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ADVERSE SELECTION WITH BI-DIMENSIONAL ASYMMETRIC INFORMATION AND GENERALIZED COST FUNCTION

Abstract. *In the paper we propose an extension of the multidimensional adverse section model of Laffont and Martimort (2002). We use a generalized cost function of the Agent, with two adverse selection parameters. We solve the nonlinear optimization problem of the Principal using informational rents and quantities as variables and we derive the optimal contracts when the adverse selection parameters are positive correlated. The final of the paper summarizes the features of the optimal contracts in multidimensional asymmetric information and we propose some possible applications of the model.*

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JEL Classification: **C61, D82**

1. Introduction

In their book, Laffont and Martimort [8] show that the most of the standard adverse selection problems were studied and applied with the major disadvantage that the adverse selection parameter is modeled as an one-dimensional parameter. There are many cases however, where the adverse selection problem involves several parts of private information that affect the results of the contractual relation (such that: the marginal cost of production and the corresponding fixed cost, risk aversion and the probability of accident for an insurer, productivity and skills for a worker etc.). These aspects generated a special class of adverse selection models – the multidimensional asymmetric information problems.

The literature on multidimensional adverse selection problems has been recently developed. Armstrong (1996) shows that it is optimal for the Principal to exclude some consumers from its products in order to extract more profit from the other higher value consumers. In their paper, Rochet and Chone (1998) provide a general analysis and show that bunching of types is always present in bi-dimensional models, such that consumers with different tastes choose the same bundle of goods. They introduce the "ironing and sweeping techniques" for analyzing the bunching contracts. Armstrong and Rochet (1999) provide a real "user's guide" for studying the multidimensional screening problems; they studied the simplest formulation of the general screening model and provide a complete solution of this problem. A special section of their paper is dedicated to some possible applications for their model such that: multiproduct nonlinear pricing, multiproduct monopoly regulation, auctions design and optimal taxation. Within the context of the nonlinear pricing problem by a multiproduct monopolist (multidimensional asymmetric information) there are also several papers to be mentioned: Sibley and Srinagesh (1997), Rochet and Stole (2000). Brighi and D'Amato (2002) derived the optimal regulatory policy of a monopolist producing two goods with two-dimensional private information about costs and studied the case of perfect and negatively correlated cost parameters.

In the paper we deal with a special case of Laffont and Martimort [8]'s model of multidimensional adverse selection problem with two cost parameters and a linear cost function. We extend this latter model to the case where the Agent has a generalized cost function, dependent on two adverse selection parameters. We study a situation where the Agent (the firm) has two-dimensional private information about his production cost. The Agent (the firm) produces two goods and has private information about his marginal cost in each production line and the adverse selection parameters are positive correlated across goods. The Principal's objective is to maximize the profit and hence he is concerned in making a *take-it or leave-it offer* of a menu of contracts specifying the quantities to be produced and the transfers to be paid to the firm.

The paper is organized as follows. Section 2 presents the standard one dimensional adverse selection model of Laffont and Martimort (2002) and the main features of the optimal contract in this situation. In Section 3 we extend the standard model to the case with two adverse selection parameters (the two-dimensional asymmetric information model); we present the Principal's optimization problem and we solve this program using as variables the informational rents and quantities. In Section 4 we derive the optimal contracts that can be obtained in this adverse selection model; two situations are discussed here - weak and strong correlation between the adverse selection parameters. The last section summarizes the features of the optimal contracts in both cases.

2. The standard model (Laffont, Martimort, 2002): two types of Agent (one-dimensional asymmetric information)

First we summarize the main assumptions of the model:
a. The Principal (a firm or consumer) wants to consume a good and delegates to an Agent (a firm) the production of q units of this good. The value for the Principal of q units is $S(q)$, where $S(\cdot)$ satisfies $S'(\cdot)>0$, $S''(\cdot)<0$ and $S(0)=0$.
b. The Agent's production cost is unobservable to the Principal. Therefore, the Agent has as private information his marginal cost of production. He can have one of two types: $\theta \in \Theta = \{\underline{\theta}, \overline{\theta}\}$, with $\underline{\theta} < \overline{\theta}$. The fixed cost is neglected.
c. The economic variables of the problem (the contractual variables) are: q – the quantity to be produced and t – the transfer received by the Agent (the Principal's payment to the Agent if the later produces q units for the Principal).
d. The timing of contracting in adverse selection situation:

$t=0$: the Agent discovers his type θ; but the Principal doesn't observe this type;

$t=1$: the Principal design and offers the contract;

$t=2$: the Agent accepts or refuses the contract;

$t=3$: the contract is executed; payoffs for Principal and Agent.

