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SOME APPROXIMATIONS USED IN THE RISK PROCESS OF INSURANCE COMPANY

Abstract: *In an insurance company, the risk process estimation and the estimation of the ruin probability are important concerns for an actuary: for researchers, at the theoretical level, and for the management of the company, as these influence the insurer strategy. We consider the evolution on an extended period of time of an insurer surplus process. In this paper, we present some processes of estimating of the ruin probability. We discuss the approximations of ruin probability with respect to the parameters of the individual claim distribution, with the load factor of premiums and with the intensity parameter of the number of claims process. We analyze the model where the premiums are computed on the basis of the mean value principle. We give numerical illustration.*

Keywords: *ruin probability, risk process, adjustment coefficient.*

JEL Classification: C020, G220, G320

1. INTRODUCTION

The mathematical model of an insurance risk business is composed of the following objects:

a) A sequence $\{X_i\}_{i=1,2,3,\ldots}$ of independent and identically distributed random variables (i.i.d.r.v.), having the common distribution function *F*. X_i is the cost of the i-th individual claim.

b) A stochastic process $N=\{N(t);t\geq 0\}$. *N* is the number of claims paid by the company in the time interval $[0,t]$. The counting process *N* and the sequence $\{X_i\}$ are independent objects.

The total amount of claims paid by the company in the time interval $[0,t]$ is $D(t)=\sum_{i=1}^{N(t)} X_i$ ($\{D(t);t\geq 0\}$ being the claim process). The risk process $Y=\{Y(t);t\geq 0\}$ is defined by $Y(t)=c\cdot t-D(t)$, where *c* is a positive real constant number corresponding to the average premium income per unit of time. We shall consider that the process *N* is a homogeneous Poisson process with intensity λ and that we will use the mean value principle in order to compute the net premiums, thus $c=(1+\theta)\cdot\lambda\cdot m_1$, where θ is the relative safety loading and m_1 is the expected value of the individual claim or the expected cost of a claim. We denote $m_k=E\left[X_i^k\right], k=1,2,3,\ldots$. We shall denote by *r* – initial capital, $C(t)$ - the capital of the company at moment *t*, hence $r=C(0)$ and $C(t)=r+Y(t)$. We define the ruin as being the situation when the capital of the company takes a negative value. The ruin moment τ is defined as $\tau=\inf\{t\geq 0 \mid C(t)<0\}$.

Let $h(r)=\int_0^\infty \left(e^{r\cdot x}-1\right)dF(x)$ and $g(\theta)=(1+\theta)\cdot\lambda\cdot m_1$.

We denote by $\Psi_n(r,\theta)$ the ruin probability up to moment *n* and by $\Psi(r,\theta)$ the ruin probability on an infinite time horizon, so

$\Psi_n(r,\theta)=P\left(\tau<n \mid C(0)=r,\, g(\theta)=c\right)$,

$\Psi(r,\theta)=P\left(\tau<\infty \mid C(0)=r,\, g(\theta)=c\right)$ and $\Psi(r,\theta)=\lim_{n\to\infty}\Psi_n(r,\theta)$.

The adjustment coefficient (or Lundberg exponent) *R* is the positive solution of the equation $\lambda\cdot h(r)-c\cdot r=0$.

A well-known result is that: if the adjustment coefficient *R* exists, the ruin probability is

$$\Psi(r,\theta)=e^{-R\cdot r}\cdot\left(E\left[e^{R\cdot S(\tau)}\right]\right)^{-1}, \tag{1}$$

where $S(\tau)=(-C(\tau)|\tau<\infty))$ represents the severity of the loss at the moment of ruin.

In case the individual claim follows an exponential distribution, $X_i\in Exp(\alpha), \alpha>0$ then

$$\Psi(r,\theta)=\frac{\lambda}{\alpha\cdot g(\theta)}\cdot e^{-\left(\alpha-\frac{\lambda}{g(\theta)}\right)\cdot r} \tag{2}$$

The integrated tail distribution is $F_I(z)=\frac{1}{m_1}\cdot\int_0^z(1-F(x))dx$. It is known the Beekman's convolution (or Pollaczek-Khinchine) formula:

$$\Psi(r,\theta)=\frac{\theta}{1+\theta}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{1+\theta}\right)^n\cdot\tilde{F}_I^{n*}(r) \tag{3}$$

where $\tilde{F}_I(r)=1-F_I(r)$. If F is the cumulative distribution function for a exponential distributed loss, then $F_I=F$.

