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LINGUISTIC INDUCED GENERALIZED AGGREGATION DISTANCE OPERATORS AND THEIR APPLICATION TO DECISION MAKING

Abstract. *We introduce a wide range of linguistic induced generalized aggregation distance operators. First, we present the linguistic induced generalized ordered weighted averaging distance (LIGOWAD) operator. It is a generalization of the OWA operator that uses linguistic variables, distance measures, order inducing variables and generalized means in order to provide a more general formulation. One of its main results is that it includes a wide range of linguistic aggregation distance operators such as the linguistic induced OWA distance (LIOWAD), the linguistic induced Euclidean ordered weighted averaging distance (LIEOWAD) operator and the linguistic generalized OWA distance (LGOWAD) operator. We further generalize the LIGOWAD operator by using quasi-arithmetic means obtaining the linguistic induced quasi-arithmetic OWAD (Quasi-LIOWAD) operator and by using hybrid averages forming the linguistic induced generalized hybrid average distance (LIGHAD) operator. We end the paper with an application of the new approach in a linguistic decision making problem concerning human resource management.*

Key words: *Linguistic variables, OWA operator, Distance measure, Decision making, Human resource management.*

JEL Classification: D81, M12, M51

1. Introduction

Different types of aggregation operators are found in the literature for aggregating the information (Beliakov et al., 2007; Calvo et al., 2002; Xu and Da, 2003). A very common aggregation method is the ordered weighted averaging (OWA) operator introduced by Yager (1988), whose prominent characteristic is the reordering step. The OWA operator provides a parameterized family of aggregation operators that includes as special cases the maximum, the minimum and the average criteria. Since its appearance, the OWA operator has been used in a wide range of applications such as (Ahn, 2009; Amin and Emrouznejad, 2006; Cheng et al., 2009; Filev and Yager, 1998; Herrera and Herrera-Viedma, 1997; Kacprzyk and Zadrozny, 2009; Karayiannis, 2000; Liu et al., 2010; Liu, 2008; Merigó, 2010; Merigó and Casanovas, 2010a; Merigó et al., 2010; Merigó and Wei, 2011; Wang et al., 2009; Xu, 2004; 2005a; 2008; Yager, 1993; 2010; Zeng and Su, 2011; Zhou and Chen, 2010; 2011).

An interesting extension of the OWA operator is the induced OWA (IOWA) operator (Yager, 2003; Yager and Feliv, 1999). The IOWA operator differs in that the reordering step is not developed with the values of the arguments but can be induced by another mechanism such that the ordered position of the arguments depends upon the values of their associated order-inducing variables. The IOWA operator has been studied by different authors in recent years (Chiclana et al., 2007; Merigó and Casanovas, 2009; 2010b; 2010d; 2011a; 2011b; Merigó et al., 2011; Merigó and Gil-Lafuente, 2009; Tan and Chen, 2010; Wei, 2010b; Xu, 2006, Yager, 2004).

A further interesting extension is the one that uses the OWA and the IOWA operator in distance measures. Recently, motivated by the idea of the OWA operator, Xu and Chen (2008) defined the ordered weighted distance (OWD) measure whose prominent characteristic is that they can alleviate (or intensify) the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. Yager (2010) generalized Xu and Chen's distance measures and provided a variety of ordered weighted averaging norms, based on which he proposed several similarity measures between fuzzy sets. Merigó and Gil-Lafuente (2010) introduced a new index for decision-making using the OWA operator to calculate Hamming distance called the ordered weighted averaging distance (OWAD) operator, and gave its application in the selection of financial products and sport management. Zeng and Su (2011) extended Xu and Chen's result to intuitionistic fuzzy environment and presented the intuitionistic fuzzy ordered weighted distance (IFOWD) operator. On the basis of the idea of the IOWA operator, Merigó and Casanovas (2010d) presented an induced ordered weighted averaging distance (IOWAD) operator that extends the OWA operator by using distance measures and a reordering of arguments that depends on order-inducing variables. The IOWAD generalizes the OWAD operator and provides a parameterized family of distance aggregation operators between the maximum and the minimum distance. Merigó and Casanovas (2011a) presented an induced Euclidean ordered weighted averaging distance (IEOWAD) operator, which uses the IOWA operator and the Euclidean distance in the same formulation. Going a step further, Merigó and Casanovas (2011b) introduced the induced generalized OWA distance (IGOWAD) (or induced Minkowski OWA distance (IMOWAD) operator), which generalizes the OWD measure, the OWAD operator, the IOWAD operator, the IEOWAD operator and a lot of other particular cases. It is very useful for decision-making problems because it can establish a comparison between an ideal, though unrealistic, alternative and available options in order to find the optimal choice. As such, the optimal choice is the alternative closest to the ideal one. The main advantage of the IGOWAD operator is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the aggregation process. Therefore, we are able to deal with more complex situations more close to the real world.

