - Q)V]^2}{V(VB - A^{2)}} \right] \qquad (2.5.1)$$

$$D_{\max} \le \frac{1}{4}\left[U - \frac{(mnp-2)^2}{V} \right] \qquad (2.5.2)$$

where $(i, j, k) \in \{1,2,...,m\} \times \{1,2...,n\} \times \{1,2...,p\}$, *only for those numbers* $u = (u_{ijk})$, $q = (q_{ijk})$ and E_0 *for which* $(\pi_{ijk}) \ge 0$ *where U and V are given by the relation (1.7).*

Remark. In the diversification problems, the inequality (2.5.2) shows that *the diversity degree is decreasing with respect to the number of constraints.*

Consequence. *If* $u_{ijk} = 1$ *for all* (i, j, k), *then the optimal distribution (2.2) is:*

$$\pi_{ijk}^0 = \alpha + \beta q_{ijk} \;;\quad \alpha = \frac{\sum_{i,j,k} q_{ijk}^2 - E_0 \sum_{i,j,k} q_{ijk}}{mnp \sum_{i,j,k} q_{ijk}^2 - [\sum_{i,j,k} q_{ijk}]^2} \;;\quad \beta = \frac{mnpE_0 - \sum_{i,j,k} q_{ijk}}{mnp \sum_{i,j,k} q_{ijk}^2 - [\sum_{i,j,k} q_{ijk}]^2} \qquad (2.6)$$

More, if the constraint 2) from (2.1) is absent, then $b = 0$ *in (2.2), implying* $\beta = 0$ *in (2.5) and as a result we get the uniform distribution (1.9).*

Proposition 2.2. *The three-dimensional probability distribution which is compatible with the constraints:*

$$1)\ \sum_i \sum_j \sum_k \pi_{ijk} = 1 \;;\quad 2)\ \sum_i \sum_j \sum_k \pi_{ijk} q_{ijk} = E_0 \qquad (2.7.1\text{-}2)$$

$$3)\ \sum_j \sum_k \pi_{ijk} = \pi_{i\bullet\bullet} \;;\ 4) \sum_i \sum_k \pi_{ijk} = \pi_{\bullet j \bullet} \;;\ 5) \sum_i \sum_j \pi_{ijk} = \pi_{\bullet\bullet k} \qquad (2.7.3\text{-}5)$$

and which maximizes the weighted diversity (1.4) is given by the following numbers:

$$\pi_{ijk}^0 = \frac{1}{2}\left[1 + (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} \right] \qquad (2.8)$$

where $(i, j, k) \in \{1,2,...,m\} \times \{1,2...,n\} \times \{1,2...,p\}$ *and* $[\lambda; \{a_i\}; \{b_j\}; \{c_k\}; \mu]$ *are the solutions of the following equations:*

$$\sum_i \sum_j \sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2 - mnp \tag{2.9.1}$$

$$\sum_i \sum_j \sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} q_{ijk} = 2E_0 - Q \tag{2.9.2}$$

$$\sum_j \sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2\pi_{i\bullet\bullet} - np \tag{2.9.3}$$

$$\sum_i \sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2\pi_{\bullet j\bullet} - mp \tag{2.9.4}$$

$$\sum_i \sum_j (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2\pi_{\bullet\bullet k} - mn \tag{2.9.5}$$

only for those fixed numbers $[(\pi_{i\bullet\bullet}), (\pi_{\bullet j\bullet}), (\pi_{\bullet\bullet k}), (u_{ijk}), (q_{ijk}), E_0]$ *for which* $(\pi_{ijk}) \geq 0$, *where the number* Q *is given by the relation (2.4).*

3. Optimal diversification decisions with pre-established diversity degree.

Note. *As an optimal solution for a three-dimensional transportation problem, in this section, we search the three-dimensional probability distributions which are compatible with certain constraints equalities, one constraint can be even (1.11) signifying a pre-established or fixed diversity degree, and optimizes (maximizes or minimizes) the indicator (1.13).*

Proposition 3.1. *The three-dimensional probability distribution which is compatible with the following constraints:*

$$1)\ \sum_i \sum_j \sum_k \pi_{ijk} = 1;\quad 2)\ \sum_i \sum_j \sum_k u_{ijk} \pi_{ijk} (1 - \pi_{ijk}) = D_0 \tag{3.1}$$

and which minimizes or maximizes the indicator $E(\pi; q)$ *defined by (1.13) is given by:*

$$\pi^0_{ijk} = \frac{1}{2}\left[1 + (a + bq_{ijk}) v_{ijk}\right] \tag{3.2}$$

$$a = \frac{2 - mnp - Ab}{VB - A^2}; \quad b^2 = \frac{UV - 4VD_0 - (mnp-2)^2}{VB - A^2} \tag{3.3}$$

where $(i,j,k) \in \{1,2,...,m\} \times \{1,2...,n\} \times \{1,2...,p\}$, *with* U *and* V *given by (1.7),* A, B *and* Q *given by (2.4) only for those fixed numbers* $u = (u_{ijk})$, $q = (q_{ijk})$ *and* D_0 *for which* $(\pi_{ijk}) \geq 0$ *and the optimal value of the indicator* $E(\pi;q)$ *is:*

$$E_{opt}(\pi;q) = [Q + Aa + Bb]/2 \tag{3.4}$$

with maximum value of $E(\pi;q)$ *if b > 0 and with minimum value of* $E(\pi;q)$ *if b < 0.*

