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POLYNOMIAL CHAOS SOLUTION TO THE BLACK SCHOLES EQUATION WITH A RANDOM VOLATILITY

Abstract. *In this study, the Black Scholes equation with uncertainty in its volatility is considered. A numerical algorithm for option pricing based on the orthonormal polynomials from the Askey scheme is derived. Then dependence of polynomial chaos on the distribution type of the volatility is investigated. Numerical experiments show that when appropriate polynomial chaos is chosen as a basis in the random space for the volatility, the solution to the Black Scholes equation converges significantly fast.*

Keywords: *polynomial chaos, option pricing, stochastic differential equation, Black Scholes equation, spectral method.*

JEL Classification: G13, C63, C02

1. Introduction

In this paper, we study the Black-Scholes partial differential equation for the option price $u(S,t,\xi)$,

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru=0, \qquad (1)$$

where S is the price of the underlying asset, r is the risk-free interest rate and the volatility $\sigma=\sigma(\xi)$ is a function of a random variable ξ. Assuming the volatility as a random process, one can resolve shortcomings of the constant volatility assumption in the Black-Scholes model. See Hou *et al*. (2006) and Lewis (2000).

By changing the variable $x = \ln(S)$, the equation (1) becomes the following equation for $v = v(x,t,\xi) = u(S,t,\xi)$,

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial x} - rv = 0. \qquad (2)$$

Without loss of generality *v* is denoted by *u* for notational simplicity.

The solution *u* is a function of the deterministic variable (x,t) and the random variable ξ. For instance, if $\sigma = dW(t)$ is a standard Gaussian with zero mean and unit standard deviation for a Brownian motion $W(t)$, the solution *u* is a function of *x*, *t* and the Brownian motion path $W_0^\xi = \{W(s) | 0 \le s \le t\}$. There has been many studies on such type of problems. The Cameron-Martin theorem (1947) separates deterministic and random variables of the solution for $\sigma = dW(t)$ by a Fourier transform with respect to the Hermite polynomials of ξ (Mikulevicius (1998)),

Theorem 1 (Cameron-Martin theorem). *Assume that for fixed x and t, u is a function of the Brownian motion W on the interval* [*0, t*] *with* $E|\mathbf{u}(x,t,\xi)|^2 < \infty$*. Then* $\mathbf{u}(x,t,\xi)$ *has the following Wiener chaos expansion* (*WCE*)

$$u(x,t,\xi) = \sum_{\alpha \in I} u_\alpha(x,t) H_\alpha, \qquad u_\alpha(x,t) = E[u(x,t,\xi)\, H_\alpha], \qquad (3)$$

where $I = \{\alpha = (\alpha_1, \alpha_2, \ldots) \mid \alpha_i \in \{0,1,2,\ldots\},\ |\alpha| < \infty\}$ *denotes the set of multi-indices for* $|\alpha| \equiv \sum_{i=1}^{\infty} \alpha_i$*, and* $H_\alpha(\xi)$ *is the multi-variate Hermite polynomial of* $\xi = (\xi_1, \xi_2, \ldots)$*,* $H_\alpha(\xi) = \prod_{i=1}^{\infty} H_{\alpha_i}(\xi_i)$*.* $H_{\alpha_i}(\xi_i)$ *is the normalized* α_i^{th} *order Hermite polynomial. First two statistical moments of* $u(x,t$ *are given by* $E[u(x,t,\xi)] = u_0(x,t)$ *and* $E[u^2(x,t,\xi)] = \sum_{\alpha \in I} |u_\alpha(x,t)|^2$*, respectively.*

Due to the randomness of the solution of this type of equation, one wants to know statistical properties of the solution such as its first, second or higher moments, instead of one particular solution corresponding to a specific realization. Theorem 1 shows that if $\sigma = dW(t)$, statistical moments of the solution can be obtained from the coefficients $\{u_\alpha\}$, which implies that the stochastic equation (2) can be interpreted as a system of deterministic equations for u_α's. Cameron and Martin (1947) show that this *Wiener chaos expansion* (WCE) represents a second-order random processes with respect to orthogonal Hermite polynomials which converges in the mean square sense. Ghanem and Spanos (1991, 1999) extend the polynomial chaos with respect to solid mechanics problems. Askey and Wilson (1985) consider various orthogonal polynomials and classify them.

Polynomials in the Askey classification can be an orthonormal basis if the probability density function of an appropriately chosen random distribution is used as the weight function of the inner product. For example, Jacobi polynomials are orthonormal if the inner product is defined in terms of the density function of the Beta distribution. Xiu and Karniadakis (2002) apply hypergeometric polynomial chaos into stochastic differential equation problems. Hou *et. al*. (2003, 2006) or Lin *et. al*. (2006) then extend the polynomial chaos to stochastic partial differential equations in fluid dynamics. But their studies are mainly focused on a randomness driven by a Brownian motion and their studies are limited to the effects of Hermite polynomial chaos. Mikulevicius and Rozovskii (1998, 2004) perform analytical approaches to the Wiener Chaos expansion.

In this paper, we construct a numerical algorithm based on the hypergeometric polynomial chaos for the Black Scholes equation (1) with a random volatility. Inspired from the work by Hou *et. al.* (2006), we solve the Black Scholes partial differential equation when σ is random and investigate the effects of the type of polynomial chaos. Numerical experiments in Section 4 show that given a certain random distribution for the volatility the option value converges substantially fast if the chosen Wiener-Askey chaos is orthonormal with respect to the probability density function. The solution to the Black Scholes equation seems to converge at a slower rate if the appropriate polynomial chaos is not used.

This paper is organized as follows: In Section 2, we outline properties of the Black Scholes equations and orthogonal polynomials. The numerical scheme based on the polynomial chaos expansion is explained in Section 3. Numerical schemes introduced in Section 3 are validated in Section 4.1 using a linear advection equation, then they are applied to the Black Scholes equation with random volatility in Section 4.2.

