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CONVERGENCE OF A COURNOT OLIGOPOLY GAME WITH EXTRAPOLATIVE EXPECTATIONS

Abstract: *The analysis of convergence to equilibrium behavior in Cournot competition has always been a crucial issue of interest. The convergence of a Cournot oligopoly game with nonlinear cost functions is investigated in this paper, where each firm is assumed able to have simple foresight with respect to its rivals' choices in the sense of extrapolative expectations. We eventually arrive at a conclusion that extrapolative expectations act as a stabilizing factor, especially when the foresight is strong enough. This seemingly rational mechanism is therefore desirable since the corresponding long-term behavior will become more predicable.*

Keywords**:** *Cournot oligopoly; convergence; equilibrium behavior; extrapolative expectations; stabilizing factor.*

JEL classification: C62, C73

1. Introduction

It always remains a question whether or not the convergence to equilibrium behavior in oligopoly competition can be achieved (Valléea and Yıldızoglub, 2009). As is well-known, under the original assumption in game theory, perfect rationality and full knowledge, the equilibrium will be reached in one shot without any adjustment. Adjustment dynamics are subsequently introduced to overcome this unrealistic hypothesis and stability analysis thus is discussed to demonstrate firms' long-run behavior (Angelini *et al.*, 2009). For example, Theocharis (1959) argues that the equilibrium under the best-reply dynamic will become unstable for more than three firms in a simple Cournot oligopoly game with linear cost functions. Fisher (1961) points out that increasing marginal costs and decreasing adjustment speeds can make his considered modified best-reply dynamic more stable.

For a fixed adjustment dynamic in an oligopoly game and other economic models, the stability is more associated with the underlying rationality which is embodied by different learning mechanisms. For example, Kamalinejad *et al.* (2010) prove that stability of the equilibrium can be achieved under their adjustment dynamic in Cournot competition by means of linear regression and recursive weighted least-squares. Naimzada and Tramontana (2010) show under a gradient-like decisional process for a boundedly rational consumer, his consumption choice may converge to the equilibrium provided that he becomes able to learn from his history by a simple least squared learning mechanism. In addition, Huck *et al.* (1999) introduce inertia into the best-reply dynamic so that the firms update their quantities with a probability of reluctance to show the adjustment process. They indicate that the firms' behavior can converge to Cournot equilibrium easily with this learning mechanism. This similar mechanism involving inertia, at the same time, is also discussed under the joint strategy fictitious play dynamic (2009).

It is worth noticing that most investigations (Agiza and Elsadany, 2004; Bischi and Kopel, 2001; Fisher, 1961; Kopel, 1996; Matsumoto, 2006; Theocharis, 1959), have been made on a myopic postulation that in every time period each oligopolistic firm determines its profit-maximizing quantity by assuming that its rivals' outputs will remain the same as in the immediately previous period, which seems far from real. Our present paper investigates how extrapolative expectations influence the convergence under a discrete adjustment process for a Cournot oligopoly game with nonlinear cost functions. The firms with extrapolative expectations usually have short-term foresight with respect to its rivals' choices, which has been discussed widely under the continuous price adjustment dynamic (Enthoven and Arrow, 1956; Arrow and Nerlove, 1958). In particular, Quandt (1967) shows that the convergence to the equilibrium can be guaranteed under a weaker condition in a Bertrand oligopoly game. That is, the firms can obtain their optimal prices easily if they are endowed with this learning mechanism. We finally conclude that extrapolative expectations act as a stabilizing factor, even though our discussion is made in the context of a discrete adjustment dynamic in a Cournot oligopoly game.

The remaining of this paper is organized as follows. We illustrate a basic model in Section 2 and introduce extrapolative expectations in Section 3. In Section 4, our main results are derived. A sophisticated model is discussed in Section 5 and Section 6 concludes this paper. We provide the technical proofs in the Appendix.