Usually, when using the IGOWAD operator and above distance measures, it is assumed that the available information is clearly known and can be assessed with exact numbers. However, this may not be the real-life situation found in the decision-making problems because often the available information is vague or imprecise, or it is not possible to analyze the situation with exact numbers. In this case, a better approach may be the use of linguistic variables (for example, when evaluating the comfort or design of a car, terms like good, fair, poor can be used; when evaluating a car's speed linguistic terms like fast, very fast, slow can be used instead of numerical values (Bordogna and Fedrizzi,1997)). The use of the fuzzy linguistic approach (Zadeh, 1975) provides a direct way to manage the uncertainty and model the linguistic assessments by means of linguistic variables. Thus it is necessary to extend the IGOWAD operator and above distance measures to accommodate these situations.

For doing so, we will develop the linguistic induced generalized ordered weighted averaging distance (LIGOWAD) operator (or linguistic induced Minkowski OWA distance (LIMOWAD) operator), which is an extension of the IGOWAD operator with linguistic variables. Thus, the LIGOWAD uses the IOWA operator, distance measures and uncertain information represented in the form of linguistic variables. The LIGOWAD includes a wide range of distance operators such as the linguistic maximum distance, the linguistic minimum distance, the linguistic normalized generalized distance (LNGD), the linguistic weighted generalized distance (LWGD), the linguistic generalized ordered weighted averaging distance (LGOWAD) operator, the linguistic induced ordered weighted averaging distance (LIOWAD) operator and the linguistic induced Euclidean ordered weighted averaging distance (LIEOWAD) operator. We study some families of the LIGOWAD operators. The main advantage of the LIGOWAD is that it is able to deal with complex reordering processes that represent a wide range of factors in an uncertain environment that can be assessed with linguistic variables. Then, we can deal with the information in situations with high degree of uncertainty. Another advantage is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the decision process by using order-inducing variables. In addition, we generalize the LIGOWAD operator by using quasi-arithmetic means and obtaining the quasi-arithmetic LIOWAD (Quasi-LIOWAD). The main advantage of this approach is that it includes the LIGOWAD as a special case and a lot of other cases. Thus, we get a more robust formulation of this model.

Moreover, we also extend this approach by using the hybrid average (Xu and

Da, 2003). By doing so, we are able to use the weighted average, distance measures and the IOWA in the same formulation and in an uncertain environment that can be assessed with linguistic variables. We call it the linguistic induced generalized hybrid averaging distance (LIGHAD) operator. One of its key features is that it includes a wide range of aggregation operators including the LGOWAD and the LWGD. We also generalize this approach by using quasi-arithmetic means obtaining the linguistic induced quasi-arithmetic hybrid average distance (Quasi-LIHAD) operator. Finally, we develop a decision making approach for human resource management based on the developed operators.

This paper is organized as follows. Section 2 presents some basic concepts. In Sect. 3, we present the LIGOWAD operator and Sect. 4 introduces the Quasi-LIOWAD operator. Sect. 5 presents the LIGHAD and the Quasi-LIHAD operators and in Sect. 6 we develop an application in decision making. Finally, in Sect. 7 we summarize the main conclusions of the paper.