2. Black Scholes equation and polynomial chaos

Black Scholes equation (1) can be solved analytically when the option is simple enough. For instance, the European vanilla call option has $\max\{S(T)-E,0\}$ as the payoff at expiry T, where $S(t)$ is the price of the underlying asset at t and E is the exercise price. Then its solution is

$$u(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \ ,$$

where $N(\cdot)$ is the $N(0,1)$ distribution function for a standard normal random variable and

$$d_1 = \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \ ,$$

$$d_2 = \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \ .$$

Let $u_E(S,t)$ denote the solution $u(S,t)$ for the European vanilla call option with the emphasis on the exercise price E in order to use in Section 4.2.

Let $\{p_n(x)\}$ represent hypergeometric orthonormal polynomials such as Jacobi, Laguerre or Hermite polynomials of degree *n*. Polynomials $p_n(x)$ can be derived from the $\mathbf{n}^{\text{th}}$order differentiation, $p_n(x) = \frac{c_n}{\omega(x)}\frac{d^n}{dx^n}[\omega(x)\phi^n(x)]$ for some $\omega(x)$ and $\phi(x)$, with a constant of normalization c_n or from a generating function as explained in Kreyszig(1989). $\{p_n(x)\}$ are orthonormal when the inner product is defined by $< p_m, p_n > \equiv \int_a^b p_m(x)p_n(x)d\mu = \delta_{mn}$, where the measure μ is expressed using a weight function $\omega(x)$, $d\mu(x) = \omega(x)dx$, and [*a,b*] is the corresponding support of the measure μ. For example, Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ are derived by

$$J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!\,(1-x)^\alpha(1+x)^\beta}\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right)$$

with $J_0^{(\alpha,\beta)}(x) = 1$. Setting $\alpha = 0,\ \beta = 0$ results in the Legendre polynomials:

$$J_1^{(0,0)}(x) = \sqrt{3}x,\ \ J_2^{(0,0)}(x) = \sqrt{5}\left(\frac{3x^2-1}{2}\right),\ \ \ J_3^{(0,0)}(x) = \sqrt{7}\left(\frac{10x^3-6x}{4}\right)\ ,\ldots$$

$\left\{J_n^{(\alpha,\beta)}(x)\right\}$ are orthonormal with respect to the probability density function of the Beta distribution, $< J_m^{(\alpha,\beta)}, J_n^{(\alpha,\beta)} > \equiv \int_{-1}^{1} J_m^{(\alpha,\beta)}(x)J_n^{(\alpha,\beta)}(x)\omega(x)dx = \delta_{mn}$, where $\omega(x) = \frac{\Gamma(\alpha+\beta+2)(1-x)^\alpha(1+x)^\beta}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$. Note also that orthogonality of $\{p_n(x)\}$ derives a recurrence relation,

$$c_n p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x), \quad n = 1,2,3,\ldots \qquad (4)$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$, where a_n, b_n and c_n are constants. For example, Jacobi polynomials satisfy

$$xJ_n^{(\alpha,\beta)}(x) = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}J_{n+1}^{(\alpha,\beta)}(x) + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}J_n^{(\alpha,\beta)}(x) + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}J_{n-1}^{(\alpha,\beta)}(x)\ .$$

Jacobi, Laguerre and Hermite polynomials will be used in this study and Appendix A summarizes the properties of Hermite and Laguerre polynomials. Table 1 outlines the definitions of those orthogonal polynomials and corresponding probability density functions with respect to which those polynomials are orthonormal. See Szego(1939) for more properties of various orthogonal polynomials.

Table 1 : Orthogonal polynomials

Polynomials $p_n(x)$	Definition	Pdf with respect to which $p_n(x)$ are orthonormal
Jacobi $J_n^{(\alpha,\beta)}(x)$	$\frac{(-1)^n}{2^n n!\,(1-x)^\alpha(1+x)^\beta} \times \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right)$	$\frac{\Gamma(\alpha+\beta+2)(1-x)^\alpha(1+x)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$
Laguerre $L_n^{(\alpha)}(x)$	$\frac{(\alpha+1)_n}{n!}F_1(-n;\alpha+1;x)$	$\frac{e^{-k}k^\alpha}{\Gamma(\alpha+1)}$
Hermite $H_n(x)$	$\frac{(-1)^n}{\sqrt{2^n n!}}e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

3. Numerical formulation

We represent the solution u of the Black Scholes equation (1) with random volatility $\sigma(\xi)$ by

$$u(x,t,\xi) = \sum_{\alpha \in I} u_\alpha(x,t) p_\alpha(\xi) \ ,$$

where $u_\alpha = \mathrm{E}[u(x,t,\xi)p_\alpha]$. $p_\alpha(\xi)$ are orthonormal polynomials such as normalized Jacobi polynomials. From $p_\alpha(\xi) = \prod_{i=1}^{\infty} p_{\alpha_i}(\xi_i)$, this $u(x,t,\xi)$ can be simply written as

$$u(x,t,\xi) = \hat{u}_0 p_0 + \sum_{i=1}^{\infty} \hat{u}_i p_1(\xi_i) + \sum_{i=1}^{\infty}\sum_{j=1}^{i} \hat{u}_{ij} p_2(\xi_i,\xi_j)$$

$$+ \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{k=1}^{j} \hat{u}_{ijk} p_3(\xi_i,\xi_j,\xi_k) + \ \dots \ , \qquad (5)$$

where $p_n(\xi_1,\xi_2,\dots,\xi_n)$ denotes the polynomial chaos of order n in the n independent and identically distributed random variables $\xi = (\xi_1,\xi_2,\dots,\xi_n)$. For notational simplicity, we follow the notation of Xiu and Karniadakis (2002). Then (5) can be rewritten as

$$u(x,t,\xi) = \sum_{\alpha=0}^{\infty} u_\alpha p_\alpha(\xi), \qquad (6)$$

where there is a one-to-one correspondence between the functions $p_n(\xi_1,\xi_2,\dots,\xi_n)$ in (5) and $p_\alpha(\xi)$ in (6) and also between $\hat{u}_{1,2,\dots,n}$ and u_α. When (6) is used, (1) can be written as