2. SOME APPROXIMATIONS OF RUIN PROBABILITY

De Vylder (1978) proposed the following approximation, which is based on the idea to replace the risk process Y by a risk process $\tilde{Y}$ with exponential distributed claims such that $E\left[\tilde{Y}^k(t)\right]=E\left[Y^k(t)\right], k=1,2,3$. The risk process $\tilde{Y}$ is determined by the parameters $\left(\tilde{\lambda},\tilde{\theta},\tilde{\alpha}\right)$. Since $\ln z=\ln r+i\cdot\theta$, where $z=r\cdot e^{i\cdot\theta}$, we have

$\ln E\left[e^{i\cdot\nu\cdot Y(t)}\right]=t\cdot\left(i\cdot\nu\cdot\theta\cdot\lambda\cdot m_1-\frac{\nu^2}{2}\cdot\lambda\cdot m_2+i\cdot\frac{\nu^3}{3!}\cdot\lambda\cdot m_3+0\left(\nu^3\right)\right)$ and we get $\tilde{m}_1=\frac{1}{\tilde{\alpha}}=\frac{m_3}{3\cdot m_2}$, $\tilde{\theta}=\frac{2}{3}\cdot\frac{m_1\cdot m_3}{m_2^2}\cdot\theta$ and $\tilde{\lambda}=\frac{9}{2}\cdot\frac{m_2^3}{m_3^2}\cdot\lambda.$ Thus, we obtain the approximation

$$\Psi(r,\theta)\approx\Psi_{DV}\left(r,\tilde{\theta}\right)=\frac{1}{1+\tilde{\theta}}\cdot e^{-\frac{\tilde{\theta}\cdot\tilde{\alpha}}{1+\tilde{\theta}}\cdot r} \tag{4}$$

$$\Psi_{DV}(r,\theta)=\frac{3\cdot m_2^2}{3\cdot m_2^2+2\cdot m_1\cdot m_3\cdot\theta}\cdot e^{-\frac{6\cdot m_1\cdot m_2\cdot\theta\cdot r}{3\cdot m_2^2+2\cdot m_1\cdot m_3\cdot\theta}} \tag{5}$$

If $X:\begin{pmatrix}n_i\\p_i\end{pmatrix}_{i\in I\subset\mathbf{N}}, p_i>0, \sum_{i\in I}p_i=1$, we introduce $b_i=\frac{n_i\cdot p_i}{m_1}$, $a_i=\frac{n_i^2\cdot p_i}{m_2}$, $X':\begin{pmatrix}n_i\\a_i\end{pmatrix}_{i\in I}$, $X'':\begin{pmatrix}n_i\\b_i\end{pmatrix}_{i\in I}$ and we obtain $\tilde{m}_1=\frac{1}{3}\cdot E(X')$, $\tilde{\theta}=\frac{2}{3}\cdot\frac{E(X')}{E(X'')}\cdot\theta$, $\tilde{\lambda}=\frac{9}{2}\cdot\frac{E(X)\cdot E(X'')}{E(X')}\cdot\lambda$ and

$$\Psi_{DV}(r,\theta)=\frac{3\cdot E(X'')}{3\cdot E(X'')+2\cdot E(X')\cdot\theta}\cdot e^{-\frac{6\cdot\theta\cdot r}{3\cdot E(X'')+2\cdot E(X')\cdot\theta}} \tag{6}$$

The most famous approximation is the Cramer-Lundberg approximation:

$$\Psi(r,\theta)\approx\Psi_{CL}(r,\theta)=\frac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R)-c}\cdot e^{-R\cdot r} \tag{7}$$

Let *SE* be the class of subexponential distribution, i.e. $F\in SE$ if $\lim\limits_{x\to\infty}\frac{\tilde{F}^{2*}(x)}{\tilde{F}(x)}=2$. It is shown by Embrechts and Veroverbeke (1982), that, if $F_I\in SE$, then

$$\Psi(r,\theta)=\frac{1}{1+\theta}\cdot\tilde{F}_I(r) \tag{8}$$

We showed that if $F_I\in SE$ then

$$\lim_{r\to\infty}\frac{\Psi(r,\theta)}{1-F_I(r)}=\frac{\lambda\cdot m_1}{g(\theta)-\lambda\cdot m_1} \tag{9}$$

Another well known result is $\lim\limits_{r\to\infty}\Psi(r,\theta)\cdot e^{R\cdot r}=C$, where

$C=\dfrac{\theta\cdot m_1}{R\cdot\int_0^\infty x\cdot e^{R\cdot x}\cdot\tilde{F}(x)dx}$ is Cramer constant and *R* is Lundberg exponent.