2. Preliminaries

The distance measures are very useful techniques that have been used in a wide range of applications such as fuzzy set theory, decision making, operational research, etc. The generalized (or Minkowski) distance is one of the most widely used distance measures which generalizes a wide range of other distances such as the Hamming distance, the Euclidean distance, etc. For two sets, $A=\{a_1,a_2,...,a_n\}$ and $B=\{b_1,b_2,...,b_n\}$, they can be described as follows.

Definition 1. A normalized generalized distance (NGD) of dimension n is a mapping NGD: $R^n \to R$, which has the following form

$$NGD(A,B)=\left(\frac{1}{n}\sum_{i=1}^{n}\left|a_i-b_i\right|^{\lambda}\right)^{1/\lambda} \tag{1}$$

where a_i and b_i is the ith arguments of the sets A and B and λ is a parameter such that $\lambda\in(-\infty,+\infty)$. If we give different values to the parameter λ, we can obtain a wide range of special cases. For example, if $\lambda=1$, we obtain the normalized Hamming distance. If $\lambda=2$, the normalized Euclidean distance.

Sometimes, when normalizing the generalized distance, we prefer to give different weights to each individual distance. In this case, the distances are known as the weighted generalized distance, which can be defined as follows, respectively:

Definition 2. A weighted generalized distance (WGD) of dimension n is a mapping WGD: $R^n \to R$ that has an associated weighting $w=(w_1,w_2,...,w_n)$ with $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$ such that:

$$WGD = \left(\sum_{i=1}^{n} w_i \left| a_i - b_i \right|^{\lambda} \right)^{1/\lambda} \quad (2)$$

where a_i and b_i is the ith arguments of the sets A and B and λ is a parameter such that $\lambda \in (-\infty, +\infty)$. If $\lambda = 1$, we obtain the weighted Hamming distance (WHD). If $\lambda = 2$, the weighted Euclidean distance (WED).

The IOWA operator is an extension of the OWA operator. The main difference is that the reordering step is not carried out with the values of the argument a_i. In this case, the reordering step is developed with order-inducing variables that reflect a more complex reordering process. The IOWA operator also includes as particular cases maximum, minimum and average criteria. It can be defined as follows:

Definition 3. An IOWA operator of dimension n is a mapping *IOWA*: $R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IOWA\left(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, ..., \langle u_n, a_n \rangle \right) = \sum_{j=1}^{n} w_j b_j \quad (3)$$

where b_j is a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

The IGOWAD (or IMOWAD) operator is a distance measure that uses the IOWA operator in the normalization process of the Minkowski distance. Then, the reordering of the individual distances is developed with order inducing variables. For two sets $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$, the IGOWAD operator can be defined as follows:

Definition 4. An IGOWAD operator of dimension n is a mapping f : $R^n \times R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$f\left((u_1, a_1, b_1), (u_2, a_2, b_2), ..., (u_n, a_n, b_n) \right) = \left(\sum_{j=1}^{n} w_j d_j^{\lambda} \right)^{1/\lambda} \quad (4)$$

where d_j is the $|a_i - b_i|$ value of the IGOWAD triplet (u_i, a_i, b_i) having the jth largest u_i, u_i is the order inducing variable, $|a_i - b_i|$ is the argument variable represented in the form of individual distances and λ is a parameter such that $\lambda \in (-\infty, +\infty)$. Especially, if $\lambda = 1$, then the IGOWAD is called the induced ordered weighted averaging distance (IOWAD) operator (Merigó and Casanovas, 2010d), and if $\lambda = 2$, then the induced Euclidean ordered weighted averaging distance (IEOWAD) operator (Merigó and Casanovas, 2011a).

When using the IGOWAD operator, it is assumed that the available information is represented in the form of exact numbers. However, this may not be the real situation found in the decision-making problem. Sometimes the available information is vague or imprecise and it is not possible to analyze it with exact numbers. In this case, it is more suitable to use linguistic variables to assess the uncertainty. In the following, we shall develop the linguistic induced generalized ordered weighted averaging distance (LIGOWAD) operator.