$$\left(\sum_{\alpha=0}^{\infty} u_\alpha p_\alpha\right)_\tau - \left(\sum_{\beta=0}^{\infty} \sigma_\beta p_\beta\right)\left[\left(\sum_{\gamma=0}^{\infty} u_\gamma p_\gamma\right)_{xx} - \left(\sum_{\gamma=0}^{\infty} u_\gamma p_\gamma\right)_x\right] - r\left(\sum_{\alpha=0}^{\infty} u_\alpha p_\alpha\right)_x + r\left(\sum_{\alpha=0}^{\infty} u_\alpha p_\alpha\right) = 0 \ ,$$

where $\tau = T - t$ and $\sigma_\alpha = \int \frac{1}{2}\sigma^2(\xi) p_\alpha(\xi)\omega(\xi)d\xi$. Since p_α's are not dependent on x or τ,

$$\left(\sum_{\alpha=0}^{\infty} (u_\alpha)_\tau p_\alpha\right) - \left(\sum_{\beta=0}^{\infty} \sigma_\beta p_\beta\right)\left[\left(\sum_{\gamma=0}^{\infty} (u_\gamma)_{xx} p_\gamma\right) - \left(\sum_{\gamma=0}^{\infty} (u_\gamma)_x p_\gamma\right)\right] - r\left(\sum_{\alpha=0}^{\infty} (u_\alpha)_x p_\alpha\right) + r\left(\sum_{\alpha=0}^{\infty} u_\alpha p_\alpha\right) = 0 \ .$$

Since p_α's are an orthonormal basis, $p_\beta p_\gamma$ can be written in terms of p_α's,

$$p_\beta p_\gamma = \sum_{\alpha=0}^{\infty} e_{\alpha\beta\gamma} p_\alpha \ ,$$

where $e_{\alpha\beta\gamma} = < p_\alpha p_\beta p_\gamma > = \int p_\alpha(x) p_\beta(x) p_\gamma(x) \omega(x) dx$. Then the Black Scholes equation can be written as

$$\sum_{\alpha=0}^{\infty}\left((u_\alpha)_\tau - \sum_{\beta,\gamma} e_{\alpha\beta\gamma}\sigma_\beta\left((u_\gamma)_{xx} - (u_\gamma)_x\right) - r(u_\alpha)_x + r u_\alpha\right) p_\alpha = 0 \ .$$

Since p_α's are orthonormal, we derive an infinite system of u_α's ,

$$(u_\alpha)_\tau - \sum_{\beta,\gamma} e_{\alpha\beta\gamma}\sigma_\beta\left((u_\gamma)_{xx} - (u_\gamma)_x\right) - r(u_\alpha)_x + r u_\alpha = 0, \qquad \forall\alpha = 0,1,2, \ldots$$

When this infinite system is truncated into a finite dimension,

$$(u_\alpha)_\tau - \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta\left((u_\gamma)_{xx} - (u_\gamma)_x\right) - r(u_\alpha)_x + r u_\alpha = 0, \forall\alpha = 0,1, \ldots, \mathrm{P}. \ (7)$$

we obtain the algorithm for the $(P+1)^{st}$ order polynomial chaos,

$$u(x,t,\xi) = \sum_{\alpha=0}^{P} u_\alpha p_\alpha(\xi) \ . \qquad (8)$$

Under simplifying assumptions, we can derive an error estimate in the Appendix B.

$$\max_x \sqrt{E\left|u(x,t) - u_{K,N}(x,t)\right|^2} \le \frac{\beta_1}{\sqrt{5K^3}} t^{3/2} + \frac{\beta_{N+1}}{\sqrt{(N+1)!}} \left(\frac{2t^3}{3}\right)^{\frac{N+1}{2}} . (9)$$

We are working on the analytical error analysis above to eliminate those simplifying assumptions. It should be noted that the polynomial chaos expansion derives a system of deterministic equations, whose solution determines statistical moments of the solution of (1). Since the resultant system is deterministic, it needs be solved only once, and thus the computational loads will be reduced.

Since the Black Scholes partial differential equation we consider in Section 4 require very high order of accuracy, the fourth order Runge Kutta method is used for temporal discretization in this study and spectral method in Trefethen (2000) is used for the spatial differentiation. From (7),

$$(u_\alpha)_\tau = \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta\big((u_\gamma)_{xx} - (u_\gamma)_x\big) - r(u_\alpha)_x + ru_\alpha,$$

and by taking the Fourier transform we obtain

$$(\hat{u}_\alpha)_\tau = \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta\big(-\kappa^2\hat{u}_\gamma - i\kappa\hat{u}_\gamma\big) - i\kappa r\hat{u}_\alpha + r\hat{u}_\alpha \ .$$

The Runge Kutta method of order 4 for the Black Scholes equation based on the spectral method and the polynomial chaos expansion is as follows:

Step 1. Compute $\hat{u}_\alpha$ at τ for each α.
Step 2. Set $K_{1,\alpha} = r(i\kappa - 1)\hat{u}_\alpha + \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta(-\kappa^2 - i\kappa)\hat{u}_\gamma$.
Step 3. Set $K_{2,\alpha} = r(i\kappa - 1)(\hat{u}_\alpha + K_{1,\alpha}/2) + \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta(-\kappa^2 - i\kappa)(\hat{u}_\gamma + K_{1,\gamma}/2)$.
Step 4. Set $K_{3,\alpha} = r(i\kappa - 1)(\hat{u}_\alpha + K_{2,\alpha}/2) + \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta(-\kappa^2 - i\kappa)(\hat{u}_\gamma + K_{2,\gamma}/2)$.
Step 5. Set $K_{4,\alpha} = r(i\kappa - 1)(\hat{u}_\alpha + K_{3,\alpha}) + \sum_{\beta,\gamma=0}^{P} e_{\alpha\beta\gamma}\sigma_\beta(-\kappa^2 - i\kappa)(\hat{u}_\gamma + K_{3,\gamma})$.
Step 6. Update $\hat{u}_\alpha$ at $\tau + d\tau$ by

$$\hat{u}_\alpha(\tau + d\tau) = \hat{u}_\alpha(\tau) + \frac{d\tau}{6}\big(K_{1,\alpha} + 2K_{2,\alpha} + 2K_{3,\alpha} + K_{4,\alpha}\big),$$

where $d\tau$ is the time step in τ.

