T. Vasudha WARRIER, PhD
Nita H. SHAH, PhD
Department of Mathematics, Gujarat University
Ahmedabad - 380009, Gujarat, INDIA
E-mail : nita_sha_h@rediffmail.com

REVIEW PERIOD REORDER POINT PROBABILISTIC INVENTORY SYSTEM FOR DETERIORATING ITEMS WITH THE MIXTURE OF BACKORDERS AND LOST SALES

Abstract. *This paper is developed to describe a review period reorder point inventory system in which units are subject to deterioration at a constant rate and shortages are allowed but only partially backlogged, i.e. rest goes as lost sales. Also the demand during prescribed scheduling period is a random variable following suitable probability distribution. The objective is to find the optimum reorder point which gives the minimum cost during the review period.*
Key words : *Review period, probabilistic inventory model, deterioration, partial backlogging, lost sales.*

JEL Classification: 90B05

1.Introduction

Since the development of Wilson's lot size inventory model in 1915, a number of papers has been written analyzing mathematical models of inventory under different situations. In these models, it was assumed that once units enter in inventory, they last forever. Afterwards, researchers showed considerable interest in developing mathematical models of inventory for describing optimal policies for deteriorating items. Raafat (1991), Shah and Shah (2000), Goyal and Giri (2001) gave an up-to- date review of inventory models for perishable items. Most of the authors have considered simultaneous obsolescence, i.e., all units remaining in inventory at the end of the planning horizon become useless. Hadley and Whitin (1961, 1962)) derived inventory models for deteriorating items under the assumption of finite planning horizon which was revisited by Murray and Silver (1966). Stochastic demand was incorporated in the aforesaid models by Barankin and Denny (1960). Brown et al (1964) established ordering policies for inventory which is subject to obsolescence. They modeled obsolescence by assuming planning horizon to be a random variable. Bulinskaya (1964) developed newsboy problem where delivery perishes immediately with probability *p* and after one period with probability (1 – *p*). Van Zyl (1964) considered the model where the product life time is exactly two periods with proportional costs of ordering and shortages. The demands in successive periods were assumed to be independent identically distributed random variable. He gave dynamic programming formulation of the said model. Nahmias and Pierskalla (1973, 1975) developed a optimal ordering policy to balance the expected outdating and expected shortages arises in blood bank management. Various authors viz. Nahmias (1975a, 1975b), Fries (1975) etc. have considered inventory models for items with fixed life time. Jani et al (1978) developed a periodic review inventory model for deteriorating items in an inventory as a random variable with some specified continuous probability distribution and the distribution of demand during the review period is stationary over time. The purpose of this paper is to develop a periodic review inventory model for deteriorating items with stochastic demand in which partial backordering and partial lost sales is allowed, with some specified continuous probability distribution for demand and assuming that the distribution of the demand during the review period is stationary over time.

2. Assumptions and Notations :

The stated mathematical model is developed with the following assumptions and notations :

- The inventory position of the system is reviewed regularly at a period of w_p time units.
- Whenever the on hand inventory is found to be less than or equal to the reorder point s, a prescribed lot size of q_p units is scheduled for replenishment.
- Lead time is zero.
- The random demand of x units during the review period w_p follows uniform distribution with the probability density function f(x), $0 \le x < \infty$ and average demand , $\mu = \int_0^\infty x\, f(x)\, dx$.
- Shortages are allowed. The α^{th} fraction of it is backlogged and $(1 - \alpha)^{th}$ fraction is lost sales at any instant of time during the review period.
- Items in inventory are subject to deterioration during the period (0, w_p) and there is no repair or replacement of deteriorated items during this period.
- The time taken for deterioration of an item follows a negative exponential distribution given by

$$g(t, \theta) = \theta \exp(-\theta t), \theta > 0, t \ge 0$$

where

$$E(t) = \theta^{-1} = \text{average life of an item.}$$

The cumulative density function (c.d.f.) of t is

$$G(t, \theta) = 1 - \exp(-\theta t), \theta > 0, t \geq 0$$

So that the age specific failure rate ϕ (t) defined by Cox (1963) is given by

$$\phi(t) = \frac{g(t,\theta)}{1-G(t,\theta)} = \theta, \text{ a constant.}$$

- The inventory carrying cost, h (per unit per time unit), the cost of backordered shortages, π (per unit per time unit) , the cost due to lost sales, τ (per unit) and the unit purchase cost C (per unit) are known and remain constant during the period under consideration.