3. Linguistic induced generalized ordered weighted averaging distance (LIGOWAD) operator

The linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables. For computational convenience, let $S = \{s_\alpha \,|\, \alpha = -t, ..., 0, 1, ..., t\}$ be a finite and totally ordered discrete term set, where s_α represents a possible value for a linguistic variable. For example, a set of nine terms S could be given as follows:

$$S = \{s_{-4} = \textit{extremely poor}; s_{-3} = \textit{very poor}; s_{-2} = \textit{poor}; s_{-1} = \textit{slightly poor}; s_0 = \textit{fair}; s_1 = \textit{slightly good}; s_2 = \textit{good}; s_3 = \textit{very good}; s_4 = \textit{extremely good}\} \quad (5)$$

In these cases, it is usually required that there exist the following (Xu, 2004):

1) A negation operator: $Neg(s_i) = s_{-i}$;

2) The set is ordered: $s_i \le s_j$ if and only if $i \le j$.

In order to preserve all the given information, Xu (2005b) extended the discrete term set S to a continuous term set $\overline{S} = \{s_\alpha \,|\, \alpha \in [-t, t]\}$, where, if $s_\alpha \in S$, then we call s_α the original term, otherwise, we call s_α the virtual term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in the actual calculation (Xu, 2006).

Consider any two linguistic terms $s_\alpha, s_\beta \in \overline{S}$, and $\mu > 0$, we define some operational laws as follows:

1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$

2) $\mu s_\alpha = s_{\mu\alpha}$

Different approaches have been suggested for dealing with linguistic information (Alonso et al., 2009; Cabrerizo et al., 2009; Herrera and Herrera-Viedma, 1997; Herrera and Martínez, 2000; Herrera et al., 2008; Kacprzyk and Zadrozny, 2009; Merigó and Casanovas, 2010c; Merigó et al., 2010; Wei, 2009; 2010a; Xu, 2004; 2005b; 2006). In order to measure the deviation between any two linguistic variables $s_\alpha, s_\beta \in \bar{S}$, Xu (2005b) defined a linguistic distance as follows:

Definition 5. Let $s_\alpha, s_\beta \in \bar{S}$, then

$$d(s_\alpha, s_\beta) = \left|s_\alpha - s_\beta\right| = \frac{\alpha - \beta}{2t} \tag{6}$$

is called a distance measure between s_α and s_β.

The linguistic induced generalized ordered weighted averaging distance (LIGOWAD) operator is an extension of the IGOWAD operator that uses uncertain information in the aggregation represented in the form of linguistic labels. The reason for using this operator is that the uncertain factors that affect our decisions are sometimes not clearly known; thus, we shall use linguistic variables in order to assess these situations with a high degree of uncertainty in the information. Note that the LIGOWAD operator can also be seen as an aggregation operator that uses the main characteristics of the IOWA, distance measures and linguistic information. Moreover, it also uses a complex reordering process by using order inducing variables. For two collections of linguistic labels $\alpha = \left(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}\right)$ and $\beta = \left(s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_n}\right)$, it can be defined as follows.

Definition 6. A LIGOWAD operator of dimension n is a mapping *LIGOWAD*: $R^n \times \bar{S}^n \times \bar{S}^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$LIGOWAD\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = \left(\sum_{j=1}^{n} w_j d_j^\lambda\right)^{1/\lambda} \tag{7}$$

where d_j is $\left|s_{\alpha_i} - s_{\beta_i}\right|$ value of the LIGOWAD pair $\left\langle u_i, s_{\alpha_i}, s_{\beta_i}\right\rangle$ having the jth largest u_i, u_i is the order inducing variable and $\left|s_{\alpha_i} - s_{\beta_i}\right|$ is the argument variable represented in the form of individual distances and λ is a parameter such that $\lambda \in (-\infty, +\infty)$.

