Lecturer Costin-Ciprian POPESCU, PhD
Department of Mathematics
The Bucharest Academy of Economic Studies

MATHEMATICAL PROGRAMMING FOR OPTIMAL DECISION MAKING

Abstract. *Data analysis and especially forecasts are important when an investor wants to place his capital. A useful mathematical tool in conducting analysis and making decisions of this type is regression. In this paper we deal with orthogonal regression, particularly focusing on two-dimensional case. On this basis, we present a comparative analysis on the evolution of some securities listed on stock exchange.*

Key words: *data analysis, orthogonal regression, forecasting, decision making.*

JEL Classification: C02, C51, C53.

1. Preliminaries

Regression, with data written in the form of real numbers or fuzzy numbers, is an useful method in areas where data analysis plays a central role (Van Huffel et al., 2007; Ramos, 2007; Petras and Podlubny, 2007; Popescu and Giuclea, 2007; Bifulco et al., 2009; Giuclea and Popescu, 2009; Iacob et al., 2010). Economy is one such research field, where various kinds of algorithms make their presence felt (Roşca and Moldoveanu, 2009; Ruxanda, 2010; Mărăcine and Ianole, 2010). In this article we use the orthogonal regression in the study of issues to improve decision making on capital investment. Otherwise, orthogonal regression has been much discussed in the literature (Van Huffel and Vanderwalle, 1991; Nievergelt, 1994; De Groen, 1996; Van Huffel et al., 2007). When it is necessary to study a large number of interrelated (statistical) variables, the method is based on matrix factorization (Golub and Van Loan, 1996). However, for two-dimensional or three-dimensional case, some geometric solutions can be found (Pozzi, 2008; etc.). For n statistical data, $(x_{1k}, x_{2k}, \ldots, x_{sk}) \in \mathbf{R}^s$, $k = \overline{1, n}$ (representing n numerical values which are obtained for s variables: $X_1, \ldots, X_s$), orthogonal regression in s dimensions involves finding

$$\Pi(a_0, \ldots, a_s) = \left\{ (x_1, \ldots, x_s) \middle| a_0 + \sum_{j=1}^{s} a_j x_j = 0 \right\},$$

which is a solution for the problem

$$\min_{(a_0, \ldots, a_s) \in \mathbf{R}^{s+1}} \left[\sum_{k=1}^{n} \mathrm{D}^2\left((x_{1k}, x_{2k}, \ldots, x_{sk}), \Pi \right) \right].$$

$\mathrm{D}^2\left(\left(x_{1k}, x_{2k}, ..., x_{sk}\right), \Pi\right)$ represents the squared distance between $\left(x_{1k}, x_{2k}, ..., x_{sk}\right)$ and Π. The hyperplane $\Pi\left(a_0, ..., a_s\right)$ can be determined based on a sequence of steps described below (Van Huffel and Vanderwalle, 1991; De Groen, 1996; Golub and van Loan, 1996). First, consider the $n \times s$ matrix A with columns a^j $\left(j = \overline{1, s}\right)$, where $a^j = \left(x_{j1} - x_{C_j}, ..., x_{jn} - x_{C_j}\right)^T$ and $x_{C_j} = n^{-1} \sum_{k=1}^{n} x_{jk}$. Using singular value decomposition approach, A can be decomposed into a matrix product as $A = Q_1 \Sigma Q_2^T$. The numbers $a_1, ..., a_s$ will be the components of the column vector from $s \times s$ matrix Q_2, corresponding to the smallest eigenvalue of the matrix A. Then, it can be shown that the point $C\left(n^{-1} \sum_{k=1}^{n} x_{1k}, ..., n^{-1} \sum_{k=1}^{n} x_{sk}\right)$ (which is called centroid) belongs to $\Pi\left(a_0, ..., a_s\right)$. Finally, a_0 can be determined using the fact that $C \in \Pi\left(a_0, ..., a_s\right)$. Under certain conditions, the obtained solution is unique. Because it is not necessary to know the role played by each variable (independent/dependent), orthogonal regression can be applied in problems where other methods do not work. Some interesting comparisons with the classical method of least squares can be found in Nievergelt (1994) or Petras and Podlubny (2007) (together with other important features of the method that make it unique among regression techniques). In particular, for two variables, X and Y, we have $s = 2$ and $\Pi\left(a_0, a_1, a_2\right)$ can be regarded as a line $\left(L^*\right)$. In this case, there are solving methods that do not require matrix factorization, but are based on geometric arguments. In the following lines, we will discuss such an approach.