2. A basic model

We focus on a Cournot oligopoly market where n firms produce homogeneous goods and simultaneously adjust their outputs at each discrete time period. Denote by $q_i(t)$ firm i's quantity at time period t. At each period any firm will form an expectation concerning its rivals' quantities in the subsequent period in order to determine the corresponding profit-maximizing choice. Generally speaking, they take the following adjustment process, referred as the best-reply dynamic (Bischi and Kopel, 2001; Kamalinejad *et al.*, 2010)

$$q_i(t+1) = \arg\max_{q_i} \pi_i(q_1^e(t+1), \cdots, q_{i-1}^e(t+1), q_i, q_{i+1}^e(t+1), \cdots, q_n^e(t+1)) \quad (1)$$

where $q_j^e(t+1)$ ($j \neq i$) denotes firm i's expectation with respect to firm j's quantity.

The market price at time period t is usually assumed linear and decreasing with respect to the total demand $\sum_{i=1}^{n} q_i(t)$,

$$p(t) = a - b\sum_{i=1}^{n} q_i(t), a, b > 0 .$$

We employ the following nonlinear cost function for firm i

$$C_i(t) = (d_i / 2)q_i^2(t) + c_i q_i(t) + g_i ,$$

where d_i, g_i and c_i are non-negative. The resulting expected profit for firm i is thus given by

$$\pi_i(q_1(t), \cdots, q_n(t)) = q_i(t)\left[a - c_i - b\sum_{i=1}^{n} q_i(t) \right] - g_i - (d_i / 2)q_i^2(t) .$$

The optimization problem for q_i can be solved when

$$\frac{\partial \pi_i (q_1^e(t+1),\cdots,q_i,\cdots,q_n^e(t+1))}{\partial q_i} = a - c_i - b\sum_{j=1,\, j\neq i}^{n} q_j^e(t+1) - (2b+d_i)q_i = 0 .$$

Solving this equation yields the best-reply dynamic for firm i,

$$q_i(t+1) = \frac{a-c_i}{2b+d_i} - \frac{b}{2b+d_i}\sum_{j=1,\, j\neq i}^{n} q_j^e(t+1) . \tag{2}$$

The expectations towards any competitor's quantity are traditionally assumed equal to corresponding adjusted ones in the immediately preceding period directly, i.e. $q_j^e(t+1) = q_j(t)$ (Agiza and Elsadany, 2004; Bischi and Kopel, 2001; Kamalinejad *et al.*, 2010; Matsumoto, 2006). Accordingly, the best-reply dynamic in the Cournot oligopoly above takes the form

$$q_i(t+1) = \frac{a-c_i}{2b+d_i} - \frac{b}{2b+d_i}\sum_{j=1,\, j\neq i}^{n} q_j(t) .$$

That is,

$$q(t+1) = (I - A)q(t) + v \tag{3}$$

where $q(t) = (q_1(t),\cdots,q_n(t))^T$, $v = \left(\frac{a-c_1}{2b+d_1},\cdots,\frac{a-c_n}{2b+d_n}\right)^T$ and I is an identity matrix. $A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & i = j \\ \dfrac{b}{2b+d_i}, & i \neq j \end{cases}, \qquad i, j = 1,\cdots,n .$$

3. Introduction of extrapolative expectations

Even though traditional expectation of the competitors' quantity above simplifies the adjustment process, it is myopic and inconsistent with reality in that rivals' quantities will not remain the same as in the immediately preceding period until the stable equilibrium is approached. To avoid this unrealistic assumption, we assume all firms share the homogeneous extrapolative expectations with respect to the competitors' quantity in the following way (Enthoven and Arrow, 1956; Arrow and Nerlove, 1958; Quandt, 1967)

$$q_j^e(t+1) = q_j(t+\lambda) = q_j(t) + \lambda \dot{q}_j(t)$$

where $0 \le \lambda < 1$ measures the level of extrapolative expectations.

Since we focus on the discrete adjustment process, the derivative of quantity over time $\dot{q}_j(t)$ can be approached by

$$\dot{q}_j(t) = q_j(t+1) - q_j(t),$$

which implies

$$q_j^e(t+1) = q_j(t+\lambda) = (1-\lambda)q_j(t) + \lambda q_j(t+1). \tag{4}$$

Hence, the discrete extrapolative expectations can be viewed as a tradeoff between the quantities to be adjusted in the coming time period and in the immediately preceding period. This type of expectations seems to represent the individual moderate rationality between bounded rationality usually discussed above and complete rationality.