4. Numerical results

Numerical schemes introduced in Section 3 are validated in Section 4.1 using a simple linear advection equation problem, then the schemes are applied to the Black Scholes equation in Section 4.2. In each section, it is shown that the polynomial chaos expansion converges very fast if the solution is expanded in terms of some appropriate polynomials, that is in terms of those polynomials, which are orthonormal with respect to the probability density function of the random variable. Then by observing the behaviors of Jacobi, Laguerre, and Hermite polynomial chaos expansions, it is shown that given a random variable following a certain random distribution, if the polynomial chaos is not chosen properly the convergence rate may be slow.

When $\bar{\Phi}_{\text{exact}}(x)$ represents the mean or variance of the random solution, if $\bar{\Phi}(x)$ is a numerical approximation to the function $\bar{\Phi}_{\text{exact}}(x)$, we use the L_2 error in this study defined by

$$\epsilon_{\Phi} = \sqrt{\frac{1}{N_x}\sum_{j=1}^{N_x}\left(\bar{\Phi}(x_j) - \bar{\Phi}_{\text{exact}}(x_j)\right)^2}\ , \qquad (10)$$

where N_x is the number of points in x.

4.1. Stochastic linear advection equation

Let us consider a following linear advection equation

$$u_t + 2ku_x = 0,\ \ (x,t) \in [-\pi,\pi] \times [0,0.5] \qquad (11)$$

with the initial condition $u(x,0) = \sin(x)$ and a periodic boundary condition. The exact solution is $u(x,t) = \sin(x - 2kt)$. Let us assume that k is a random variable. Using the polynomial chaos expansion (8) for $u = \sum_{\alpha=0}^{P} u_\alpha p_\alpha$ and $k = \sum_{\alpha=0}^{P} k_\alpha p_\alpha$, the advection equation (11) becomes

$$\sum_{\alpha=0}^{P} \frac{\partial u_\alpha}{\partial t} p_\alpha = -2\sum_{\beta=0}^{P}\sum_{\gamma=0}^{P} k_\beta (u_\gamma)_x p_\beta p_\gamma = -2\sum_{\alpha=0}^{P}\sum_{\beta=0}^{P}\sum_{\gamma=0}^{P} e_{\alpha\beta\gamma} k_\beta (u_\gamma)_x p_\alpha \quad .$$

Since $\{p_\alpha\}$ are orthonormal, we obtain a system of $P+1$ equations,

$$\frac{\partial u_\alpha}{\partial t} = -2\sum_{\beta=0}^{P}\sum_{\gamma=0}^{P} e_{\alpha\beta\gamma} k_\beta (u_\gamma)_x, \qquad \alpha = 0,1,\ldots,P\ . \quad (12)$$

Equation (12) is solved using the Runge Kutta method and the spectral method as explained in Section 3.

Let us first assume that k is a Beta random variable with the probability density function $f^{(\alpha,\beta)}(k) = \frac{\Gamma(\alpha+\beta+2)(1-k)^{\alpha}(1+k)^{\beta}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}, -1 < k < 1,\ \alpha,\beta > 1$. Section 2 suggests that Jacobi polynomials are optimal for the Beta distribution. Figure 1 (Left) represents a semi-log plot of the errors of mean and variance for a Beta random variable in terms of P when $\alpha = \beta = 0$. The figure shows that those mean and variance converge at an exponential rate as the length P of the expansion increases when the Jacobi polynomial chaos is used as a basis.

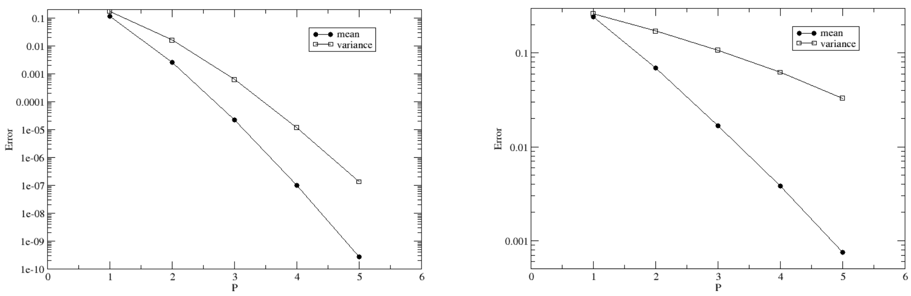

Figure 1: Errors of mean and variance at $t = 0.5$ (Left) when the Jacobi polynomial chaos is used for the Beta forcing and (Right) when the Laguerre polynomial chaos is used for the Gamma forcing.

In case of the Gamma distribution with a parameter α, the probability density function is $f_{\alpha}(k) = \frac{e^{-k}k^{\alpha}}{\Gamma(\alpha+1)}$, $0 \le k < \infty, \alpha > -1$, and the Laguerre polynomial chaos from the Askey polynomial chaos family is orthonormal with respect to $f_{\alpha}(k)$. Figure 1 (Right) shows the exponential convergence of the Laguerre polynomial chaos for the mean and variance when k follows the Gamma distribution with $\alpha = 0$ and it corroborates the fact that a proper choice of the polynomial basis results in significantly fast convergence.

In order to validate the importance of the proper selection of the polynomial chaos, let us consider the situation when ξ for the polynomial basis $\{p_{\alpha}(\xi)\}$ and k in the equation belong to two different probability spaces (Ω, A, P) with different event spaces Ω, σ—algebras A and probability measures P. That is, k follows a certain random distribution and the polynomial basis $\{p_{\alpha}(\xi)\}$ is not orthonormal with respect to the probability density function of k as the weight function of the inner product for $\{p_{\alpha}(\xi)\}$. For example, k follows the Beta distribution and $\{p_{\alpha}(\xi)\}$ can be Laguerre polynomials. Let ϕ and ψ be the probability density functions of k and ξ, respectively, and let Φ and Ψ be their distribution functions. When $k_{\alpha} = \int k p_{\alpha}(\xi)\psi(\xi)d\xi$ is being computed, k and ξ