2.1. First best solution (complete information optimal contract)

We suppose that there is no asymmetry of information between the Principal and the Agent: the Principal knows exactly the Agent's type. Therefore, he makes a contractual offer to the Agent and this offer corresponds to the solution of the following optimization problem:

$$
\begin{aligned}
&\left(\max_{t,q}\right)\{S(q)-t\} \\
&s.t. \\
&t \geq \theta q \\
&q \geq 0, t \geq 0
\end{aligned}
$$

The optimal solution for the above problem is given by the first order conditions:

$$
\begin{cases}
S'(q^*) = \theta \Rightarrow q^* = (S')^{-1}(\theta) \\
t^* = \theta q^*
\end{cases}
$$

Hence, if the Agent has low marginal cost $\left(\underline{\theta}\right)$, the optimal contract entails a production $\underline{q}^* = \left(S'\right)^{-1}\left(\underline{\theta}\right)$ and an optimal transfer $\underline{t}^* = \underline{\theta}\underline{q}^*$. If the Agent has high marginal cost $\left(\overline{\theta}\right)$, the optimal quantity he must produce is given by $\overline{q}^* = \left(S'\right)^{-1}(\overline{\theta})$ and he receives the optimal transfer $\overline{t}^* = \overline{\theta}\overline{q}^*$.

Remark: The first best contract yields to a higher optimal production for the efficient type than the corresponding production for the inefficient type, i.e. $\underline{q}^* > \overline{q}^*$.

2.2. The case of asymmetric information

Now, the Principal doesn't know the Agent's type. Suppose that the probability that the Agent is efficient ($\underline{\theta}$) is ν; then, the probability that the Agent is inefficient ($\overline{\theta}$) is $1-\nu$.

It is optimal for the Principal to design a menu of two contracts, hoping that each type of Agent chooses the contract designed for him. We denote by $\left\{\left(\underline{t},\underline{q}\right),\left(\overline{t},\overline{q}\right)\right\}$ the menu of contracts being derived such that the Principal's expected profit is maximized:

$$\left(\max_{\overline{t},\overline{q},\underline{t},\underline{q}}\right)\left\{\nu\left[S\left(\underline{q}\right)-\underline{t}\right]+(1-\nu)\left[S\left(\overline{q}\right)-\overline{t}\right]\right\}$$

The *incentive feasible contracts* must satisfy the *participation constraints*:

$$\underline{t}-\underline{\theta}\underline{q} \geq 0 \text{ and } \overline{t}-\overline{\theta}\overline{q} \geq 0$$

and the incentive compatibility constraints:

$$\underline{t}-\underline{\theta}\underline{q} \geq \overline{t}-\underline{\theta}\overline{q} \text{ and } \overline{t}-\overline{\theta}\overline{q} \geq \underline{t}-\overline{\theta}\,\underline{q}$$

Theorem (Laffont, Martimort, 2002). In the situation of asymmetric information, the features of the optimal contract (the second best solution) are:

i) $S'\left(\underline{q}\right)=\underline{\theta}$ (the efficient Agent produces the first best quantity)

ii) $S'\left(\overline{q}^{SB}\right)=\overline{\theta}+\Delta\theta\dfrac{\nu}{1-\nu}$, with $\Delta\theta=\overline{\theta}-\underline{\theta}$ (the inefficient Agent produces a reduced quantity with respect to the first best production), $\overline{q}^{SB} < \overline{q}^*$.

iii) The type $\overline{\theta}$ of Agent gets no rent (he obtains exactly the outside opportunity level), $\overline{U}^{SB}=0$.

iv) The efficient Agent gets a positive informational rent, $\underline{U}^{SB} = \Delta\theta \overline{q}^{SB}$.