2. A geometric approach

Consider two statistical variables, X and Y, for which we can not state the type of dependence (which one clearly influences the other). However, similar to ordinary regression, we wish to establish a connection that results in an equation that includes both. Suppose that after n observations we know the numerical values $\left(x_k, y_k\right)$, $k = \overline{1, n}$. Let it be $C\left(x_C, y_C\right)$ with coordinates given by $x_C = n^{-1} \sum_{k=1}^{n} x_k$ and $y_C = n^{-1} \sum_{k=1}^{n} y_k$. As we said before, C is the centroid for the set consisting of points $P_k\left(x_k, y_k\right)$, $k = \overline{1, n}$. A line $\left(L_m\right)$ (which is not vertical), with slope equal to m, and passing through a point $P_0\left(x_0, y_0\right)$ has the equation

$$mx - y - mx_0 + y_0 = 0 .$$

The sum of squared distances from the points $P_k(x_k, y_k)$, $k=\overline{1,n}$ to (L_m) is calculated by the formula

$$\sum_{k=1}^{n} \mathrm{D}^2(P_k,(L_m)) = \sum_{k=1}^{n} \frac{(mx_k - y_k - mx_0 + y_0)^2}{m^2+1}.$$

On the other hand, we have

$$\sum_{k=1}^{n}(mx_k - y_k - mx_0 + y_0)^2 = \sum_{k=1}^{n}[m(x_k - x_C) - (y_k - y_C)]^2 -$$

$$-2[m(x_0 - x_C) - (y_0 - y_C)] \cdot \sum_{k=1}^{n}[m(x_k - x_C) - (y_k - y_C)] +$$

$$+\sum_{k=1}^{n}[m(x_0 - x_C) - (y_0 - y_C)]^2 =$$

$$= \sum_{k=1}^{n}[m(x_k - x_C) - (y_k - y_C)]^2 + \sum_{k=1}^{n}[m(x_0 - x_C) - (y_0 - y_C)]^2.$$

Thus

$$\sum_{k=1}^{n} \mathrm{D}^2(P_k,(L_m)) = \frac{1}{m^2+1}\sum_{k=1}^{n}[m(x_k - x_C) - (y_k - y_C)]^2 +$$

$$+\frac{1}{m^2+1}\sum_{k=1}^{n}[m(x_0 - x_C) - (y_0 - y_C)]^2 \geq \frac{1}{m^2+1}\sum_{k=1}^{n}[m(x_k - x_C) - (y_k - y_C)]^2,$$

with equality when $x_0 = x_C$ and $y_0 = y_C$. Also, for a vertical line (L_∞) we have

$$\sum_{k=1}^{n} \mathrm{D}^2(P_k,(L_\infty)) = \sum_{k=1}^{n}(x_k - x_0)^2 = \sum_{k=1}^{n}(x_k - x_C - x_0 + x_C)^2 =$$

$$= \sum_{k=1}^{n}(x_k - x_C)^2 + \sum_{k=1}^{n}(x_0 - x_C)^2 \geq \sum_{k=1}^{n}(x_k - x_C)^2,$$

with equality if $x_0 = x_C$. In conclusion, for all line with a slope $m \in \overline{\mathbf{R}}$, the sum of squared distances is minimized for that which passes through $C(x_C, y_C)$ (see also De Groen, 1996). So the feasible set will consist only of the lines (L) passing through the centroid. The next step is to calculate the angle between the optimum line, (L^*), and one of the coordinate axis (for a Cartesian coordinate system xOy). Therefore, consider a line (L) (with $C \in (L)$), and an arbitrary point $P(x_P, y_P)$, such that $P(x_P, y_P) \neq C(x_C, y_C)$, $P \notin (L)$ (Fig. 1). Assume that α is the value of the angle between CP and Ox. Similarly, β ($\beta \neq 0, \frac{\pi}{2}$) is the value of the angle between (L) and Oy. If $T = \mathrm{pr}_{(L)}P$, then the distance from P to (L) is equal to the length of segment PT. For triangle QPC, we can write

$$\cos\alpha = \frac{|x_P - x_C|}{PC},\ \sin\alpha = \frac{|y_P - y_C|}{PC}.$$

For triangle TPC we have $\cos\left(T\hat{P}C\right) = \frac{PT}{PC}$. Due to the fact that $(L) \perp PT$ and $PS \perp Oy$ it results that the measure of $T\hat{P}S$ is equal to β. Also, we have $S\hat{P}C \equiv P\hat{C}Q$. Thus

$$\left(T\hat{P}C\right) = \left(T\hat{P}S\right) + \left(S\hat{P}C\right) = \alpha + \beta.$$

From $\cos(\alpha + \beta) = \frac{PT}{PC}$, it follows that

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{PT}{PC}.$$

Then the distance from P to (L) is

$$PT = |x_P - x_C|\cos\beta - |y_P - y_C|\sin\beta = (x_P - x_C)\cos\beta - (y_P - y_C)\sin\beta.$$

Further, we make the following notation: $\gamma = \frac{\pi}{2} - \beta$. For other possible positions of P over (L), the results is similar (except for a possible change of sign).