Example 1. Let $\alpha = \left(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_5}\right) = \left(s_2, s_{-1}, s_3, s_4, s_{-3}\right)$ and $\beta = \left(s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_5}\right) = \; = \left(s_{-2}, s_1, s_0, s_2, s_4\right)$ be two collections of linguistic labels taken from the linguistic label set (5), then

$$d(s_{\alpha_1}, s_{\beta_1}) = \frac{|2-(-2)|}{2 \times 4} = 0.5$$

Similarly, we have

$$d(s_{\alpha_2}, s_{\beta_2}) = 0.25, \quad d(s_{\alpha_3}, s_{\beta_3}) = 0.375, \quad d(s_{\alpha_4}, s_{\beta_4}) = 0.25,$$
$$d(s_{\alpha_5}, s_{\beta_5}) = 0.875$$

Assume that both sets have the same order-inducing variables $U = (6, 7, 3, 9, 4)$. Assume the following weighting vector $W = (0.15, 0.2, 0.2, 0.35, 0.1)$ and without loss of generality, let $\lambda = 2$, then we can calculate the distance between α and β by using the LIGOWAD operator:

$$LIGOWAD(\alpha, \beta) = \left(0.15 \times 0.25^2 + 0.2 \times 0.25^2 + 0.2 \times 0.5^2 + \right.$$
$$\left. + 0.35 \times 0.875^2 + 0.1 \times 0.375^2\right)^{1/2} = 0.59$$

From a generalized perspective of the reordering step, we can distinguish between the descending LIGOWAD (DLIGOWAD) operator and the ascending LIGOWAD (ALIGOWAD) operator by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DLIGOWAD and w^*_{n-j+1} the jth weight of the ALIGOWAD operator.

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the LIGOWAD operator can be expressed as:

$$LIGOWAD\left(\langle u_1, s_{\alpha_1}, s_{\beta_1}\rangle, \ldots, \langle u_n, s_{\alpha_n}, s_{\beta_n}\rangle\right) = \left(\frac{1}{W}\sum_{j=1}^{n} w_j d_j^{\lambda}\right)^{1/\lambda} \quad (8)$$

Similar to the IGOWAD operator, the LIGOWAD operator is commutative, monotonic, bounded and idempotent. Another interesting issue is the problem of ties in the order inducing variables. As it was explained by Yager and Filev (1999), the easiest way to solve this problem consists in replacing each argument of the tied inducing variables by its linguistic normalized linguistic generalized distance.

Analyzing the applicability of the LIGOWAD operator, we can see that it is applicable to similar situations already discussed in other types of induced aggregation operators where it is possible to use linguistic information. For example, we could use it in different decision making problems, etc.

The LIGOWAD operator provides a parameterized family of aggregation operators. Basically, we distinguish between the families found in the weighting vector W and those found in the parameter λ.

If we analyze the parameter λ, we can find a wide range of distance measures such as the LIOWAD, the LIEOWAD, the linguistic induced ordered weighted geometric distance (LIOWGD) operator, the linguistic induced ordered weighted harmonic averaging distance (LIOWHAD) operator and a lot of other cases.

Remark 1. If $\lambda = 1$, then, we get the LIOWAD operator.

$$f\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = \sum_{j=1}^{n} w_j d_j \tag{9}$$

Note that if $w_j = 1/n$ for all j, we get the linguistic normalized Hamming distance (LNHD). The linguistic weighted Hamming distance (LWHD) is obtained if $u_i > u_{i+1}$ for all i, and the linguistic ordered weighted averaging distance (LOWAD) is obtained if the ordered position of u_i is the same as the ordered position of d_j such that d_j is the jth largest of $\left|s_{\alpha_i} - s_{\beta_i}\right|$.