For small θ we have $\frac{m_1}{m_2}\cdot\theta<R<2\frac{m_1}{m_2}\cdot\theta$, so that we may have $R\approx\frac{3m_1}{2m_2}\cdot\theta$.

Let $H(r)=P\left(-\inf\limits_{t\ge0}Y(t)\le r\;\middle|\;-\inf\limits_{t\ge0}Y(t)>0\right)$.

We have $\Psi(r,\theta)=\frac{1}{1+\theta}\cdot(1-H(r))$, $\mu_H=\frac{(1+\theta)\cdot m_2}{2\cdot\theta\cdot m_1}$ and

$\sigma_H^2=\frac{(1+\theta)\cdot m_2}{2\cdot\theta\cdot m_1}\left(\frac{2m_3}{3m_2}+\frac{(1-\theta)m_2}{2\cdot\theta\cdot m_1}\right)$, where μ_H and σ_H^2 are the mean and the variance corresponding to *H*. The idea of Beekman-Bowers approximation is to replace $H(r)$ with a Γ-distribution function $G(r)$, such that the two first moments of *H* and *G* should match. It this way, it is obtained the approximation formula

$$\Psi(r,\theta) \approx \Psi_{\mathrm{BB}}(r,\theta) = \frac{1}{(1+\theta)\cdot\Gamma(a)} \int_{br}^{\infty} x^{a-1}\cdot e^{-x} dx \tag{10}$$

where $a = \dfrac{3(1+\theta)\cdot m_2^2}{3m_2^2 + \theta\left(4m_1 m_3 - 3m_2^2\right)}$ and $b = \dfrac{6m_1 m_2 \theta}{3m_2^2 + \theta\left(4m_1 m_3 - 3m_2^2\right)}$.

In the case of exponentially distributed claims we have $a = 1$, $b = \dfrac{\alpha\theta}{1+\theta}$ and $\Psi_{\mathrm{BB}}(r,\theta) = \dfrac{1}{1+\theta}\cdot e^{-\frac{\alpha\theta r}{1+\theta}} = \Psi(r,\theta)$.

The simplest approximation, which only depends on some moments of F, seems to be the diffusion approximation:

$$\Psi(r,\theta) \approx \Psi_{\mathrm{D}}(r,\theta) = e^{-2\cdot\frac{m_1}{m_2}\cdot\theta\cdot r} \tag{11}$$

As the Lundberg exponent R is small for small values of θ we have $h(R) = m_1\cdot R + \dfrac{m_2}{2}\cdot R^2 + o\left(R^3\right)$ which leads to $R = 2\dfrac{m_1}{m_2}\cdot\theta + o\left(\theta^2\right)$ and $\Psi_{\mathrm{CL}}(r,\theta) = \dfrac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R) - c}\cdot e^{-R\cdot r} = \dfrac{1}{1+o\left(\theta^2\right)}\cdot e^{-2\frac{m_1}{m_2}\cdot\theta\cdot r + o\left(\theta^2\right)}$.

Thus the diffusion approximation may be regarded as a simplified Cramer-Lundberg approximation. Since the diffusion approximation is not very accurate, Grandell (2000) proposed to use for $h(r)$ three moments. Thus $R = 2\dfrac{m_1}{m_2}\cdot\theta - \dfrac{4m_1^2 m_3\theta^2}{3m_2^3} + o\left(\theta^3\right)$,

$\dfrac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R) - c} = \dfrac{3m_2^2}{3m_2^2 + 2m_1 m_3\theta} + o\left(\theta^2\right)$ and

$$\Psi_{\mathrm{G}}(r,\theta) = \frac{3m_2^2}{3m_2^2 + 2m_1 m_3\theta}\cdot e^{-\left(2\frac{m_1}{m_2}\cdot\theta - \frac{4m_1^2 m_3\theta^2}{3m_2^3}\right)\cdot r} \tag{12}$$

Since $2\dfrac{m_1}{m_2}\cdot\theta - \dfrac{4m_1^2 m_3\theta^2}{3m_2^3} = \dfrac{6m_1 m_2\theta}{3m_2^2 + 2m_1 m_3\theta} + o\left(\theta^2\right)$ the De Vylder approximation may be regarded as a simplified Grandell approximation.