may not have the same support. Then we need to change the measures similarly to the way in Xiu and Karniadakis (2002). Let us first define a uniformly distributed random variable y such that $y = \Phi(k) = \Psi(\xi)$. If $k^*(y)$ and $\xi^*(y)$ are defined by $\Phi^{-1}(y) = k^*(y)$ and $\Psi^{-1} = \xi^*(y)$, $k_\alpha = \int k p_\alpha(\xi)\psi(\xi)d\xi = \int_0^1 k^*(y)\, p_\alpha(\xi^*(y))dy$. Then we can estimate the probability density function of k using $\{p_\alpha(\xi)\}$. k will be represented in the form of $k = \sum_{\alpha=0}^{p} k_\alpha p_\alpha(x)$. From the transformation of variables, $dy = \phi(k)dk = \psi(\xi)d\xi$, the probability density functions of k of different orders are approximated by $\phi(k) = \psi(\xi)\frac{d\xi}{dk}$. Similar procedures will be performed for (1) in Section 4.2.

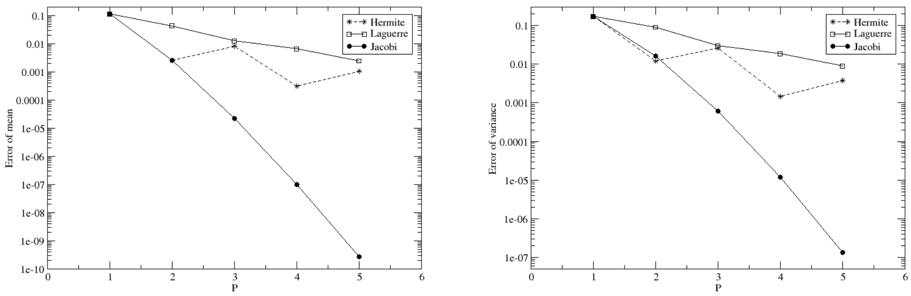

Figure 2: Convergence from the Hermite, Laguerre and Jacobi polynomial chaos for (Left) mean and (Right) variance when k follows the Beta distribution

Figure 2 (Left) compares the convergences of the mean for (11) from the Hermite (star), Laguerre (square) and Jacobi (filled circle) polynomial expansions when k follows the Beta distribution. Figure 2 (Right) compares the convergence of the variance. The figures show that when the Hermite or Laguerre polynomials are used as a basis for the Beta distribution, the error still decreases but the convergence using the Jacobi polynomials is substantially faster than those using the other polynomials.

4.2. Black Scholes equations

Let us solve the Black Scholes equation (1) where the volatility σ is random. For the numerical experiments, we consider the butterfly-spread option, that is, for the same asset and expiry date, we hold a European call option with exercise price E_1 and another with exercise price E_3 and write two calls with exercise price $E_2 = (E_1 + E_3)/2$. Figure 3 shows its payoff diagram at expiry.

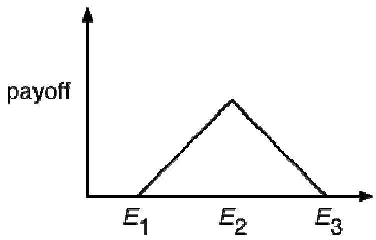

Figure 3: The payoff of the butterfly spread call option at expiry

Given the price $u_E(S,t)$ of the European vanilla call option with the exercise price E in Section 2, the exact price of the butterfly-spread option above is

$$u_{\text{exact}}(S,t) = u_{E_1}(S,t) + u_{E_3}(S,t) - 2u_{E_2}(S,t). \qquad (13)$$

$E_1 = 15$ and $E_3 = 25$ are used in the numerical experiments with time to maturity $T = 0.5$ and risk-free interest rate $r = 0.05$. Let us first assume that the volatility σ follows the Beta distribution with $\alpha = 0$ and $\beta = 0$, that is, the uniform distribution,

$$\sigma = 0.1 + 0.4\xi \qquad (14)$$

where ξ has a standard uniform distribution $U(0,1)$, so that σ ranges between 0.1 and 0.5. Since the exact option value $u_{\text{exact}}(S,t)$ is known from (13), we can estimate for each S the error between the exact option price and the numerical approximation from the finite-order polynomial chaos. Figure 4 shows the errors of mean and variance for a range of S values between the exact option values and polynomial chaos approximations as the index P increases. For each S value, the magnitude of errors decreases in P.

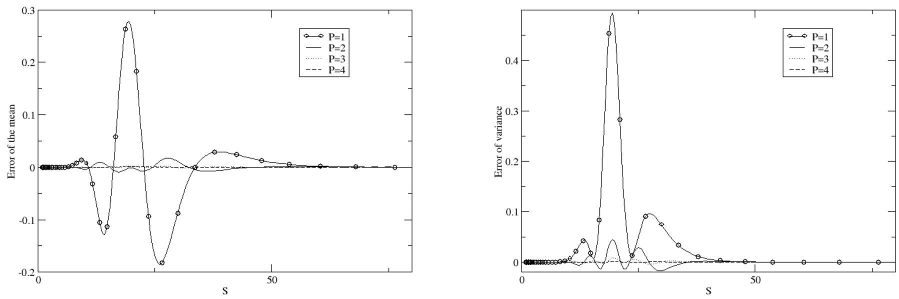

Figure 4: Errors of mean and variance for a range of *S* values when the Jacobi polynomials chaos is used for the Gamma volatility

Figure 5 is a semi-log plot of the L_2 errors (10) of mean and variance when the number of grids N_x is 128 and 256. The figure shows that the errors converge exponentially when the Jacobi polynomial chaos is used as a basis for the Beta distribution.

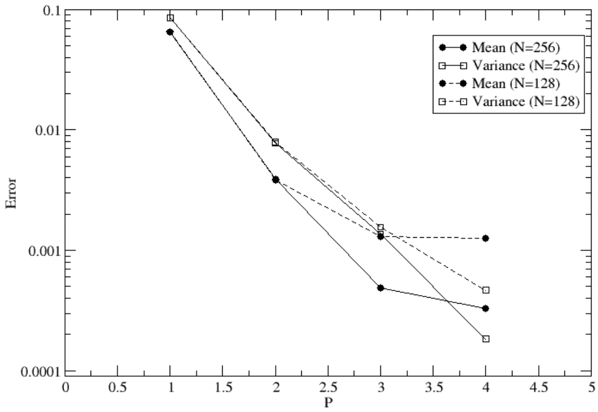

Figure 5: Convergence for the mean (filled circle) and variance (square) with $N_x = 256$ (solid) and 128 (dashed) grid points with respect to the Jacobi polynomial chaos when the volatility σ follows the Beta distribution.