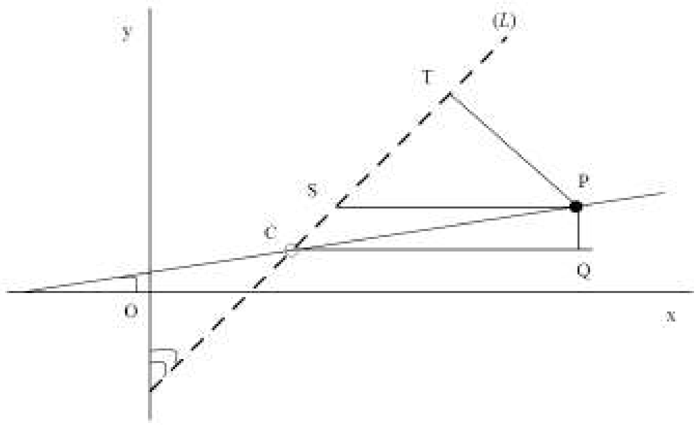

Figure 1. Graphical representation of (L)

If (L) is a vertical line, the distance from P to (L) is $|x_P - x_C|$. Similarly, for a horizontal line (L), the distance from P to (L) is $|y_P - y_C|$. In conclusion, we have

$$\mathrm{D}^2(P,(L)) = [(x_P - x_C)\cos\beta - (y_P - y_C)\sin\beta]^2.$$

Thus minimizing the sum of squared distances from P_k ($k=\overline{1,n}$) to (L) is equivalent to solving the problem $\left[\min\limits_{\beta} u(\beta)\right]$, where

$$u(\beta)=\sum_{k=1}^{n}(a_k\cos\beta-b_k\sin\beta)^2=$$
$$=\cos^2\beta\sum_{k=1}^{n}a_k^2+\sin^2\beta\sum_{k=1}^{n}b_k^2-2\sin\beta\cos\beta\sum_{k=1}^{n}a_kb_k\ .$$

For all $k=\overline{1,n}$, we used the notation: $a_k=x_k-x_C$ and $b_k=y_k-y_C$. After equating to zero the derivative of $u(\beta)$ we obtain

$$\sum_{k=1}^{n}(a_k\cos\beta-b_k\sin\beta)(a_k\sin\beta+b_k\cos\beta)=0$$
$$\Rightarrow\sin\beta\cos\beta\sum_{k=1}^{n}\left(a_k^2-b_k^2\right)+\left(\cos^2\beta-\sin^2\beta\right)\sum_{k=1}^{n}a_kb_k=0$$
$$\Rightarrow(\sin 2\beta)\sum_{k=1}^{n}\left(a_k^2-b_k^2\right)+2(\cos 2\beta)\sum_{k=1}^{n}a_kb_k=0$$
$$\Rightarrow d\sin 2\beta+2c\cos 2\beta=0\ ,$$

where $c=\sum\limits_{k=1}^{n}a_kb_k$ and $d=\sum\limits_{k=1}^{n}\left(a_k^2-b_k^2\right)$. Consider the following two equations

$$d\sin 2\beta+2c\cos 2\beta=0\ ,\ \sin^2 2\beta+\cos^2 2\beta=1\ .$$

If $c\neq 0$, $d\neq 0$, we get that

$$\sin^2 2\beta=4c^2\left(4c^2+d^2\right)^{-1},\ \cos^2 2\beta=d^2\left(4c^2+d^2\right)^{-1},$$
$$\operatorname{sgn}(\sin 2\beta\cos 2\beta)=-\operatorname{sgn}(cd)\ .$$

Thus $\tan 2\beta=-2cd^{-1}$. Using the relationship $\tan 2\beta=\tan(\pi-2\gamma)=-\tan 2\gamma$ we obtain $\tan 2\gamma=2cd^{-1}$. Since $\tan 2\gamma=2\tan\gamma\left(1-\tan^2\gamma\right)^{-1}$ then $m\left(1-m^2\right)^{-1}=cd^{-1}$. Thus $cm^2+dm-c=0$. For this last equation, we have $\Delta=d^2+4c^2>0$. This means that we get two distinct real solutions, m_1 and m_2. Also using the second derivative of $u(\delta)$, it can be shown that one and only one of the two solutions gives a minimum point. For its determination, we can actually compare the values of the sum of squared distances in the two cases and with those given by the horizontal and vertical lines through centroid. The line leading to a lower value of the mentioned sum is chosen as the optimal solution, $\left(L^*\right)$.