Another approximation is obtained using Renyi's theorem about the p-thinning of the point process. Thus

$$\Psi_{\mathrm{R}}\left(r,\theta\right)=\frac{1}{1+\theta}\cdot e^{-\frac{2m_1\theta r}{m_2\cdot(1+\theta)}} \tag{13}$$

Kalashnikov (1997) showed that $\sup_{r}\left|\Psi_{\mathrm{R}}\left(r,\theta\right)-\Psi\left(r,\theta\right)\right|\le\frac{4m_1m_3\theta}{3m_2^2(1+\theta)}$ for all $\theta>0$.

3. NUMERICAL RESULTS AND CONCLUSIONS

In this section, our purpose is to compare the different approximations listed above, through a numerical example. We have to deal with the absolute error (δ) and the relative error (ε). Thus, for approximation Ψ_{A} of Ψ we have $\delta_{\mathrm{A}}\left(r,\theta\right)=\left|\Psi_{\mathrm{A}}\left(r,\theta\right)-\Psi\left(r,\theta\right)\right|$ and $\varepsilon_{\mathrm{A}}\left(r,\theta\right)=\frac{\delta_{\mathrm{A}}\left(r,\theta\right)}{\Psi\left(r,\theta\right)}$. In order to compare an approximation Ψ_{A} with another approximation Ψ_{B}, we will use $\delta_{\mathrm{AB}}\left(r,\theta\right)=\left|\Psi_{\mathrm{A}}\left(r,\theta\right)-\Psi_{\mathrm{B}}\left(r,\theta\right)\right|$ and $\varepsilon_{\mathrm{AB}}\left(r,\theta\right)=\frac{\delta_{\mathrm{AB}}\left(r,\theta\right)}{\Psi_{\mathrm{A}}\left(r,\theta\right)}$.

Let $X:\begin{pmatrix}1 & 5\\ 0.875 & 0.125\end{pmatrix}$ be a discrete random variable describing a claim (a loss) which takes the value of 1 monetary unit a high probability, and a relatively large value of 5 monetary units (in the case of a natural disaster, for example) with a relatively low probability. In the following, we list the approximations of ruin probability, ARP (see table 1-4). For graphic illustration (see figure 1-4) we compute $\mathrm{MARP}=10^6\cdot\mathrm{ARP}$, which is shown in figures.

Table 1

ARP for $\theta = 0.2$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.732078	0.733542	0.735414	0.860708
5	0.445179	0.444494	0.446051	0.472367
10	0.239060	0.238377	0.238754	0.223130
20	0.068937	0.068845	0.068480	0.049787
30	0.019879	0.019917	0.019598	0.011109
40	0.005732	0.005767	0.005615	0.002479
50	0.001653	0.001670	0.001609	0.000553