3. Two-dimensional asymmetric information

3.1. The two-dimensional adverse selection model

We extend the analysis from the previous section to the case of two-dimensional asymmetric information. First we present the main assumptions of the model (some of them are common with those from Laffont and Martimort).

We consider that the Principal wants to delegate to the Agent two tasks (two production activities). Each of these tasks involves a constant marginal cost that represents the Agent's private information.

We assume that the Agent produces two goods in the quantities q_1 and q_2, at the respective marginal cost θ_1 and θ_2. For every cost parameter there are two possible values (low and high marginal cost), that is $\theta_i \in \{\underline{\theta}, \overline{\theta}\}$ for $i = 1,2$ and with $\Delta\theta = \overline{\theta} - \underline{\theta}$. For convenience, the fixed costs are neglected so that the firm's cost function can be written as:

$$C(q,\theta) = C(\theta_1, q_1) + C(\theta_2, q_2)$$

with the assumptions $C_q > 0, C_\theta > 0, C_{qq} > 0, C_{q\theta} > 0$ and $C_{qq\theta} > 0$ (It is obvious that this form of cost function implies that the Agent utility function satisfies the Spence-Mirrlees property).

The production activities are independently performed such that there is no externality between them and the Principal's profit (payoff) function is given by:

$$V(t,q) = S(q_1) + S(q_2) - t$$

where *t* is the transfer paid by the Principal to the Agent.

In the same way as we did before, the *utility function* (*informational rent*) for each type $\theta = (\theta_1, \theta_2)$ of Agent is given by the following expression:

$$U^{(\theta_1,\theta_2)}(t, q_1, q_2) = t - C(q_1, \theta_1) - C(q_2, \theta_2)$$

The adverse selection vector of cost parameters $\theta = (\theta_1, \theta_2)$ can have one of the four values, with the respective probabilities:

$$\underline{\gamma} = \Pr\left(\theta_1 = \underline{\theta}, \theta_2 = \underline{\theta}\right),$$
$$\frac{\tilde{\gamma}}{2} = \Pr\left(\theta_1 = \underline{\theta}, \theta_2 = \overline{\theta}\right) = \Pr\left(\theta_1 = \overline{\theta}, \theta_2 = \underline{\theta}\right),$$
$$\overline{\gamma} = \Pr\left(\theta_1 = \overline{\theta}, \theta_2 = \overline{\theta}\right)$$

This distribution represents common knowledge to the Principal and the Agent.

We consider that the types are positive correlated and this implies the condition $\rho = \underline{\gamma}\overline{\gamma} - \frac{\tilde{\gamma}^2}{4} > 0$. We give in Appendix a complete analysis of this types' correlation.

We also impose, without loss of generality, symmetry of transfers (for the mixed types) so that, the informational rent for each type of Agent becomes:

- the rent of the type $\left(\theta_1, \theta_2\right) = \left(\underline{\theta}, \underline{\theta}\right)$ is $\underline{U} = \underline{t} - 2C\left(\underline{q}, \underline{\theta}\right)$;
- the rent of the types $\left(\theta_1, \theta_2\right) = \left(\underline{\theta}, \overline{\theta}\right)$ and $\left(\theta_1, \theta_2\right) = \left(\overline{\theta}, \underline{\theta}\right)$ is, by symmetry:

$$\tilde{U} = \tilde{t} - C\left(\tilde{q}_2, \underline{\theta}\right) - C\left(\tilde{q}_1, \overline{\theta}\right);$$

- the rent of the type $\left(\theta_1, \theta_2\right) = \left(\overline{\theta}, \overline{\theta}\right)$ is $\overline{U} = \overline{t} - 2C\left(\overline{q}, \overline{\theta}\right)$.