Next, let us consider the case when the volatility σ is not uniformly distributed but it follows the Gamma distribution. As explained in Section 4.1, the Gamma distribution with a parameter α has the probability density function $f_\alpha(k) = \frac{e^{-k}k^\alpha}{\Gamma(\alpha+1)}$, $0 \le k < \infty$ and the Laguerre polynomial chaos from the Askey polynomial chaos family is orthonormal with respect to the density function $f_\alpha(k)$. Figure 6 shows the errors of the mean and variance with respect to the Laguerre polynomial chaos when the volatility σ is given by

$$\sigma = 0.05 + 0.55\left(\frac{\xi}{20}\right)^4,$$

where ξ follows the Gamma distribution with $\alpha = 0$, so that σ values mostly range over [0.05, 0.6]. It is shown that exponential convergence rate is also obtained for both mean and variance.

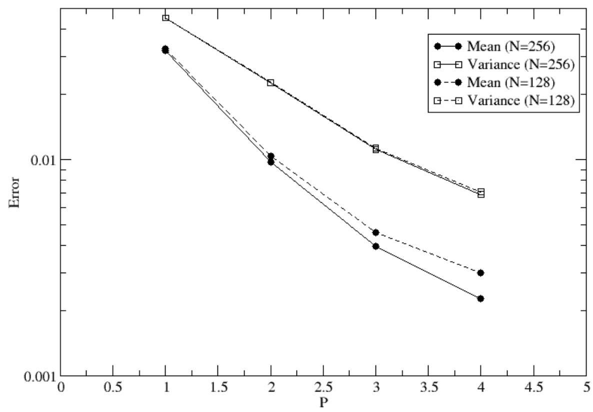

Figure 6: Convergence for the mean (filled circle) and variance (square) with $N = 256$ (solid) and 128 (dashed) grid points with respect to the Laguerre polynomial chaos when the volatility σ follows the Gamma distribution

In order to see the effectiveness of the appropriate selection of the polynomial chaos for the Black Scholes equation, let us compare again how the solution converges with respect to Jacobi, Laguerre and Hermite polynomial bases when the volatility σ follows a certain random distribution. Figure 7 compares the convergences of the option values in terms of the Jacobi (filled circle), Laguerre (square), and Hermite (star) polynomial chaos when the random σ follows the Beta distribution (14). The Jacobi polynomial chaos results in significantly faster convergence than the Laguerre or Hermite polynomials for both mean and variance similarly to the results observed in Section 4.1.

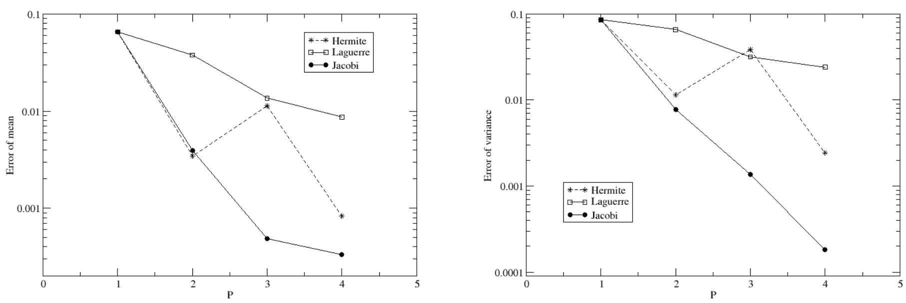

Figure 7: Convergence for the (Left) mean and (Right) variance with respect to the Jacobi (solid), Hermite (filled circle), and Laguerre (square) polynomial chaos when the volatility σ follows the Beta distribution.

Table 2 and Table 3 compare the convergence when two different volatility functions, $\sigma = 0.1 + 0.4\xi$ and $\sigma = 0.1 + 0.4\xi^3$ are considered, respectively. ξ is the standard uniform random variable $\xi \sim U(0,1)$. Both tables confirm that when appropriate polynomial chaos is chosen as a basis in the random space for the volatility, the solution to the Black Scholes equation converge significantly fast, but other polynomial chaos, however, may result in a slower convergence.

Table 2: Errors in mean (and errors in variance inside the parentheses) with respect to Jacobi, Laguerre, and Hermite polynomial chaos for various orders P when the volatility $\sigma = 0.1 + 0.4\xi$, for a standard uniform distribution $\xi \sim U(0,1)$.

	P=1	P=2	P=3	P=4
Hermite	0.065454 (0.085752)	0.003437 (0.011514)	0.011311 (0.038713)	0.000828 (0.002437)
Laguerre	0.065454 (0.085752)	0.037745 (0.065843)	0.013580 (0.031650)	0.008696 (0.024110)
Jacobi	0.065454 (0.085752)	0.003904 (0.007744)	0.000484 (0.001367)	0.000331 (0.000183)

Table 3: Errors in mean (and errors in variance inside the parentheses) with respect to Jacobi, Laguerre, and Hermite polynomial chaos for various orders P when the volatility $\sigma = 0.1 + 0.4\xi^3$, for a standard uniform distribution $\xi \sim U(0,1)$.

	P=1	P=2	P=3	P=4
Hermite	0.093967 (0.107185)	0.043596 (0.118877)	0.010517 (0.010216)	0.005559 (0.015068)
Laguerre	0.093967 (0.107185)	0.029228 (0.054360)	0.003983 (0.005413)	0.002736 (0.007506)
Jacobi	0.093967 (0.107185)	0.036413 (0.101943)	0.004124 (0.003747)	0.000763 (0.001684)

5. Conclusions

In this paper, the polynomial chaos expansion has been extended to the analysis of the stochastic linear advection equation and the Black Scholes equation with a random volatility. Statistical moments of the solution can be easily obtained from its Fourier coefficients with respect to the polynomial chaos. The substantially fast convergence is obtained when appropriate polynomial basis is used to estimate Fourier coefficients. For instance, when Jacobi polynomial chaos is chosen as a basis in the Beta random space, numerical option price converges exponentially. Usage of non-proper polynomial chaos may lead to lowering of its convergence rate.