Proposition 3.2. *The three-dimensional probability distribution which are compatible with the following constraints:*

$$1)\ \sum_i\sum_j\sum_k \pi_{ijk} = 1; \quad 2)\ \sum_i\sum_j\sum_k u_{ijk}\pi_{ijk}(1-\pi_{ijk}) = D_0 \tag{3.5.1-2}$$

$$3)\ \sum_j\sum_k \pi_{ijk} = \pi_{i\bullet\bullet}; \quad 4)\ \sum_i\sum_k \pi_{ijk} = \pi_{\bullet j\bullet}; \quad 5)\ \sum_i\sum_j \pi_{ijk} = \pi_{\bullet\bullet k} \tag{3.5.3-5}$$

and which minimizes or maximizes the indicator $E(\pi;q)$ *defined by (1.13) is given by:*

$$\pi^0_{ijk} = \frac{1}{2}\left[1 + (\lambda + a_i + b_j + c_k + dq_{ijk})v_{ijk}\right] \tag{3.6}$$

where $(i,j,k) \in \{1,2,...,m\} \times \{1,2...,n\} \times \{1,2...,p\}$ *and* $[\lambda;\{a_i\};\{b_j\};\{c_k\};d]$ *are the solutions of the following equations:*

$$\sum_i\sum_j\sum_k (\lambda + a_i + b_j + c_k + dq_{ijk})v_{ijk} = 2 - mnp \tag{3.7.1}$$

$$\sum_i\sum_j\sum_k (\lambda + a_i + b_j + c_k + dq_{ijk})^2 v_{ijk} = U - 4D_0 \tag{3.7.2}$$

$$\sum_j\sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk})v_{ijk} = 2\pi_{i\bullet\bullet} - np \tag{3.7.3}$$

$$\sum_i \sum_k (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2\pi_{\bullet j \bullet} - mp \qquad (3.7.4)$$

$$\sum_i \sum_j (\lambda + a_i + b_j + c_k + \mu q_{ijk}) v_{ijk} = 2\pi_{\bullet\bullet k} - mn \qquad (3.7.5)$$

only for those fixed numbers $[(\pi_{i\bullet\bullet}), (\pi_{\bullet j \bullet}), (\pi_{\bullet\bullet k}), (u_{ijk}), (q_{ijk}), D_0]$ *for which we have* $(\pi_{ijk}) \geq 0$ *and the optimal distribution (3.6) maximizes* $E(\pi;q)$ *for* $d > 0$ *and minimizes* $E(\pi;q)$ *for* $d < 0$.

4. Some conclusions.

Similarly to the presented results in this paper, using the weight diversity measure as an objective function or as a constraint, many other interesting nonlinear three-dimensional transportation models can be analyzed. In each concrete situation, which illustrate a *diversity management problem*, we have an optimal solution named *optimal diversification decision* in transportation or allocation which correspond to an optimal parametrical probability distribution, if and only if this probability distribution exist. In the presented results, if we have $m \geq 2$, $n \geq 2$ and $p = 1$, then we obtain the two-dimensional optimal distributions or diversifications and if $m \geq 2$ and $n = p = 1$, then we have the one-dimensional optimal distributions or optimal diversifications for a transportation or for an allocation economical problem.

5. Annex. Proof of Proposition 1.1.

Proof. From the definition of the index (1.4), we have the first inequality. If λ is the Lagrange multiplier, we are looking for a stationary point of Lagrange function:

$$L(\pi;\lambda) = D(\pi;u) + \lambda\left[\sum_i \sum_j \sum_k \pi_{ijk} - 1\right]$$

which is obtained by solving the following system of (mnp+1) equations:

$$\frac{\partial L}{\partial \pi_{ijk}} = u_{ijk} - 2u_{ijk}\pi_{ijk} + \lambda = 0; \quad \frac{\partial L}{\partial \lambda} = \sum_i \sum_j \sum_k \pi_{ijk} - 1 = 0$$

for $(i,j,k)\in\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\}$. From the first $m\times n\times p$ equations of the system, we get:

$$\pi_{ijk}=\frac{1+\lambda v_{ijk}}{2}$$

which, introduced into the last equation, give the distribution (1.6). Because we have the following relations:

$$\frac{\partial^2 L}{\partial \pi_{ijk}^2}=-2u_{ijk} \quad \text{and} \quad \frac{\partial^2 L}{\partial \pi_{ijk}\partial \pi_{rst}}=0$$

for $(i,j,k)\neq(r,s,t)$, $\{(i,j,k),(r,s,t)\}\subseteq\{1,2,...,m\}\times\{1,2...,n\}\times\{1,2...,p\}$ and the following differential constraint:

$$\sum_i\sum_j\sum_k d\pi_{ijk}=0$$

corresponding to the constraint (1.1), we get the inequality:

$$d^2L(\pi^0)=-2\sum_i\sum_j\sum_k u_{ijk}d\pi_{ijk}^2<0$$

implying that the distribution (1.6) maximizes the diversity $D(\pi,u)$, with the maximum value given by the relation (1.5), and the proposition is proved.

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