3.1.1. Participation and incentive compatibility constraints

The set of incentive feasible allocations (contracts) satisfies the following *constraints*:

The participation constraints are:

$$\underline{U} = \underline{t} - 2C\left(\underline{q}, \underline{\theta}\right) \geq 0$$
$$\tilde{U} = \tilde{t} - C\left(\tilde{q}_2, \underline{\theta}\right) - C\left(\tilde{q}_1 . \overline{\theta}\right) \geq 0$$
$$\overline{U} = \overline{t} - 2C\left(\overline{q}, \overline{\theta}\right) \geq 0$$

The incentive compatibility constraints are the following:

a) For the type $\left(\theta_1, \theta_2\right) = \left(\underline{\theta}, \underline{\theta}\right)$ we have:

$$U^{(\underline{\theta},\underline{\theta})}\left(\underline{t}, \underline{q}, \underline{q}\right) = \underline{U} \geq U^{(\underline{\theta},\underline{\theta})}\left(\tilde{t}, \tilde{q}_1, \tilde{q}_2\right) = \tilde{t} - C\left(\tilde{q}_2, \underline{\theta}\right) - C\left(\tilde{q}_1, \underline{\theta}\right)$$
$$U^{(\underline{\theta},\underline{\theta})}\left(\underline{t}, \underline{q}, \underline{q}\right) = \underline{U} \geq U^{(\underline{\theta},\underline{\theta})}\left(\overline{t}, \overline{q}, \overline{q}\right) = \overline{t} - 2C\left(\overline{q}, \underline{\theta}\right)$$

and

$$\underline{U} \geq U^{(\underline{\theta},\underline{\theta})}\left(\tilde{t}, \tilde{q}_1, \tilde{q}_2\right) = \tilde{t} - C\left(\tilde{q}_1, \underline{\theta}\right) - C\left(\tilde{q}_2, \underline{\theta}\right)$$

(this latter constraint is identical with the first one).

In order to solve easier the Principal's optimization problem, we transform all the constraints into a "friendly form" and this is done using the following function.

Proposition 1. Let $f:(0,\infty)\to R$, $f(x)=C(x,\overline{\theta})-C(x,\underline{\theta})$ be the *incremental cost function* for each quantity value *x*. Then $f(\cdot)$ satisfies the properties:

i) $f(x)>0, \forall x>0$;
ii) $f'(x)>0$.

Proof

From the assumptions made on the cost function $C(x,\theta)$ we have $C_x>0$, $C_\theta>0$, $C_{xx}>0$, $C_{x\theta}>0$ and $C_{xx\theta}>0$.

It is obvious that $f(x)>0$ for every $x>0$ (the marginal cost is strictly positive).

Then, differentiating the function $f(\cdot)$ with respect to *x* we get:

$$f'(x)=C_x(x,\overline{\theta})-C_x(x,\underline{\theta})$$

and from $\overline{\theta}>\underline{\theta}$ and the single-crossing property (Spence-Mirrlees property) it follows that $f'(x)>0$.

The above incentive compatibility constraints, expressed in terms of the new function $f(\cdot)$ become:

$$\underline{U}\geq\widetilde{U}+f(\widetilde{q}_1) \tag{1}$$
$$\underline{U}\geq\overline{U}+2f(\overline{q}) \tag{2}$$

b) Next, for the type $(\theta_1,\theta_2)=(\underline{\theta},\overline{\theta})$, the incentive compatibility constraints are:

$$\widetilde{U}\geq U^{(\underline{\theta},\overline{\theta})}(\widetilde{t},\widetilde{q}_1,\widetilde{q}_2)=\widetilde{t}-C(\widetilde{q}_1,\underline{\theta})-C(\widetilde{q}_2,\overline{\theta})=\widetilde{U}-f(\widetilde{q}_2)+f(\widetilde{q}_1)$$

or

$$f(\widetilde{q}_2)\geq f(\widetilde{q}_1)\Rightarrow\widetilde{q}_2\geq\widetilde{q}_1 \qquad (*)$$