Substituting (4) into (2) yields

$$q_i(t+1) = \frac{a-c_i}{2b+d_i} - \frac{(1-\lambda)b}{2b+d_i}\sum_{j=1, j\ne i}^{n} q_j(t) - \frac{\lambda b}{2b+d_i}\sum_{j=1, j\ne i}^{n} q_j(t+1),$$

namely,

$$q(t+1) = (1-\lambda)(I-A)q(t) + \lambda(I-A)q(t+1) + v.$$

We can determine λ such that the spectral radius of the matrix $\lambda(I-A)$ is less than one, namely, $\lambda < 1/\rho(I-A)$. So we have

$$q(t+1) = (1-\lambda)[I-\lambda(I-A)]^{-1}(I-A)q(t) + [I-\lambda(I-A)]^{-1}v. \tag{5}$$

4. Main results

We now turn to discuss the long-term behavior if all firms determine their quantities following dynamic (5). It is well-understood that the stability of dynamic (5) depends completely on the eigenvalues of corresponding coefficient matrix $(1-\lambda)[I-\lambda(I-A)]^{-1}(I-A)$. The firms' behavior under dynamic (5) will converge to Nash equilibrium if and only if these eigenvalues are less than one in the absolute value. This coefficient matrix is associated with matrix A, whose fundamental properties can be given by Proposition 1. (The proof is provided in the Appendix).

Without loss of generality, assume that $d_1 \le d_2 \le \cdots \le d_n$. Let $\mu_1, \cdots, \mu_n$ be characteristic roots of matrix A. For convenience, denote $\Delta_i = (b+d_i)/(2b+d_i)$.

Proposition 1.

(a) $\Delta_i \le \mu_i \le \Delta_{i+1}, \quad (i = 1, 2, \cdots, n-1)$

both equalities holding if and only if $d_i = d_{i+1}$.

(b)
$$n-(n-1)\Delta_n \le n-\sum_{i=2}^{n}\Delta_i \le \mu_n \le n-\sum_{i=1}^{n-1}\Delta_i \le n-(n-1)\Delta_1,$$

these equalities holding if and only if $d_1 = d_2 = \cdots = d_n$.

It is worth noticing that $\mu_1 \le \cdots \le \mu_n$ since $\Delta_i < 1$. Noticing that

$$[I-\lambda(I-A)]^{-1}(I-A) = [I+\lambda(I-A)+\lambda^2(I-A)^2+\cdots](I-A),$$

we can conclude that

$$\delta_i = (1-\lambda)[1+\lambda(1-\mu_i)+\cdots](1-\mu_i) = (1-\lambda)(1-\mu_i)/[1-\lambda(1-\mu_i)]$$

is a characteristic root of $(1-\lambda)[I-\lambda(I-A)]^{-1}(I-A)$.

By Proposition 1 we know δ_i is real because μ_i is real (and further positive). It can be derived that $|\delta_i| < 1$ if and only if $\mu_i(1-2\lambda) < 2(1-\lambda)$, which implies the following claim.

Proposition 2. For $\lambda < \max\{1/\rho(I-A), 1\}$,

(a) if $\lambda \ge 1/2$, dynamic (5) is always stable.

(b) if $\lambda < 1/2$,

(i) a sufficient condition for the stability of dynamic (5) is

$$n-\sum_{i=1}^{n-1}\Delta_i < \frac{2(1-\lambda)}{1-2\lambda}$$

and it is also sufficient that $n-(n-1)\Delta_1 < \dfrac{2(1-\lambda)}{1-2\lambda}$;

(ii) a necessary condition for the stability of dynamic (5) is

$$n-\sum_{i=2}^{n}\Delta_i < \frac{2(1-\lambda)}{1-2\lambda},$$

and it is also necessary that $n-(n-1)\Delta_n < \dfrac{2(1-\lambda)}{1-2\lambda}$;

(iii) if $d_1 = d_2 = \cdots = d_n$, a necessary and sufficient condition for the stability of dynamic (5) is $n-(n-1)\Delta_i < \frac{2(1-\lambda)}{1-2\lambda}$.