3. Numerical application

Models that allow analysis of developments in the stock markets are a subject of great interest in Financial Mathematics (Ruxanda and Stoenescu, 2009; Fulga et al., 2009; Kavkler and Festić, 2010; Shavandi and Alizadeh, 2010; Popescu and Fulga, 2011). In this paper we apply the method discussed in previous sections, for the study of some stock exchange listed assets. Consider two securities (which will be formally marked by S1 and S2) whose yields are given in Table 1 (the displayed data were taken from the source: Deutsche Bundesbank, Prices and Yields of Listed Federal Securities, January 2010).

Table 1. Prices and yields of listed securities

Date	S1		S2	
	Price 1	Yield 1 (x_i)	Price 2	Yield 2 (y_i)
04.01.2010	100.477	0.31	100.750	0.30
05.01.2010	100.472	0.30	100.746	0.28
06.01.2010	100.472	0.26	100.745	0.25
07.01.2010	100.440	0.31	100.715	0.28
08.01.2010	100.435	0.30	100.690	0.34
11.01.2010	100.432	0.27	100.690	0.31
12.01.2010	100.430	0.24	100.690	0.28
13.01.2010	100.420	0.25	100.660	0.37
14.01.2010	100.400	0.24	100.650	0.31
15.01.2010	100.390	0.25	100.650	0.28
18.01.2010	100.385	0.24	100.635	0.31
19.01.2010	100.375	0.25	100.625	0.32
20.01.2010	100.360	0.31	100.615	0.32
21.01.2010	100.345	0.25	100.590	0.33
22.01.2010	100.330	0.31	100.585	0.31
25.01.2010	100.330	0.25	100.570	0.35
26.01.2010	100.320	0.28	100.565	0.33
27.01.2010	100.315	0.25	100.561	0.31
28.01.2010	100.290	0.28	100.535	0.33
29.01.2010	100.290	0.21	100.528	0.32

These data can be represented in three space dimensions (where, for a more illustrative representation, Yield 1 and Yield 2 are multiplied by 10^2) (Fig. 2). Such a representation is useful because it outlines a pattern of simultaneous evolution. But a rigorous decision by an investor requires an analysis based on an equation established between the two financial instruments. Such a mathematical relationship can be established, for example, between Yield 1 (represented by variable X) and Yield 2 (represented by Y), using orthogonal regression.

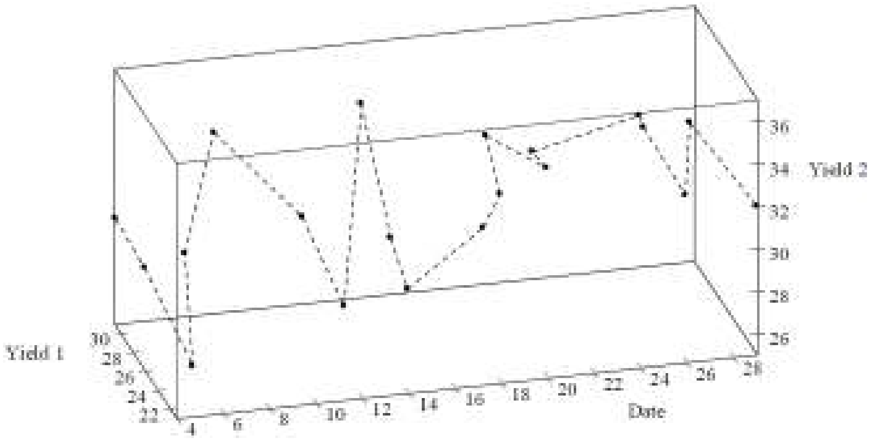

Figure 2. The initial values depending on time

Based on data from Table 1, we find that the centroid for (x_i, y_i), $i = \overline{1,20}$ is the point $C(0.268, 0.3115)$. Finally, we obtain the regression line