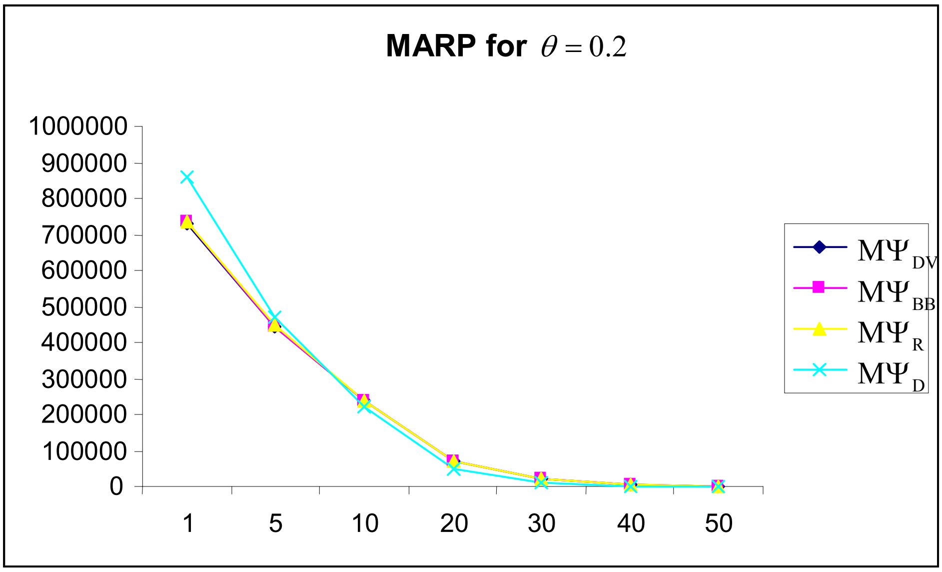

Figure 1

Table 2

ARP for $\theta = 0.3$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.643143	0.644393	0.646979	0.798516
5	0.323441	0.322402	0.323761	0.324652
10	0.136979	0.136533	0.136268	0.105399
20	0.024568	0.024610	0.024140	0.011109
30	0.004406	0.004447	0.004276	0.001171
40	0.000790	0.000805	0.000758	0.000123
50	0.000142	0.000146	0.000134	0.000013

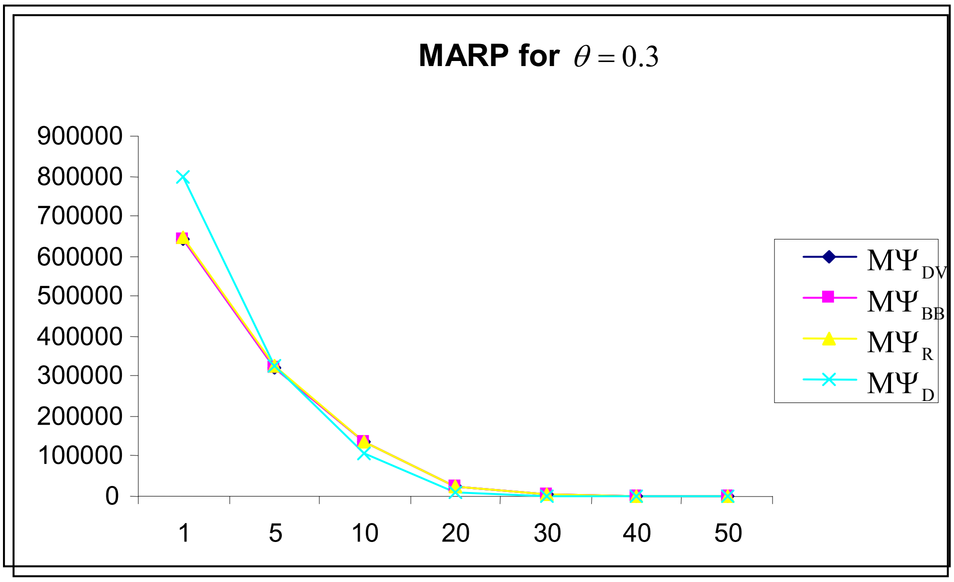

Figure 2

Table 3

ARP for $\theta = 0.5$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.515174	0.516071	0.519201	0.687289
5	0.191486	0.190638	0.191003	0.153355
10	0.055573	0.055507	0.054723	0.023518
20	0.004681	0.004740	0.004492	0.000553
30	0.000394	0.000407	0.000369	0.000013
40	0.000033	0.000035	0.000030	$0.3 \cdot 10^{-6}$
50	0.000003	0.000003	0.000002	$0.7 \cdot 10^{-8}$