Since dynamic (5) reduces to dynamic (3) if $\lambda = 0$, stability analysis for dynamic (3) can be derived directly. We now turn to discuss the effect of parameter λ on the convergence to equilibrium behavior. Given that the sufficient condition for the convergence of dynamic (3) holds, it is also sufficient for the stability of dynamic (5) since $2(1-\lambda)/(1-2\lambda)$ increases with respect to λ. Therefore, extrapolative expectations indeed act as a stabilizing factor.

We can further conclude that larger λ rises with great improvement on the convergence, that is, strong foresight leads to the long-term convergent behavior easily. In particular, if $\lambda \geq 1/2$, all firms with moderate rationality characterized by extrapolative expectations can interact with each other like completely rational ones.

Some instinctive explanations are provided here to explain why extrapolative expectations can help to stabilize a Cournot oligopoly market. Each firm's quantity will remain almost unchanged when the equilibrium is being approached for any adjustment dynamic. Under dynamic (3), each firm adjusts its quantity based on the quantities in the immediately preceding period, which implies strict restrictions are needed for a very close approximation to every firm's choice becoming invariable. On the other hand, under dynamic (5), extrapolative expectations relaxes this myopic assumption partially, decreasing the variations of two successive decision-making time periods and leading to a better approximation to the equilibrium even with weaker conditions.

5. A sophisticated model

In dynamic (5), just a simple linear method is used to deal with extrapolative expectations. However, more appropriate technology can be employed to approach these expectations provided that necessary historical outputs are available. To be exact, the following second order Taylor approximation is introduced

$$q_j^e(t+1) = q_j(t+\lambda) \approx q_j(t) + \lambda \dot{q}_j(t) + \frac{\lambda^2}{2}\dot{q}_j(t)$$
$$\approx q_j(t) + \lambda[q_j(t+1) - q_j(t)] + \frac{\lambda^2}{2}[q_j(t+1) - 2q_j(t) + q_j(t-1)]$$
$$= \frac{\lambda^2}{2}q_j(t-1) + (1-\lambda-\lambda^2)q_j(t) + \frac{\lambda^2+2\lambda}{2}q_j(t+1)$$

where the approximation transformation $\dot{q}_j(t) \approx q_j(t+1) - 2q_j(t) + q_j(t-1)$ is used as well. Obviously, historical output $q_j(t-1)$ is needed to form this sophisticated foresight. With all firms behaving with these expectation, dynamic (2) becomes

$$q_i(t+1) = \frac{a-c_i}{2b+d_i}$$
$$-\frac{b}{2b+d_i}\left[\frac{\lambda^2}{2}\sum_{j=1,j\neq i}^{n} q_j(t-1) + (1-\lambda-\lambda^2)\sum_{j=1,j\neq i}^{n} q_j(t) + \frac{\lambda^2+2\lambda}{2}\sum_{j=1,j\neq i}^{n} q_j(t+1)\right]$$

namely,

$$q(t+1) = \frac{\lambda^2}{2}(I-A)q(t-1) + (1-\lambda-\lambda^2)(I-A)q(t) + \frac{\lambda^2+2\lambda}{2}(I-A)q(t+1) + v.$$

Similarly, for the parameter λ ensuring that the spectral radius of the matrix $(\lambda^2/2+\lambda)(I-A)$ is less than one, i.e. $\lambda < \sqrt{1+2/\rho(I-A)} - 1$, we have

$$q(t+1) = \frac{\lambda^2}{2}\left[I - \frac{\lambda^2+2\lambda}{2}(I-A)\right]^{-1}(I-A)q(t-1) +$$
$$(1-\lambda-\lambda^2)\left[I - \frac{\lambda^2+2\lambda}{2}(I-A)\right]^{-1}(I-A)q(t) + \left[I - \frac{\lambda^2+2\lambda}{2}(I-A)\right]^{-1} v \qquad (6)$$

It is extremely difficult to derive analytical results about the stability of dynamic (6). Therefore, we investigate the effect of parameter λ on the convergence to the equilibrium by numerical simulations. Letting $a = 6, b = 0.14$, $c_1 = c_2 = c_3 = c_4 = c_5 = 1, d_1 = 0.1, d_2 = 0.2, d_3 = 0.3, d_4 = 0.4, d_5 = 0.5$, we can get the non-convergent trajectory of dynamic (3) in Fig. 1. On the other hand, when $\lambda = 0.1, 0.2, 0.3$ respectively, with the same parameters we are able to obtain the convergent trajectories of dynamic (6), shown in Fig. 2, Fig. 3 and Fig. 4. As a sequence, sometimes unstable quantities emerging in dynamic (3) can be stabilized by introducing extrapolative expectations. Moreover, it can be observed that a larger parameter λ gives rise to a faster convergence speed.