$$(L^*):\ 0.48777x + 0.87297y - 0.40265 = 0 \text{ (Fig. 3).}$$

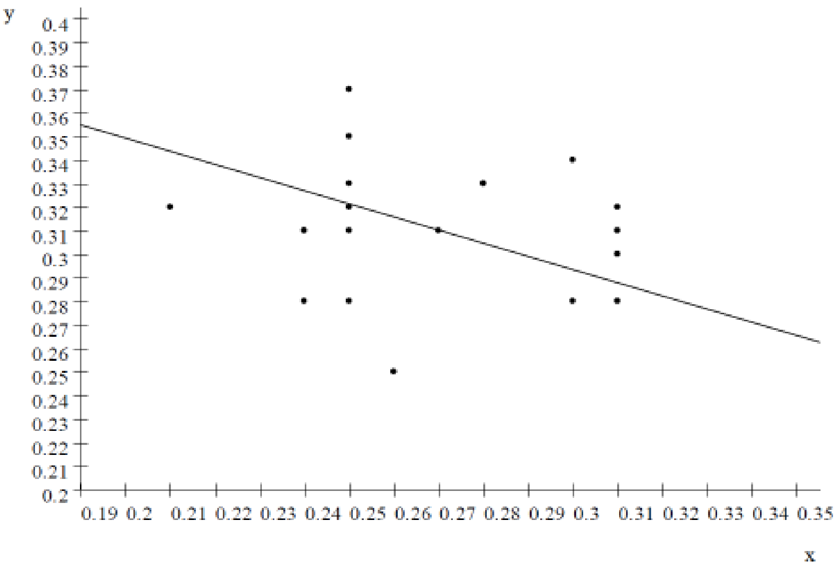

Figure 3. The datapoints (x_i, y_i), $i = \overline{1,20}$ and the line (L^*)

Remark

The least squares approach leads to the following two solutions:

$$(L_Y):\ y = -0.10748x + 0.3403 \text{ (if } Y \text{ is the dependent variable),}$$

(L_X): $x = -0.12386y + 0.30658$ (when X is the dependent variable).
Instead, for the method discussed in this paper, regardless of dependency, the result is the same, (L^*). The lines (L^*), (L_Y) and (L_X) are plotted in Figure 4. Note that all three pass through the centroid, the first being included within the angle formed by the last two. These results can be added to those obtained by Nievergelt (1994), or Petras and Podlubny (2007). There, (L_Y) and (L_X) were called conjugate lines.

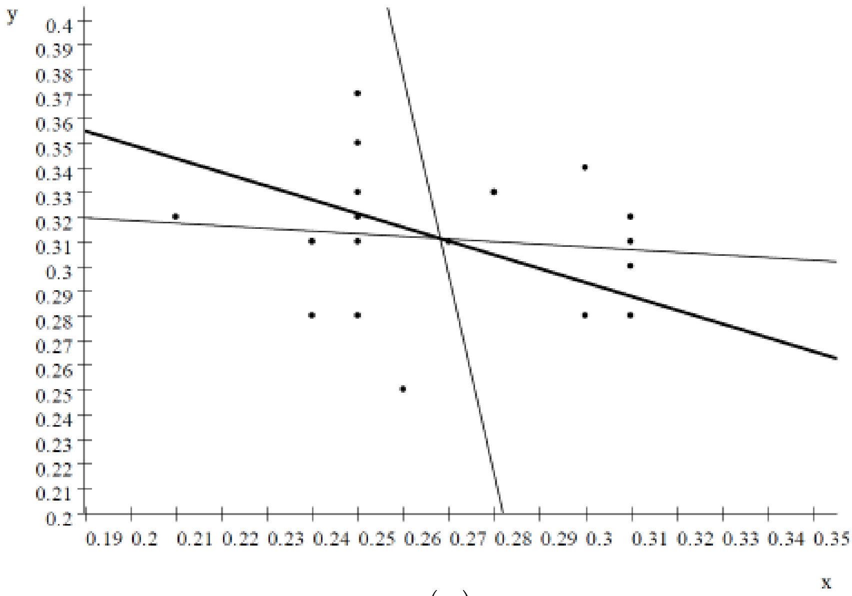

Figure 4. (L_X), (L_Y) and (L^*) (the last as a thick line)

4. Conclusions

Unlike other methods of regression, orthogonal regression allow the development of a model even when the direction of causality is not known with precision. Moreover, the discussion may be taken for an arbitrary number of parameters that characterize a particular process. Thus, bringing a unified formula of the various variables allows the possibility of the analysis, making predictions and optimal decision-making.