3. Mathematical Model :

Let S denote the inventory level at the beginning of each review period (after replenishment, if any), then S is a random variable following the p.d.f. given by

$$h(S) = \begin{cases} \frac{1}{q_p} & , s \leq S \leq s + q_p \\ 0 & , \text{otherwise} \end{cases} \tag{1}$$

Two cases may arise :

Case 1 : When shortages do not occur.

The inventory level at the beginning of the review period is S; x units is the demand during the review period w_p ; the items in the inventory are subject to deterioration during the period (0, w_p) at the constant rate θ. We assume that S units can satisfy the total demand during the scheduling period including deteriorated units.

Let Z(t | x, S) denote the inventory level of the system at time t, $0 \leq t \leq w_p$, when a demand of x units occurs during the review period w_p and the initial inventory level is S units. Then the differential equation that describes the instantaneous state of the inventory level during the review period w_p is given by

$$\frac{dZ(t \mid x,S)}{dt} + \theta Z(t \mid x,S) = -\frac{x}{w_p}, \; 0 \leq t \leq w_p .$$

Since, Z(0 | x, S) = S, the solution of the above first order linear differential equation is

$$Z(t \mid x, S) = \exp(-\theta t) \{S - \frac{x}{\theta w_p}(\exp(\theta t) - 1)\}, \; 0 \leq t \leq w_p . \tag{2}$$

From (2), the number of units that remain in inventory at the end of the review period w_p is given by

$$Z(w_p \mid x, S) = \exp(-\theta w_p)\{S - \frac{x}{\theta w_p}(\exp(\theta w_p) - 1)\}.$$

Now $Z(w_p \mid x, S) \geq 0$ gives $x \leq S_1$ where S_1 is given by

$$S_1 = \frac{S\theta w_p}{\exp(\theta w_p) - 1} \tag{3}$$

The number of units that deteriorate during the review period w_p is given by

$$D_1(x \mid x \leq S_1) = S - x - Z(w_p \mid x, S)$$

$$= (S + \frac{x}{\theta w_p})(1 - \exp(-\theta w_p)) - x \tag{4}$$

The average number of units in inventory during the review period w_p is given by

$$I_{11}(x \mid x \leq S_1) = \frac{1}{w_p}\int_0^{w_p} Z(t \mid x, S)\, dt$$

$$= \frac{(x + S\theta w_p)}{\theta^2 w_p}(1 - \exp(-\theta w_p)) - \frac{x}{\theta} \tag{5}$$

The costs due to backordering and lost sales do not arise as there are no shortages occurring during the review period in this case.

Case 2 : When shortages do occur.

When $t \leq t_1$, i.e. shortages do not occur, expressions for inventory in the system, number of units deteriorated during the review period w_p is same as those derived in case 1.

We assume that S units, the on hand inventory at the beginning of the review period, satisfies the demand only up to t_1 ($t \leq t_1$) time units. Among the shortages, occurring after t_1 time units till the end of the review period only α^{th} fraction is backordered at any instant and $(1 - \alpha)^{th}$ fraction comes under lost sales. The differential equation governing the system is given by

$$\frac{dZ(t \mid x, S)}{dt} + \theta Z(t \mid x, S) = -\frac{x}{w_p}, \quad 0 \leq t \leq t_1.$$

and

$$\frac{dZ(t \mid x, S)}{dt} = -\frac{\alpha x}{w_p}, \quad t_1 \leq t \leq w_p$$

The solution of above equations is respectively given by

$$Z(t \mid x, S) = \exp(-\theta t)\left\{S - \frac{x}{\theta w_p}(\exp(\theta t) - 1)\right\},\ 0 \le t \le t_1. \qquad (6)$$

and

$$Z(t \mid x, S) = \frac{\alpha x}{w_p}(t_1 - t),\ t_1 \le t \le w_p \qquad (7)$$

Since, $Z(t_1 \mid x, S) = 0$, we get (from (6)),

$$S = \frac{x}{\theta w_p}(\exp(\theta t_1) - 1).$$

which gives

$$t_1 = \frac{1}{\theta}\log\left(1 + \frac{S\theta w_p}{x}\right) \qquad (8)$$

Note that $t_1 < w_p$ gives $x > S_1$.