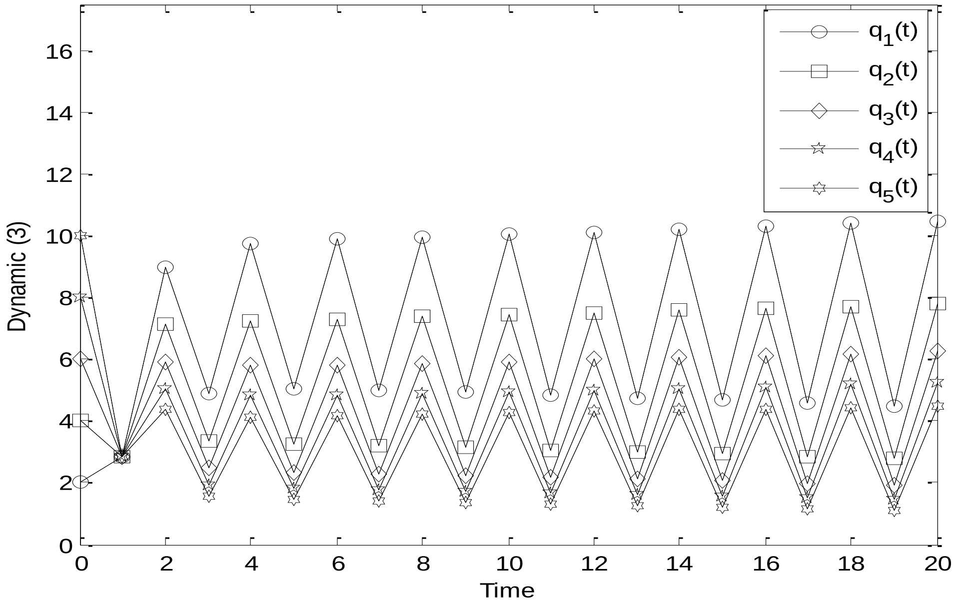

Figure 1. Dynamic (3)