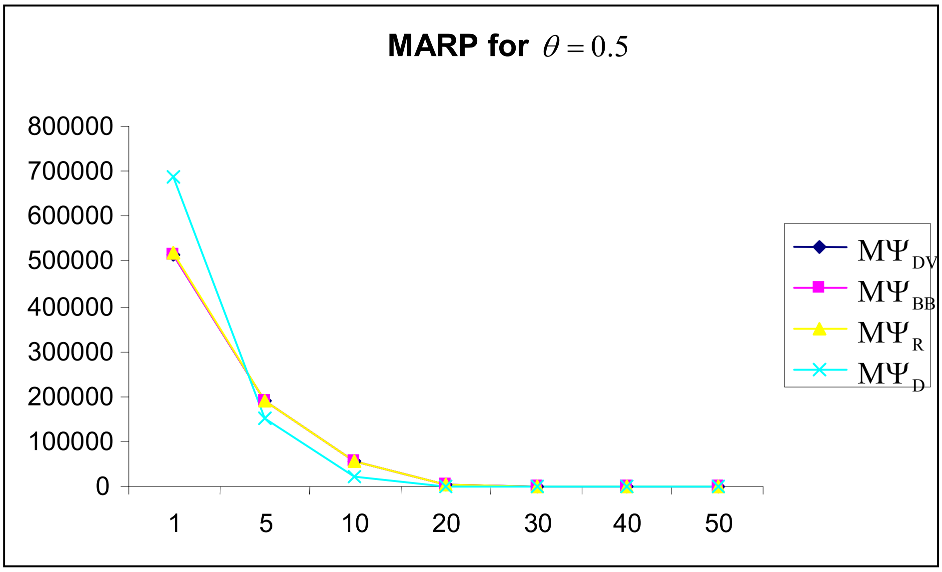

Figure 3

Table 4

ARP for $\theta = 0.8$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.394417	0.394444	0.398073	0.548812
5	0.105883	0.105335	0.104931	0.049787
10	0.020461	0.020516	0.019819	0.002479
20	0.000764	0.000789	0.000707	0.000006
30	0.000029	0.000030	0.000025	$0.15 \cdot 10^{-7}$
40	0.000001	0.000001	$0.9 \cdot 10^{-6}$	$0.38 \cdot 10^{-10}$
50	$0.4 \cdot 10^{-7}$	$0.5 \cdot 10^{-7}$	$0.3 \cdot 10^{-7}$	$0.9 \cdot 10^{-13}$

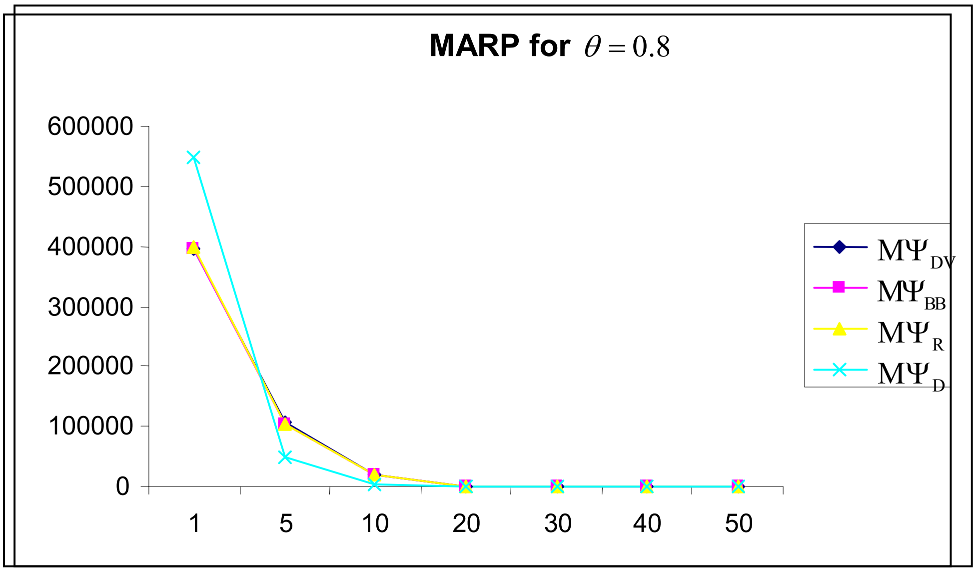

Figure 4

As we illustrate relative error (see tables 5-6) we compute $\mathrm{MERR1} = 10^5 \cdot \varepsilon_{DV,BB}$, $\mathrm{MERR2} = 10^5 \cdot \varepsilon_{DV,R}$ and $\mathrm{MERR3} = 10^5 \cdot \varepsilon_{DV,D}$ which are shown in figures 5-6.

Table 5

$\varepsilon_{\mathrm{DV},\bullet}$ **for** $\theta = 0.3$

r	$\varepsilon_{\mathrm{DV,BB}}$	$\varepsilon_{\mathrm{DV,R}}$	$\varepsilon_{\mathrm{DV,D}}$
1	0.00194	0.00596	0.24158
5	0.00321	0.00099	0.00374
10	0.00326	0.00519	0.23055
20	0.00171	0.01742	0.54783
30	0.00931	0.02951	0.73423
40	0.01899	0.04051	0.84430
50	0.02817	0.05634	0.90845