The number of units that deteriorate during the review period w_p is given by

$$D_2(x \mid x > S_1) = S - \frac{x}{w_p}t_1 = S - \frac{x}{\theta w_p}\log\left(1 + \frac{S\theta w_p}{x}\right) \qquad (9)$$

The average number of units carried during the review period w_p is given by

$$I_{12}(x \mid x > S_1) = \frac{1}{w_p}\int_0^{t_1} Z(t / x, S)\,dt$$

$$= \frac{S}{\theta} - \frac{x}{\theta^2 w_p}\log\left(1 + \frac{S\theta w_p}{x}\right) \qquad (10)$$

Also, total number of backordered units carried during the review period w_p is given by

$$I_{21}(x \mid x > S_1) = \frac{\alpha}{w_p}\int_{t_1}^{w_p} \frac{x}{w_p}(t - t_1)\,dt$$

$$= \frac{\alpha x}{2w_p}\left(w_p - \frac{1}{\theta}\log(1 + \frac{S\theta w_p}{x})\right)^2 \qquad (11)$$

Finally, the total number of units lost during time w_p is

$$I_{22}(x \mid x > S_1) = (1 - \alpha)\,\frac{x}{w_p}\,(w_p - t_1)$$

$$= (1-\alpha)\frac{x}{w_p}\left(w_p - \frac{1}{\theta}\log(1+\frac{S\theta w_p}{x})\right) \qquad (12)$$

Combining both the cases, the expected number of units that deteriorate during the review period w_p for a given S (using eqs. (4) and (9))

$$D(S) = \int_0^{S_1} D_1 (x \mid x \le S_1) f(x)\, dx + \int_{S_1}^{\infty} D_2 (x \mid x > S_1) f(x)\, dx \qquad (13)$$

, the average expected number of units in inventory during the review period w_p is (using eqs. (5) and (10)) for a given S

$$I_1 (S) = \int_0^{S_1} I_{11} (x \mid x \le S_1) f(x)\, dx + \int_{S_1}^{\infty} I_{12} (x \mid x > S_1) f(x)\, dx \qquad (14)$$

,the average expected number of units backordered during the review period w_p is (using eq. (11)) for a given S

$$I_{21} (S) = \int_{S_1}^{\infty} I_{21} (x \mid x > S_1) f(x)\, dx \qquad (15)$$

and the expected number of units lost during the review period w_p (using eq. (12)) for a given S

$$I_{22} (S) = \int_{S_1}^{\infty} I_{22} (x \mid x > S_1) f(x)\, dx \qquad (16)$$

For a given S, the expected cost during the review period w_p is

$$K_1 (S) = Cq_p + C\, D(S) + hI_1(S) + \pi I_{21}(S) + \tau\, I_{22}(S)$$

Hence, the expected total cost during the review period w_p is

$$K(s) = \int_s^{s+q_p} K_1 (S)\, h(S)\, dS \qquad (17)$$

To obtain the optimum value of s, we have to solve $d\,K(s) / ds = 0$ for s and verify that $d^2\, K(s) / ds^2 > 0$. In practice, the resulting equation is very complicated and involves several integrals which can not be explicitly evaluated.

We can however, find the approximate solution by considering the first order approximation as follows :

Let us assume that $\theta << w_p$ so that for $x > S_1$, $S\theta w_p / x < 1$. Hence we can take the power series expansion of exp (θ w_p) and log ($1 + S\theta w_p / x$) upto first degree approximation in θ for $x > S_1$. With this approximation, we can take

$$S_1 = \frac{S\theta w_p}{\exp(\theta w_p) - 1} \approx (1 - \theta w_p / 2) \approx \eta S \qquad (18)$$

where $\eta = (1 - \theta w_p / 2)$. Also note that $\eta^r = (1 - r\theta w_p / 2)$. For computation see Appendix 1.

Using above relations and following Jani et al (1978), the expected total cost of the system during the review period w_p is given by

$$K(s) = \frac{1}{q_p} \int_s^{s+q_p} K_1(S)\, dS \qquad (19)$$

4. Observations :

For a given x and S considering t_1 as a function of θ, we have $t_1(0) = Sw_p / x$ and $t_1(\theta) = \frac{1}{\theta}\log\left(1 + \frac{S\theta w_p}{x}\right) < Sw_p / x = t_1(0)$.

Thus, for $\theta > 0$, $(0, t_1(\theta)) \subset (0, t_1(0))$ which means that when $\theta > 0$, the shortages will sufficiently more.

When $\alpha = 1$, i.e. when all units are backordered then this model reduces to the model given by Jani et. al. (1978). When $\alpha = 1$ and $\theta = 0$ i.e. when all units are backordered and there is no deterioration, the model developed reduces to that of Naddor (1966).

5. An Example :

Let the p.d.f. of the demand x be given by

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0 \;, & \text{otherwise} \end{cases}$$

The average demand of this distribution is $\mu = 0.75$.

Substituting this in the cost function (eq. (19)) and simplifying it, we get the following expression for cost function :

$$K(s) = As^4 + Bs^3 + Ds^2 + Es + F \qquad (20)$$

where A, B, D, E, and F are constants. (see Appendix 2).