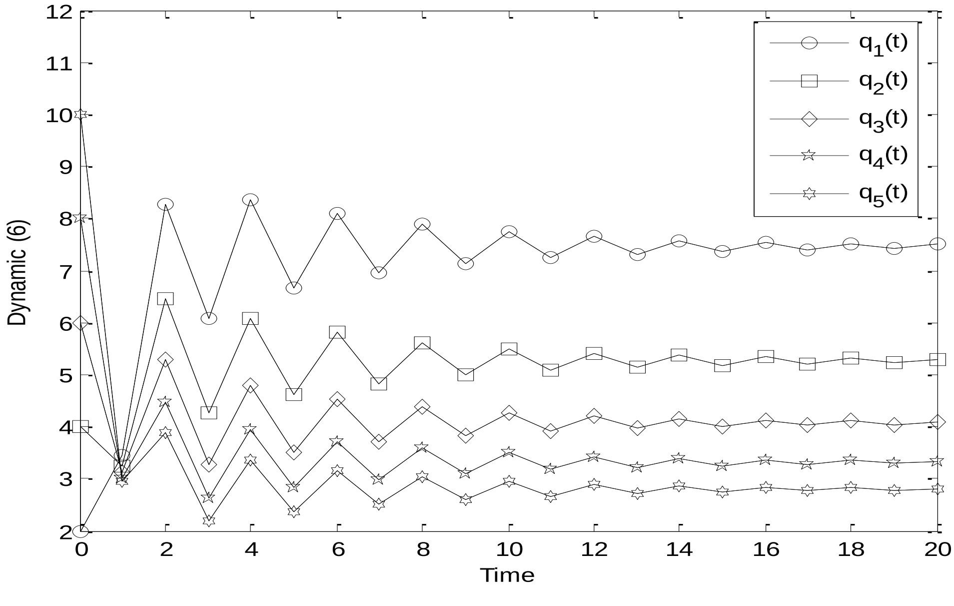

Figure 2. Dynamic (6) with $\lambda = 0.1$

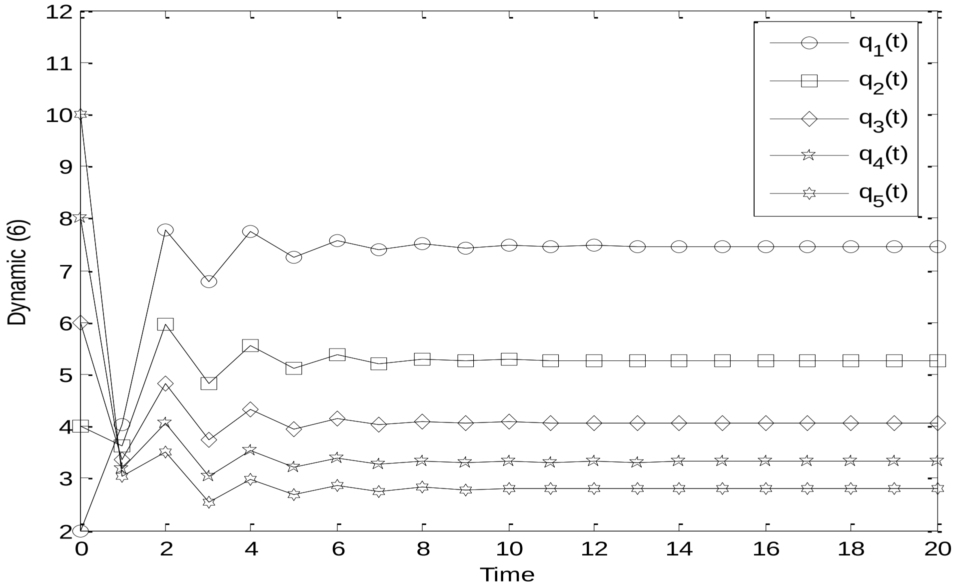

Figure 3. Dynamic (6) with $\lambda = 0.2$

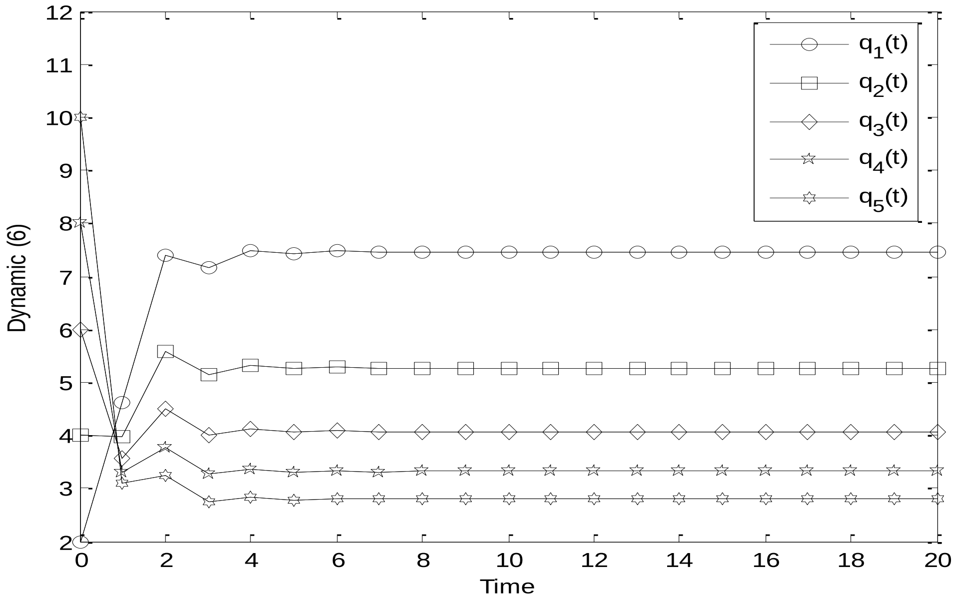

Figure 4. Dynamic (6) with $\lambda = 0.3$

6. Conclusions

One of the most important issues in a Cournot oligopoly model is the convergence analysis of the equilibrium, especially when learning mechanisms are taken into account. In this paper, introducing extrapolative expectations into every time period as an estimation method for the firms in a Cournot oligopoly game with incomplete information, we demonstrate that extrapolative expectations act as a stabilizing factor, especially for the strong foresight. We further indicate that convergence to equilibrium can be always guaranteed provided that the foresight characterized by extrapolative expectations with a linear method is strong enough.

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Appendix

Proof of Proposition 1:

(a) We suppose generally that there are k different elements in the set $\{\Delta_i | i=1,\cdots n\}$, i.e. $\Delta_1=\cdots=\Delta_{p_1}<\Delta_{p_1+1}=\cdots=\Delta_{p_2}<\cdots<\Delta_{p_{k-1}+1}=\cdots=\Delta_{p_k}$, which implies that $d_1=\cdots=d_{p_1}<d_{p_1+1}=\cdots=d_{p_2}<\cdots<d_{p_{k-1}+1}=\cdots=d_{p_k}$ and $p_k=n$. For later convenience, let $p_0=0$. The characteristic polynomial for matrix A can be calculated as follows,

$$
\begin{aligned}
F(\mu) &= \prod_{i=1}^{n}(\mu-\Delta_i)-\sum_{i=1}^{n}\left[\frac{b}{2b+d_i}\prod_{j\neq i}(\mu-\Delta_j)\right] \\