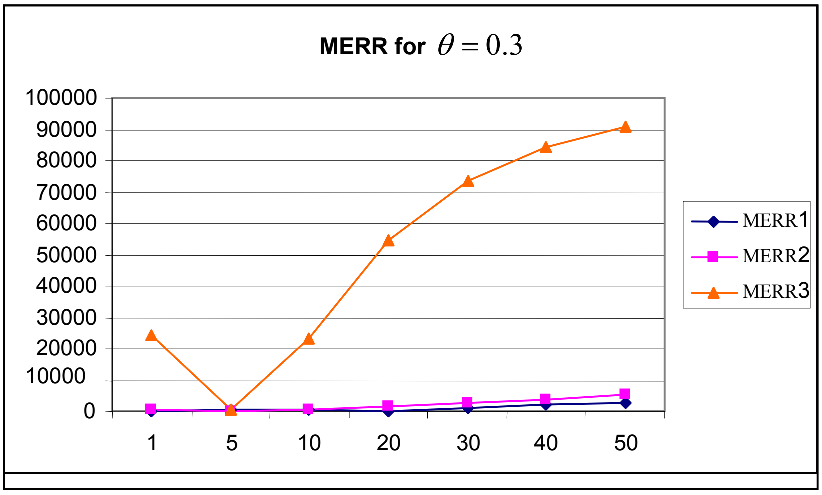

Figure 5

Table 6

$\varepsilon_{\mathrm{DV},\bullet}$ **for** $\theta = 0.5$

r	$\varepsilon_{\mathrm{DV,BB}}$	$\varepsilon_{\mathrm{DV,R}}$	$\varepsilon_{\mathrm{DV,D}}$
1	0.00174	0.00782	0.33409
5	0.00443	0.00252	0.19913
10	0.00119	0.01529	0.57681
20	0.01260	0.04038	0.88186
30	0.03299	0.06345	0.96700
40	0.06061	0.09090	0.99091
50	0.03682	0.17200	0.99767

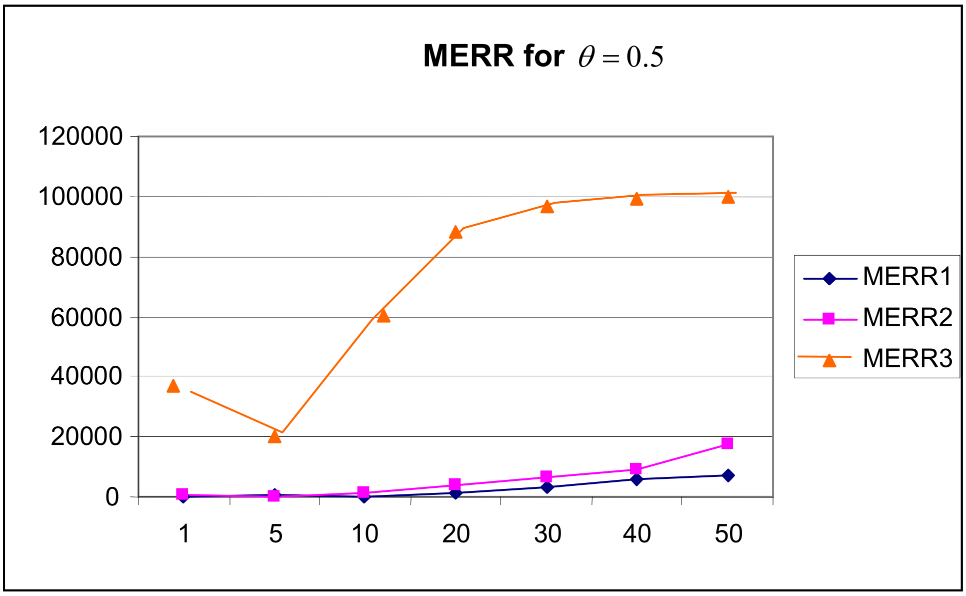

Figure 6

We consider that the approximations Ψ_{DV}, Ψ_{BB} and Ψ_{R} are better. They are maxim absolute errors up to 0.0015 for $\theta = 0.2$, up to 0.0013 for $\theta = 0.3$, up to 0.0009 for $\theta = 0.5$ and maxim relative errors up to 2.7% for $\theta = 0.2$, up to 5.5% for $\theta = 0.3$ etc. We observe that the relative errors increase when initial capital increases while the ruin probability decreases. The diffusion approximation is not very accurate because its relative errors are very high, up to 99% if $\theta = 0.5$. For the repartition of claim, which we used there are no exact ruin probabilities available.

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