The second constraint is written as:

$$\widetilde{U}\geq U^{(\underline{\theta},\overline{\theta})}(\overline{t},\overline{q},\overline{q})=\overline{t}-C(\overline{q},\underline{\theta})-C(\overline{q},\overline{\theta})$$

or

$$\tilde{U} \geq \overline{U} + f(\overline{q}) \tag{3}$$

The last incentive constraint for the type $(\underline{\theta}, \overline{\theta})$ is:

$$U^{(\underline{\theta},\overline{\theta})}(\tilde{t}, \tilde{q}_2, \tilde{q}_1) = \tilde{U} \geq U^{(\underline{\theta},\overline{\theta})}(\underline{t}, \underline{q}, \underline{q}) = \underline{t} - C(\underline{q}, \underline{\theta}) - C(\underline{q}, \overline{\theta})$$

or

$$\tilde{U} \geq \underline{U} - f(\underline{q}) \tag{4}$$

c) For the type $(\theta_1, \theta_2) = (\overline{\theta}, \overline{\theta})$ the incentive constraints are:

$$U^{(\overline{\theta},\overline{\theta})}(\overline{t}, \overline{q}, \overline{q}) = \overline{U} \geq U^{(\overline{\theta},\overline{\theta})}(\tilde{t}, \tilde{q}_2, \tilde{q}_1) = \tilde{t} - C(\tilde{q}_2, \overline{\theta}) - C(\tilde{q}_1, \overline{\theta})$$

or

$$\overline{U} \geq \tilde{U} - f(\tilde{q}_2) \tag{5}$$

And the second constraint becomes:

$$\overline{U} \geq U^{(\overline{\theta},\overline{\theta})}(\underline{t}, \underline{q}, \underline{q}) = \underline{t} - C(\underline{q}, \overline{\theta})$$

or

$$\overline{U} \geq \underline{U} - 2f(\underline{q}) \tag{6}$$

3.1.2. The Principal's objective function

Knowing that the Agent if of different types with the respective probabilities, the Principal's objective is to maximize his expected profit:

$$H(\cdot) = \underline{\gamma}\left[2S(\underline{q}) - \underline{t}\right] + \tilde{\gamma}\left[S(\tilde{q}_2) + S(\tilde{q}_1) - \tilde{t}\right] + \overline{\gamma}\left[2S(\overline{q}) - \overline{t}\right]$$

In the next section, we transform this objective function, expressing the transfer variables in terms of informational rents and quantities.

3.2. The problem in asymmetric information

With all the assumptions made in the previous section, the Principal's optimization program can therefore be written as:

$$\max_{\underline{q}, \tilde{q}_2, \tilde{q}_1, \overline{q}, \underline{U}, \tilde{U}, \overline{U}} H(\cdot) = \underline{\gamma}\left[2S(\underline{q}) - 2C(\underline{q}, \underline{\theta})\right] + \tilde{\gamma}\left[S(\tilde{q}_2) + S(\tilde{q}_1) - C(\tilde{q}_2, \underline{\theta}) - C(\tilde{q}_1, \overline{\theta})\right] +$$
$$+ \overline{\gamma}\left[2S(\overline{q}) - 2C(\overline{q}, \overline{\theta})\right] - \left[\underline{\gamma}\underline{U} + \tilde{\gamma}\tilde{U} + \overline{\gamma}\overline{U}\right]$$

subject to:

$$\underline{U} \geq \tilde{U} + f(\tilde{q}_1) \tag{1}$$
$$\underline{U} \geq \overline{U} + 2f(\overline{q}) \tag{2}$$

(P)

$$\widetilde{U} \geq \overline{U} + f(\overline{q}) \quad (3)$$
$$\widetilde{U} \geq \underline{U} - f(\underline{q}) \quad (4)$$
$$\overline{U} \geq \widetilde{U} - f(\widetilde{q}_2) \quad (5)$$
$$\overline{U} \geq \underline{U} - 2f(\underline{q}) \quad (6)$$
$$\underline{U} \geq 0,\ \widetilde{U} \geq 0,\ \overline{U} \geq 0$$

We will adopt a standard procedure for solving this program [4, 8]: first we ignore the last three incentive constraints (the local and global downward incentive constraints) and then we check that there are indeed satisfied.