&= \prod_{i=1}^{k}(\mu-\Delta_{p_i})^{p_i-p_{i-1}-1}\left\{\prod_{i=1}^{k}(\mu-\Delta_{p_i})-\sum_{i=1}^{k}\left[\frac{(p_i-p_{i-1})b}{2b+d_{p_i}}\prod_{j\neq i}(\mu-\Delta_{p_j})\right]\right\} \\
&= G(\mu)\prod_{i=1}^{k}(\mu-\Delta_{p_i})^{p_i-p_{i-1}-1}
\end{aligned}
$$

where $G(\mu)=\prod_{i=1}^{k}(\mu-\Delta_{p_i})-\sum_{i=1}^{k}\left[\frac{(p_i-p_{i-1})b}{2b+d_{p_i}}\prod_{j\neq i}(\mu-\Delta_{p_j})\right]$. It is obvious that $G(\Delta_{p_s})\neq 0$ .Thus we can conclude that if $p_s-p_{s-1}>1$,then Δ_{p_s} is a characteristic root with multiplicity $p_s - p_{s-1} - 1$. That is, $\mu_{p_{s-1}+1} = \cdots = \mu_{p_s - 1} = \Delta_{p_s}$. Further, note $G(\Delta_{p_s})$ has the same sigh with $(-1)^{k-s+1}$. Thus, $G(\Delta_{p_s})G(\Delta_{p_{s+1}}) < 0$ for $s = 1, \cdots k-1$. Consequently, there exists a characteristic root μ_{p_s} such that $\Delta_{p_s} < \mu_{p_s} < \Delta_{p_{s+1}}$ by the existence theorem of zero point ($s = 1, 2, \cdots, k-1$). In sum, $\Delta_i \le \mu_i \le \Delta_{i+1}$ for $i = 1, 2, \cdots, n-1$.

(b) Since the sum of characteristic roots is equal to the trace of corresponding matrix, we can estimate the last characteristic root μ_n by

$$n - \sum_{i=2}^{n} \Delta_i \le \mu_n = \mathrm{tr}(A) - \sum_{i=1}^{n-1} \mu_i \le n - \sum_{i=1}^{n-1} \Delta_i$$

and further

$$n - (n-1)\Delta_n \le n - \sum_{i=2}^{n} \Delta_i \le \mu_n \le n - \sum_{i=1}^{n-1} \Delta_i \le n - (n-1)\Delta_1,$$

where the order of Δ_i is considered.