Remark 2. If $\lambda = 2$, then we get the LIEOWAD operator.

$$f\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = \left(\sum_{j=1}^{n} w_j d_j^2\right)^{1/2} \tag{10}$$

Note that if $w_j = 1/n$ for all j, we get the linguistic normalized Euclidean distance (LNED). The linguistic weighted Euclidean distance (LWED) is obtained if $u_i > u_{i+1}$ for all i, and the linguistic Euclidean ordered weighted averaging distance (LEOWAD) is obtained if the ordered position of u_i is the same as the ordered position of d_j such that d_j is the jth largest of $\left|s_{\alpha_i} - s_{\beta_i}\right|$.

Remark 3. When $\lambda = 0$, we get the LIOWGD operator.

$$f\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = \prod_{j=1}^{n} d_j^{w_j} \tag{11}$$

Remark 4. When $\lambda = -1$, we get the LIOWHAD operator.

$$f\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{d_j}} \tag{12}$$

By choosing a different manifestation of the weighting vector in the LIGOWAD operator, we are able to obtain different types of distance aggregation operators. For example, we can obtain the linguistic maximum distance, the linguistic minimum distance, the LNGD, the LWGD, the LGOWAD, the Step-LIGOWAD and the Olympic-LIGOWAD

- If $w_j = 1/n$, we get the LNGD.
- The linguistic maximum distance is obtained if $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\left\{\left|s_{\alpha_i} - s_{\beta_i}\right|\right\}$.
- The linguistic minimum distance is obtained if $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\left\{\left|s_{\alpha_i} - s_{\beta_i}\right|\right\}$.
- The LWGD is obtained if $u_i > u_{i+1}$ for all i.
- The LGOWAD operator is obtained if the ordered position of u_i is the same as the ordered position of d_j such that d_j is the j th largest of $\left|s_{\alpha_i} - s_{\beta_i}\right|$.
- Step-LIGOWA: If $w_k = 1$ and $w_j = 0$ for all $j \neq k$.
- Olympic-LIGOWAD: If $w_1 = w_n = 0$ and for all others $w_j = 1/(n-2)$.

Remark 5. Using a similar methodology, we could develop numerous other families of LIGOWAD operators. For more information, refer to (Beliakov et al., 2007; Merigó and Casanovas, 2010c; 2010d; 2011a; 2011b; Xu and Da, 2008; Yager, 2010).

4. Quasi-LIOWAD Operators

The LIGOWAD can be generalized by using quasi-arithmetic means in a similar way as it was done in Ref. Merigó and Casanovas (2011b) and Merigó and Gil-Lafuente (2009). We call it the Quasi-LIOWAD operator. Its main advantage is that it provides a more general formulation because it includes the LIGOWAD operator as a particular case. It can be defined as follows.

Definition 7. A Quasi-LIOWAD operator of dimension *n* is a mapping *QLIOWAD*: $R^n \times \bar{S}^n \times \bar{S}^n \to R$ that has an associated weighting vector *W* of dimension *n* such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$QLIOWAD\left(\langle u_1, s_{\alpha_1}, s_{\beta_1}\rangle, ..., \langle u_n, s_{\alpha_n}, s_{\beta_n}\rangle\right) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(d_j\right)\right) \tag{13}$$

where d_j is $\left|s_{\alpha_i} - s_{\beta_i}\right|$ value of the QLIOWAD pair $\langle u_i, s_{\alpha_i}, s_{\beta_i}\rangle$ having the j th largest u_i, u_i is the order inducing variable and $\left|s_{\alpha_i} - s_{\beta_i}\right|$ is the argument variable represented in the form of individual distances and g is a general continuous strictly monotonic function.

As we can see, the LIGOWAD operator is a particular case of the QLIOWAD when $g\left(d_j\right) = d_j^{\lambda}$. Note that all the properties and particular cases commented in the LIGOWAD operator are also included in this generalization. For example, we could study different families of QLIOWAD operators such as the Quasi-LNGD, the Quasi-LWGD and the Quasi-Olympic-LIOWAD.