The optimum reorder point s_0 which gives minimum cost is obtained by solving $d\,K(s) / ds = 0$ for s and verifying that $d^2\,K(s) / ds^2 > 0$.

Consider an inventory model which has the following parameters :

Scheduling period, w_p = 1 month.
Lot size, q_p = 1 unit (= 1000 items).
Purchase cost, C = 50,000 $ / unit = 50 $ / item.
Inventory holding cost, h = 9,000 $ / unit / month.
Shortage cost for backorders, π = 25, 000 $ / unit / month.
Shortage cost for lost sales, τ = 20,000 $ / unit.

Table : Effect of variation in α and θ on Optimal reorder point and optimal cost

α	θ	s	K(s)
0.75	0.000	- 0.798	14881
	0.005	- 0.844	15353
	0.010	- 0.867	16076
	0.015	- 0.890	16860
0.80	0.000	- 0.775	14563
	0.005	- 0.823	15217
	0.010	- 0.843	15909
	0.015	- 0.867	16658
0.85	0.000	- 0.761	14336
	0.005	- 0.804	15126
	0.010	- 0.825	15793
	0.015	- 0.846	16512
1.00	0.000	- 0.747	14109
	0.005	- 0.762	14887
	0.010	- 0.778	15643
	0.015	- 0.803	16299

From the above table it can be observed that increase in the deterioration of units decreases the value of reorder point, s, resulting in more shortages and as a result increases the total cost during the review period w_p. Keeping deterioration rate constant, increase in the backordering rate increases the value of reorder point, s, and decreases the total cost during w_p.

6. Conclusions :

A review period reorder point inventory model is developed in which units are subject to deterioration at a constant rate and shortages are allowed but only partially backlogged, i.e. rest goes as lost sales. Also the demand during the prescribed scheduling period is a random variable following suitable probability distribution. The model can be applied to super malls where the stuff like dairy products, vegetables, fruits, cosmetics etc is subject to deterioration. On the other hand out dated fashion goods goes as a lost sales. The model can be extended to time dependent deterioration of units, in cash discount flow approach i.e. optimizing the present value of all future cash – out - flows, etc.

Appendix 1

$$S_1 = \frac{S\theta w_p}{\exp(\theta w_p) - 1} = \frac{S\theta w_p}{1 + \theta w_p + \frac{\theta^2 w_p^2}{2!} + \ldots - 1} = \frac{S\theta w_p}{\theta w_p + \frac{\theta^2 w_p^2}{2!} + \ldots}$$

$$= \frac{S\theta w_p}{\theta w_p (1 + \frac{\theta w_p}{2!} + \ldots)}$$

$$= S\,(1 + \frac{\theta w_p}{2})^{-1}$$

$$= S\,(1 - \frac{\theta w_p}{2})$$

$$= \eta\, S$$

(All expansions are done under assumption that $\theta << w_p$.).

Appendix 2

A =

$$\frac{-9C\theta w_p}{8} - \frac{hw_p}{8} + \frac{3h\theta w_p^2}{4} - \frac{\pi\alpha w_p}{8} + \frac{3\pi\theta w_p^2}{4} + \frac{\tau(1-\alpha)}{4} - \frac{3\tau(1-\alpha)\theta w_p}{4}$$

$$\frac{9C\theta w_p q_p}{4} + C\theta w_p - \frac{hw_p q_p}{4} + \frac{3h\theta w_p^2 q_p}{4} + h\theta w_p^2 - \frac{\pi\alpha w_p q_p}{4}$$

B =

$$+ \frac{3\pi\alpha\theta w_p^2 (q_p - 1)}{2} - \frac{\tau(1-\alpha) q_p (1 + 3\theta w_p)}{2}$$

$$D = \begin{aligned} &\frac{C\theta w_p(3+6q_p-q_p^4)}{4} + \frac{hw_p(3-q_p^2)}{4} + \frac{3h\theta w_p^2 q_p(q_p-1)}{2} \\ &+ \frac{3\pi\alpha\theta w_p^2(2q_p^2-3q_p+1)}{4} + \frac{\pi\alpha w_p(3-q_p^2)}{4} \\ &+ \frac{\tau(1-\alpha)q_p^2(1-3\theta w_p)}{2} + \frac{3\tau(1-\alpha)\theta w_p}{4} \end{aligned}$$