Proposition 2 (*The Implementability Condition or Monotonicity Condition*). If the set of incentive feasible solutions fro the program (P) is nonempty, then the following inequalities hold:

$$\underline{q} \geq \max(\widetilde{q}_1, \overline{q})$$
$$\widetilde{q}_2 \geq \max(\widetilde{q}_1, \overline{q})$$

Proof

We consider the pairs of upward and downward constraints as follows:

We simply sum the constraints $\underline{U} \geq \widetilde{U} + f(\widetilde{q}_1)$ (1) and $\widetilde{U} \geq \underline{U} - f(\underline{q})$ (4) and we get:

$$f(\underline{q}) \geq f(\widetilde{q}_1)$$

Using the monotonicity property of the function $f(\cdot)$ it turns that:

$$\underline{q} \geq \widetilde{q}_1 \quad (7)$$

Summing the constraints (2) and (6) implies that:

$$2f(\underline{q}) \geq 2f(\overline{q})$$

or

$$\underline{q} \geq \overline{q} \quad (8)$$

Together, the relations (7) and (8) yield to $\underline{q} \geq \max(\widetilde{q}_1, \overline{q})$

Using a similar argument, from $\widetilde{U} \geq \overline{U} + f(\overline{q})$ (3) and $\overline{U} \geq \widetilde{U} - f(\widetilde{q}_2)$ (5) we have $f(\widetilde{q}_2) \geq f(\overline{q})$ or:

$$\tilde{q}_2 \geq \bar{q} \tag{9}$$

We also have $\tilde{q}_2 \geq \tilde{q}_1$ (*) and this relation combined with (9) can be summarized in the following condition:

$$\tilde{q}_2 \geq \max(\tilde{q}_1, \bar{q})$$

We try to reduce further the optimization problem of the Principal. Just in the two-type case, among all participation constraints, only one is binding, i.e. that one corresponding to the type $(\bar{\theta}, \bar{\theta})$; the other ones hold automatically at the optimum.

Proposition 3. If the participation constraint of the type $(\bar{\theta}, \bar{\theta})$ is satisfied, then all other constraints are satisfied at the optimum.

Proof

Using the incentive compatibility constraints and the participation constraint of the type $(\bar{\theta}, \bar{\theta})$ it follows immediately that:

$$\underline{U} \geq \tilde{U} + f(\tilde{q}_1) \geq \tilde{U} \geq \bar{U} + f(\bar{q}) \geq \bar{U} \geq 0$$

With all these results, the Principal's program becomes:

$$\max_{\underline{q}, \tilde{q}_2, \tilde{q}_1, \bar{q}, \underline{U}, \tilde{U}, \bar{U}} H(\cdot) = \underline{\gamma}\left[2S(\underline{q}) - 2C(\underline{q}, \underline{\theta})\right] + \tilde{\gamma}\left[S(\tilde{q}_2) + S(\tilde{q}_1) - C(\tilde{q}_2, \underline{\theta}) - C(\tilde{q}_1, \bar{\theta})\right] +$$
$$+ \bar{\gamma}\left[2S(\bar{q}) - 2C(\bar{q}, \bar{\theta})\right] - \left[\underline{\gamma}\underline{U} + \tilde{\gamma}\tilde{U} + \bar{\gamma}\bar{U}\right]$$

subject to:

(P^red^)

$$\underline{U} \geq \tilde{U} + f(\tilde{q}_1) \tag{1}$$
$$\underline{U} \geq \bar{U} + 2f(\bar{q}) \tag{2}$$
$$\tilde{U} \geq \bar{U} + f(\bar{q}) \tag{3}$$
$$\bar{U} \geq 0$$

and the monotonicity conditions on quantities.

The next question is whether the set of incentives constraints can be reduced still further.
One can find the answer in the following proposition.

Proposition 4. At the optimum the following constraints and conditions are binding:

$\bar{U} = 0$ and $\tilde{U} = f(\bar{q})$ and

$$\underline{U} = \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$$

Proof

Suppose that the participation constraint is not binding. Then, the optimal solution $(\underline{U}, \tilde{U}, \overline{U}, \underline{q}, \tilde{q}_2, \tilde{q}_1, \overline{q})$ satisfies strictly the participation constraint of the type $(\overline{\theta}, \overline{\theta})$, i.e. $\overline{U} > 0$.