5. Using the Hybrid Average in the LIGOWAD Operator

A further generalization can be developed by using hybrid averages. Thus, we obtain the linguistic induced generalized hybrid average distance (LIGHAD) operator. This operator uses generalized means in the HA operator, distance measures and uncertain situations where the available information can not be represented with exact numbers but it is possible to use linguistic information. By using the HA operator, the LIGHAD considers the WA and the IOWA (or the OWA) in the same problem. In decision making problems, this implies that the LIGHAD operator considers the subjective probability and the attitudinal character of the decision maker in the same formulation. It can be defined as follows.

Definition 8. A LIGHAD operator of dimension n is a mapping *LIGHAD*: $R^n \times \bar{S}^n \times \bar{S}^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$LIGHAD\left(\langle u_1, s_{\alpha_1}, s_{\beta_1}\rangle, ..., \langle u_n, s_{\alpha_n}, s_{\beta_n}\rangle\right) = \left(\sum_{j=1}^{n} w_j d_j^{\lambda}\right)^{1/\lambda} \tag{14}$$

where d_j is $\hat{d}_i$ value ($\hat{d}_i = n\omega_i \left|s_{\alpha_i} - s_{\beta_i}\right|, i = 1, 2, ..., n$) of the LIGHAD pair $\langle u_i, s_{\alpha_i}, s_{\beta_i}\rangle$ having the *j*-th largest u_i, u_i is the order inducing variable, $\omega = \left(\omega_1, \omega_2, ..., \omega_n\right)$ is the weighting vector of the $\left|s_{\alpha_i} - s_{\beta_i}\right|$, with $\omega_i \in [0,1]$

and the sum of the weights is 1, and λ is a parameter such that $\lambda \in (-\infty, +\infty)$.

As we can see, if $w_i = 1/n$, for all *i*, then, the LIGHAD operator becomes the LWGD and if $\omega_i = 1/n$, for all i, it becomes the LIGOWAD operator. Note that a lot of other families could be studied following the methodology explained in Sect. 3. Moreover, it is possible to further extend this approach by using quasi-arithmetic means obtaining the linguistic induced quasi-arithmetic HA distance (Quasi-LIHAD) operator. The Quasi-LIHAD operator includes the LIGHAD as a particular case. It can be defined as follows.

Definition 9. A Quasi-LIHAD operator of dimension n is a mapping *QLIHAD*: $R^n \times \overline{S}^n \times \overline{S}^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$QLIHAD\left(\left\langle u_1, s_{\alpha_1}, s_{\beta_1}\right\rangle, \ldots, \left\langle u_n, s_{\alpha_n}, s_{\beta_n}\right\rangle\right) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(d_j\right)\right) \tag{15}$$

where d_j is $\hat{d}_i$ value ($\hat{d}_i = n\omega_i \left|s_{\alpha_i} - s_{\beta_i}\right|, i = 1, 2, \ldots, n$) of the QLIHAD pair $\left\langle u_i, s_{\alpha_i}, s_{\beta_i}\right\rangle$ having the *j*-th largest u_i, u_i is the order inducing variable, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of the $\left|s_{\alpha_i} - s_{\beta_i}\right|$, with $\omega_i \in [0,1]$ and the sum of the weights is 1, g is a general continuous strictly monotonic function.

6. Illustrative Example

The LIGOWAD and the LIGHAD operator can be applied in a wide range of problems including statistics, economics and engineering. In the following, we are going to focus on an application of the LIGHAD operator in decision-making since it generalizes the LIGOWAD operator. We will consider a decision-making problem about human resource management.

Assume that an enterprise wants to acquire a person for a new position in the company. After an application period, the company has evaluated the applications received. After careful analysis of the information, the group of experts of the enterprise considers five possible human resource.

- A_1 = Candidate 1.
- A_2 = Candidate 2.

- A_3 = Candidate 3.
- A_4 = Candidate 4.
- A_5 = Candidate 5.

When analyzing the candidates, the experts have considered the following general characteristics:

- C_1 = Experience in similar jobs.
- C_2 = Intelligence.
- C_3 = Knowledge about the job.
- C_4 = Motivation.
- C_5 = Skills of the worker.
- C_6 = Other aspects.