$$E = \begin{aligned} &\frac{C\theta w_p q_p(6+8q_p-9q_p^2)}{8} + \frac{hw_p q_p(6-q_p^2)}{8} + \frac{h\theta w_p^2 q_p^2(3q_p-4)}{4} \\ &+ \frac{3\pi\alpha\theta w_p^2 q_p(1-2q_p+q_p^2)}{4} + \frac{\pi\alpha w_p q_p(6-q_p^2)}{8} - \pi\alpha w_p \\ &+ \frac{\tau(1-\alpha)(q_p^3-4)}{4} + \frac{3\tau(1-\alpha)\theta w_p q_p(1-q_p^2)}{4} \end{aligned}$$

$$F = \begin{aligned} &\frac{C\theta w_p q_p^2(10q_p+10-9q_p^2)}{40} + \frac{hw_p q_p^2(10-q_p^2)}{40} + \frac{h\theta w_p^2 q_p^3(3q_p-5)}{20} \\ &+ \frac{\pi\alpha\theta w_p^2 q_p^2(6q_p^2-15q_p+10)}{40} + \frac{\pi\alpha w_p(10q_p^2-q_p^4-20q_p+15)}{40} \\ &+ \frac{\tau(1-\alpha)\theta w_p q_p^2(5-3q_p^2)}{20} + + \frac{\tau(1-\alpha)(5-10q_p-q_p^4)}{20} \end{aligned}$$

REFERENCES

[1]**Barankin, E. and Denny, J. (1960),** ***Examination of an inventory model incorporating probabilities of obsolescence***, Logistic Review &. Miltetary Logistic., 1, 11 – 25;
[2]**Brown, G., Lu, J. and Wolfson, R. (1964),** ***Dynamic modeling of inventory subject to obsolescence***, Management Science, Vol. 11, 51 – 63;
[3]**Bulinskaya, E. (1964),** ***Some results concerning optimal inventory policies,*** Theoretical Probability Applications, Vol. 9, 389 – 403;
[4]**Cox, D. R., (1963),** ***Renewal Theory,*** John Wiley and Sons, Inc.
[5]**Fries, B. (1975), Optimal ordering policy for a perishable commodity with fixed life - time**, Operational Research, Vol. 23, 46 – 61;
[6]**Goyal, S. K. and Giri, B. C. (2001),** ***Recent trends in modeling of deteriorating inventory,*** European Journal of Operations Research, Vol.134,1-16;
[7]**Hadley, G. and Whitin, T. M. (1961),** ***An optimal final inventory model,*** Management Science, Vol. 7;

[8]**Hadley, G. and Whitin, T. M. (1962),** ***Generalization of the optimal final inventory model*****,** Management Science, Vol. 8;
[9]**Jani, B. B., Jaiswal, M. C. and Shah, Y. K. (1978), (*****s, q_p*****) *- system inventory model for deteriorating items*,** International Journal of Production Research, Vol. 16 (1), 1 – 9;
[10]**Murray, G. R. Jr. and Silver, E. A. (1966),** ***A bayesian analysis of the style goods inventory problem,*** Management Science, Vol. 12;
[11]**Naddor, E. (1966),** ***Inventory Systems***, John Wiley Publications, NY;
[12]**Nahmias, S. (1975a),** ***Optimal ordering policies for perishable inventory - II,*** Operations Research, Vol. 23, 735 – 749;
[13]**Nahmias, S. (1975b),** ***On ordering perishable inventory under erlang demand***, Naval Research Logistics (Quarterly), Vol. 22, 415 – 425;
[14]**Nahmias, S. and Pierskalla, W. P. (1973),** ***Optimal ordering policies for a product that perishes in two periods subject to stochastic demands***, Naval Research Logistics (Quarterly) Vol. 22, 207 - 229;
[15]**Nahmias, S. and Pierskalla, W. P. (1975*), Optimal ordering policies for perishable inventory - I,*** Proceedings of XX International meeting of the Institute of Management Science, Vol. II, 485 – 493;
[16]**Raafat, F. (1991) ,** ***Survey of literature on continuously deteriorating inventory models,*** Journal of Operational Research Society, Vol. 42 (1), 27 – 37;
[17]**Shah, Nita. H. and Shah, Y. K. (2000),** ***Literature survey on inventory model for deteriorating items,*** Economic Annals (Yugoslavia) Vol. XLIV, 221 – 237;
[18]**Nita H. Shah (2008),** ***One time only incentives and inventory policies for deteriorating items****; Journal of Economic Computation and Economic Cybernetics Studies a*nd Research, vol. 42. no. 3-4, Bucharest;
[19]**Van Zyl, G. J. J. (1964),** ***Inventory control for perishable commodities,*** Doctoral dissertation, University of North Carolina.