When the volatility of the Black Scholes equation is stochastic, characteristics of the solution may be affected. The analytical study of these problems, especially of error analysis will be postponed to our future research. We will also work on the Black Scholes equation with a more general type of randomness in volatility.

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Appendix A. Orthogonal polynomials

Here are brief summary of Hermite and Laguerre polynomials. See Szego(1939) for more. Normalized Hermite polynomials $H_n(x)$ of degree n is defined by $H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!}} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$ with $H_0(x) = 1$. For example, $H_1(x) = x,\ H_2(x) = \frac{x^2-1}{\sqrt{2}}, H_3(x) = \frac{x^3-3x}{\sqrt{6}}, \ldots$ $\{H_n(x)\}$ are orthonormal with respect to the probability density function of the Gaussian distribution as the weight function of the inner product, $< H_m, H_n > \equiv \int_{-\infty}^{\infty} H_m(x) H_n(x) \omega(x)\, dx = \delta_{mn}$, where $\omega(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Normalized Hermite polynomials satisfy the recurrence relation, $\sqrt{n+1} H_{n+1}(x) - x H_n(x) + \sqrt{n} H_{n-1}(x) = 0$.

Laguerre polynomials $L_n^{(\alpha)}(x)$ of degree n with the parameter α is defined by $L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n;\alpha+1;x)$. For example, when $\alpha = 0,\ L_n(x) = L_n^{(0)}(x)$ is defined by $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^n)$ and $L_0(x) = 1, L_1(x) = -x + 1, L_2(x) = \frac{x^2-4x+2}{2},\ L_3(x) = \frac{-x^3+9x^2-18x+6}{6}, \ldots$ $\{L_n^{(\alpha)}(x)\}$ are orthonormal with respect to the probability density function of the Gamma distribution as the weight function of the inner product. For instance, when $\alpha = 0,\ < L_m, L_n > \equiv \int_0^{\infty} L_m(x) L_n(x) \omega(x) dx = \delta_{mn}$, where $\omega(x) = e^{-x}$, and the recurrence relation $(n+1) L_{n+1}(x) - (2n+1-x) L_n(x) + n L_{n-1}(x) = 0$ is satisfied.

Appendix B. Error analysis

Let the function $v(x,t)$ be the solution of the Black Scholes equation when the volatility σ is a constant, $\sigma = \sigma^*$. Under simplifying assumptions, we can derive a following error estimate in Theorem 2.

Theorem 2. *Let $\sigma = \dot{W}(t)$ for a Brownian motion $W(t)$ and suppose that u can be derived from v by*

$$u(x,t) = v\left(x - \int_0^t W(s)ds, t\right) + W(t).$$

If $u_{K,N}(x,t) = \sum_{\alpha\in I_{K,N}} u_\alpha(x,t)H_\alpha$ *denotes the truncation of the expansion of the solution* (1) *over the truncated index set*

$$I_{K,N} = \{\, \alpha = (\alpha_1, \dots, \alpha_K) | \alpha_i \in \{0,1,2,\dots\}, \qquad |\alpha| \le N\},$$

where $H_\alpha(\xi)$ *is the multi-variate Hermite polynomial of* ξ, *then the error can be estimated by*

$$\max_x \sqrt{E|u(x,t) - u_{K,N}(x,t)|^2} \le \frac{\beta_1}{\sqrt{5K^3}} t^{3/2} + \frac{\beta_{N+1}}{\sqrt{(N+1)!}} \left(\frac{2t^3}{3}\right)^{\frac{N+1}{2}} \quad (B.1)$$

where $\beta_n = \sup_x \left|\frac{\partial^n v(x,t)}{\partial x^n}\right|$.

Proof. Suppose $\{m_k(s)\}$ are an orthonormal basis in $L_2([0,t])$ defined by

$$m_1(s) = \frac{1}{\sqrt{t}}, \qquad m_k(s) = \sqrt{\frac{2}{t}} \cos\left(\frac{(k-1)\pi s}{t}\right)$$

for $k \ge 2$. Define $z(t) = \int_0^t W(s)ds$ and $\xi_i = \int_0^t m_i(s)dW(s)$. For $s \in [0,t], W(s)$ satisfies

$$W(s) = \int_0^t \chi_{[0,s]}(\tau) dW(\tau),$$

where $\chi_{[0,s]}(\tau)$ is the characteristic function, $\chi_{[0,s]}(\tau) = 1$ if $\tau \in [0,s]$ and 0 otherwise. Then

$$W(s) = \sum_{i=1}^{\infty} \xi_i \int_0^s m_i(\tau)d\tau = \frac{s}{\sqrt{t}}\xi_1 + \sum_{k=2}^{\infty} \xi_k \frac{\sqrt{2t}}{(k-1)\pi} \sin\left(\frac{(k-1)\pi s}{t}\right).$$