Acknowledgements

This work was supported by CNCSIS-UEFISCSU, project number 844 PN II-IDEI, code 1778/2008.

REFERENCES

[1] **Bifulco, I; Raiconi, G; Scarpa R. (2009),** ***Computer algebra software for least squares and and total least norm inversion of geophysical models***, Computers & Geosciences 35, 1427-1438;

[2] **De Groen, P. (1996),** ***An introduction to total least squares***, Nieuw Archief voor Wiskunde, Vierde serie 14, 237-253;

[3] **Fulga, C.; Dedu, S.; Şerban, F. (2009),** ***Portfolio Optimization with Prior Stock Selection***, Economic Computation and Economic Cybernetics Studies and Research 43(4), ASE Publishing House, Bucharest, 157-172;

[4] **Giuclea, M.; Popescu, C. C. (2009),** ***On statistical pattern with fuzzy data***, Economic Computation and Economic Cybernetics Studies and Research, 4, ASE Publishing House, Bucharest, 187-198;

[5] **Golub, G. H; Van Loan, C. H. (1996),** ***Matrix Computations***, Johns Hopkins University Press;

[6] **Iacob, A. I.; Popescu, C. C.; Dimian, G. C. (2010),** ***Tracking changes in the regional social-economic activity by a geometric fitting model: the Romanian case***, Selected topics in Economy & Management Transformation**,** WSEAS Press, 428-433;

[7] **Kavkler, A.; Festić, M (2010),** ***Smooth transition regression model for Slovene Stock Exchange index returns*****,** Economic Computation and Economic Cybernetics Studies and Research 4, ASE Publishing House, Bucharest, 147-164;

[8] **Mărăcine, V.; Ianole, R. (2010)**, ***How knowledge determines demand dynamics-a new perspective for evaluation. Study case for healthcare services***, Economic Computation and Economic Cybernetics Studies and Research 4, ASE Publishing House, Bucharest, 5-22;

[9] **Nievergelt, Y. (1994),** ***Total least squares: state of the art regression in numerical analysis***, SIAM Review 36, 258-264;

[10] **Petras, I.; Podlubny, I. (2007),** ***State space description of national economies: The V4 countries***, Computational Statistics & Data Analysis 52, 1223-1233;

[11] **Popescu, C; Giuclea, M. (2007),** ***A model of multiple linear regression***, Proceedings of the Romanian Academy Series A: Mathematics, Physics, Technical Sciences, Information Science, 2 (8), 137-144;

[12] **Popescu, C. C.; Fulga, C. (2011),** ***Possibilistic Optimization with Applications to Portfolio Selection*****,** Proceedings of the Romanian Academy Series A: Mathematics, Physics, Technical Sciences, Information Science 2, 88-94;

[13] **Pozzi F. (2008),** ***Orthogonal linear least squares on a two-dimensional plane***, Matlab Exchange Files;

[14] **Ramos, J. A. (2007),** ***Applications of TLS and related methods in the environmental sciences***, Computational Statistics & Data Analysis 52, 1234-1267;

[15] **Roşca, I., G.; Moldoveanu, G. (2009),** ***Management in turbulent conditions***, Economic Computation and Economic Cybernetics Studies and Research 2, ASE Publishing House, Bucharest, 5-12;

[16] **Ruxanda, G.; Stoenescu, S. (2009),** ***Bivariate and multivariate cointegration and their application in stock markets***, Economic Computation and Economic Cybernetics Studies and Research 4, ASE Publishing House, Bucharest, 17-32;

[17] **Ruxanda, G. (2010),** ***Learning perceptron neural network with backpropagation algorithm*****,** Economic Computation and Economic Cybernetics Studies and Research 4, ASE Publishing House, Bucharest, 37-54;

[18] **Shavandi, H.; Alizadeh, P. (2010),** ***A hybrid intelligent model using technical and fundamental analysis to forecasting stock price index***, Economic Computation and Economic Cybernetics Studies and Research 2, ASE Publishing House, Bucharest, 95-112;

[19] **Van Huffel, S; Vanderwalle, J. (1991),** ***The total least squares problem: Computational aspects and analysis***, SIAM, Philadelphia;

[20] **Van Huffel, S.; Cheng, C. L.; Mastronardi, L.; Paige , C.; Kukush A. (2007),** ***Total least squares and errors-in-variables modeling***, Computational Statistics & Data Analysis 52, 1076-1080.