Due to the fact that the general characteristics are very imprecise because they contain a lot of particular aspects, the experts cannot use numerical values in the analysis. Instead, they use linguistic variables to evaluate the general results obtained for each candidate depending on the characteristic considered. In order to do so, they establish the following linguistic scale.

$$S = \{s_{-4} = \textit{extremely poor}; s_{-3} = \textit{very poor}; s_{-2} = \textit{poor}; s_{-1} = \textit{slightly poor}; s_0 = \textit{fair}; s_1 = \textit{slightly good}; s_2 = \textit{good}; s_3 = \textit{very good}; s_4 = \textit{extremely good}\} \tag{18}$$

After careful analysis of these characteristics, the experts have given the following information shown in Table 1.

Table 1. Available information about the candidates

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	S_3	S_0	S_4	S_{-1}	S_{-2}	S_1
A_2	S_1	S_2	S_{-1}	S_0	S_1	S_1
A_3	S_{-2}	S_3	S_{-1}	S_{-1}	S_4	S_1
A_4	S_2	S_1	S_2	S_{-1}	S_1	S_0
A_5	S_3	S_0	S_4	S_{-2}	S_0	S_0

According to their objectives, the enterprise establishes the following ideal candidate shown in Table 2.

Table 2. Ideal worker

	C_1	C_2	C_3	C_4	C_5	C_6
I	S_3	S_4	S_4	S_3	S_4	S_3

In order to aggregate the information, the group of experts calculates the attitudinal character of the candidate. Due to the fact that the attitudinal character depends upon the opinion of several members of the board of directors, it is very complex. Therefore, they need to use order inducing variables in the reordering process. The results are shown in Table 3.

Table 3. Order-inducing variables

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	17	13	9	12	10	7
A_2	12	6	24	17	8	30
A_3	16	14	12	10	9	8
A_4	14	17	20	12	16	8
A_5	15	13	11	17	8	19

With this information, it is possible to use the LIGHAD to select a candidate according to the interests of the company. Suppose that, without loss of generality, $\lambda = 2$, and the weighting vector $W = (0.09, 0.17, 0.24, 0.24, 0.17, 0.09)$, which is derived by the Gaussian distribution based method Xu (2005), and $\omega = (0.15, 0.17, 0.12, 0.14, 0.22, 0.3)$, then we get

$$LIGHAD(A_1, I) = 0.485, LIGHAD(A_2, I) = 0.374, LIGHAD(A_3, I) = 0.420,$$
$$LIGHAD(A_4, I) = 0.371, \ LIGHAD(A_5, I) = 0.442$$

Note that in these cases, the result indicates the distance between the linguistic variables of the candidate and the ideal one. Note that the lowest value is the optimal result because we are using distances. Thus, we get the ranking of all the candidates $A_i \ \ (i = 1, 2, 3, 4, 5)$

$$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$$

7. Conclusions

We have presented a wide range of linguistic induced generalized aggregation distance operators. First, we have introduced the LIGOWAD operator. It is a generalization of the OWA operator that uses order inducing variables in order to assess complex reordering processes, distance measure, linguistic information and generalized means. We have analyzed some of its main properties. We have seen that it generalizes a wide range of distance aggregation operators such as the LNGD, the LGOWAD and the LOWAD operator.

Moreover, we have developed a further generalization by using quasi-arithmetic means, obtaining the Quasi-LIOWAD operator. It includes the LIGOWAD as a particular case and a lot of other situations. Thus, we obtain a more robust formulation of the linguistic aggregation operators.

Furthermore, we have presented the LIGHAD and the Quasi-LIHAD operators. The main advantage of these models is that they are able to deal with the OWA and the weighted average in the same formulation in an uncertain environment that can be assessed with linguistic variables.

We have focused on an application in decision making regarding human resource management. The result shows that the approaches are feasible and effective providing a more robust formulation of the previous models.

In future research, we expect to develop further improvements by adding more characteristics in the model such as the use of other types of aggregation operators and apply it in other decision making problems.

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