In particular, $W(t) = \sqrt{t}\xi_1$. Thus,

$$
\begin{aligned}
z(t) &= \frac{\xi_1}{\sqrt{t}}\int_0^t s\,ds + \sum_{k=2}^{\infty} \xi_k \frac{\sqrt{2t}}{(k-1)\pi}\int_0^t \sin\left(\frac{(k-1)\pi s}{t}\right) ds \\
&= \frac{t^{3/2}}{2}\xi_1 + \sum_{k=2}^{\infty} c_k \frac{t^{\frac{3}{2}}}{(k-1)^2}\xi_k \\
&= \frac{t^{3/2}}{2}\xi_1 + \sum_{k=2}^{K} c_k \frac{t^{3/2}}{(k-1)^2}\xi_k + \sum_{k=K+1}^{\infty} c_k \frac{t^{3/2}}{(k-1)^2}\xi_k,
\end{aligned}
$$

where $c_k = \sqrt{2}(1+(-1)^k)/\pi^2$. Set $z_1 = \frac{t^{3/2}}{2}\xi_1 + \sum_{k=2}^{K} c_k \frac{t^{\frac{3}{2}}}{(k-1)^2}\xi_k$ and $z_2 = z(t) - z_1$. Then z_1 and z_2 are orthogonal, $E[z_1 z_2] = 0$. From $W(t) = \sqrt{t}\xi_1$, $u(x,t) = v(x - z_1 - z_2, t) + \sqrt{t}\xi_1$. Expanding v in Taylor's series with respect to z_2 gives

$$
u(x,t) = v(x - z_1, t) - \frac{\partial v}{\partial x}(x - z_1 - \theta_1, t) z_2 + \sqrt{t}\xi_1,
$$

for some θ_1. Expanding this in Taylor's series with respect to z_1 gives

$$
u(x,t) = v(x,t) + \sum_{n=1}^{N} \frac{(-z_1)^n}{n!}\frac{\partial^n v}{\partial x^n}(x,t) - \frac{\partial v}{\partial x}(x - z_1 - \theta_1, t) z_2
$$
$$
+ \frac{(-z_1)^{N+1}}{(N+1)!}\left(\frac{\partial^{N+1} v}{\partial x^{N+1}}(x - \theta_2, t)\right) + \sqrt{t}\xi_1,
$$

for some θ_2. Let $I_1 = -\frac{\partial v}{\partial x}(x - z_1 - \theta_1, t) z_2$ and $I_2 = \frac{(-z_1)^{N+1}}{(N+1)!}\left(\frac{\partial^{N+1} v}{\partial x^{N+1}}(x - \theta_2, t)\right)$. If $\tilde{u}_{K,N}$ is defined by

$$
\tilde{u}_{K,N} = v(x,t) + \sum_{n=1}^{N} \frac{(-z_1)^n}{n!}\frac{\partial^n v}{\partial x^n}(x,t) + \sqrt{t}\xi_1,
$$

$\tilde{u}_{K,N}(x,t)$ is a polynomial of $\xi_1, \ldots, \xi_K$ with maximum order N. Since $u_{K,N}(x,t)$ is a Hermite polynomial expansion, which is an orthogonal projection with respect to Gaussian measure,

$$
\sqrt{E\left|u(x,t) - u_{K,N}(x,t)\right|^2} \le \sqrt{E\left|u(x,t) - \tilde{u}_{K,N}(x,t)\right|^2} = \sqrt{E|I_1 + I_2|^2}.
$$

Using the Minkowski's inequality,

$$\sqrt{E\left|u(x,t)-u_{K,N}(x,t)\right|^2} \le (\mathrm{E}|\mathrm{I}_1|^2)^{1/2} + (\mathrm{E}|\mathrm{I}_2|^2)^{1/2}$$
$$\le sup_x\left|\frac{\partial v}{\partial x}\right|(E|z_2^2|)^{1/2} + sup_x\left|\frac{\partial^{N+1}v}{\partial x^{N+1}}\right|\frac{1}{(N+1)!}(\ E|z_1{}^{2N+2}|)^{1/2},$$

Next, $E|z_2^2|$ satisfies

$$E|z_2^2| = E\left|\left(\sum_{k=K+1}^{\infty} c_k\frac{t^{\frac{3}{2}}}{(k-1)^2}\xi_k\right)^2\right| = \sum_{k=K+1}^{\infty} E\left(c_k\frac{t^{\frac{3}{2}}}{(k-1)^2}\xi_k\right)^2,$$

because ξ_k's are orthogonal. Thus, $E|z_2^2|$ is bounded by

$$E|z_2^2| = \sum_{k=K+1}^{\infty}\frac{|c_k|^2t^3}{(k-1)^4}E(\xi_k^2) \le \frac{8t^3}{\pi^4}\left(\frac{1}{K^4}+\int_K^{\infty}\frac{dx}{x^4}\right) = \frac{8t^3}{\pi^4}\left(\frac{1}{K^4}+\frac{1}{3K^3}\right) < \frac{t^3}{5K^3}.$$

Given a Gaussian random variable $X\sim N(\mu,\sigma^2)$, $E[(X-\mu)^{2k}] = \frac{(2k)!}{2^k k!}\sigma^{2k}$. Since $\xi_k \sim N(0,1)$, z_1 is a linear combination of centered Gaussian ξ_k's so that it is a centered Gaussian with mean 0 and variance $E|z_1^2|$. Thus,

$$E[z_1^{2N+2}] = E\left[z_1^{2(N+1)}\right] = \frac{(2N+2)!}{2^{N+1}(N+1)!}(E|z_1^2|)^{N+1} = (2N+1)!!\,(E|z_1^2|)^{N+1}.$$

Since $E|z|^2 = E|z_1+z_2|^2 \ge E|z_1|^2$,

$$E\left|z_1^{2N+2}\right| < (2N+1)!!\,(E|z^2|)^{N+1} = \left(\frac{t^3}{3}\right)^{N+1}(2N+1)!!$$

because

$$E|z|^2 = E\left|\left(\frac{t^{3/2}}{2}\xi_1 + \sum_{k=2}^{\infty} c_k\frac{t^{\frac{3}{2}}}{(k-1)^2}\xi_k\right)^2\right| = \frac{t^3}{4} + \sum_{k=1}^{\infty}\frac{8}{\pi^4}\frac{t^3}{(2k-1)^4}$$
$$= \frac{t^3}{4} + \frac{8t^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{t^3}{4} + \frac{8t^3}{\pi^4}\frac{\pi^4}{96} = \frac{1}{3}t^3$$

Now

$$(2N+1)!! = (2N+1)(2N-1)\cdots 5\cdot 3\cdot 1$$
$$= 2^{N+1}\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(\frac{5}{2}\right)\cdot\left(\frac{3}{2}\right)\cdot\left(\frac{1}{2}\right)$$
$$\le 2^{N+1}(N+1)N(N-1)\cdots 3\cdot 2\cdot 1$$
$$= 2^{N+1}(N+1)!$$

and (B.1) is derived.

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