In this case, we consider a small positive amount $\varepsilon > 0$ so as to make the preceding constraint binding $\overline{U} - \varepsilon \geq 0$. The feasible solution $(\underline{U} - \varepsilon, \tilde{U} - \varepsilon, \overline{U} - \varepsilon, \underline{q}, \tilde{q}_2, \tilde{q}_1, \overline{q})$ will leave unaffected all other relevant constraints while improving the optimal value of objective function and this is a contradiction. We have therefore: $\overline{U} = 0$.

The upward incentive constraints can be written as:

$$\underline{U} \geq \tilde{U} + f(\tilde{q}_1)$$
$$\underline{U} \geq 2f(\overline{q})$$
$$\tilde{U} \geq f(\overline{q})$$

Suppose now that $\tilde{U} > f(\overline{q})$. We can reduce by a small positive amount $\varepsilon > 0$ the informational rent $\tilde{U}$ such that:

$$\underline{U} \geq \tilde{U} + f(\tilde{q}_1) \geq \tilde{U} - \varepsilon + f(\tilde{q}_1)$$
$$\underline{U} \geq 2f(\overline{q})$$
$$\tilde{U} - \varepsilon \geq f(\overline{q})$$

Distorting downward the informational rent of the type $(\underline{\theta}, \overline{\theta})$, the Principal's payoff is increased with respect to the previous solution (optimal solution) and this is a contradiction. Hence, we must have that $\tilde{U} = f(\overline{q})$.

The incentive constraints written for the type $(\underline{\theta}, \underline{\theta})$ become:

$$\underline{U} \geq f(\overline{q}) + f(\tilde{q}_1) \text{ and}$$
$$\underline{U} \geq 2f(\overline{q})$$

And this two last relations yield to the following condition:

$$\underline{U} \geq \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$$

Suppose now that the last condition holds strictly. But this means that the Principal would pay more than it is necessary.

Suppose that $\underline{U} > \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$.

If this is the case, we could choose a small positive number $\varepsilon > 0$ such that:

$$\underline{U} - \varepsilon \geq \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$$

We could therefore increase the optimal value of the objective function and this is also a contradiction.

We can conclude that the optimal informational rents satisfy:

$$\underline{U} = \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$$
$$\tilde{U} = f(\overline{q})$$
$$\overline{U} = 0$$

It is easy now to derive the optimal transfer for each type of Agent:

$$\underline{t} = \underline{U} + 2C(\underline{q}, \underline{\theta})$$
$$\tilde{t} = f(\overline{q}) + C(\tilde{q}_2, \underline{\theta}) + C(\tilde{q}_1, \overline{\theta})$$
$$\overline{t} = 2C(\overline{q}, \overline{\theta})$$

4. The optimal contracts in asymmetric information (the second best solution)

With all the above results, substituting the informational rents into the objective function, we obtain a reduced unconstrained program with quantities as the only choice variables:

$$\underset{\underline{q}, \tilde{q}_1, \tilde{q}_2, \overline{q}}{Max}\ H\left(\underline{q}, \tilde{q}_1, \tilde{q}_2, \overline{q}\right)$$

where

$$\begin{aligned} H(\cdot) = \underline{\gamma}&\left[2S(\underline{q}) - 2C(\underline{q}, \underline{\theta}) - \max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}\right] + \\ +\tilde{\gamma}&\left[S(\tilde{q}_2) + S(\tilde{q}_1) - C(\tilde{q}_2, \underline{\theta}) - C(\tilde{q}_1, \overline{\theta}) - f(\overline{q})\right] + \\ +\overline{\gamma}&\left[2S(\overline{q}) - 2C(\overline{q}, \overline{\theta})\right] \end{aligned}$$

We can now solve the reduced optimization problem, deriving the optimal quantities (the second best solution). There are three cases to be discussed, depending on the term $\max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\}$ from the objective function.

Case I

We first assume that:

$$\max\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\} = f(\overline{q}) + f(\tilde{q}_1)$$

or $\tilde{q}_1 \geq \overline{q}$.

In this case, we substitute the expression into the objective function and we optimize the objective function with respect to the choice variables. From the assumptions made on the functions $S(\cdot)$ and $f(\cdot)$, the program satisfies the sufficient conditions. The first order conditions (which are also sufficient conditions) are:

$$\frac{\partial H}{\partial \underline{q}} = 0 \qquad \text{or} \qquad \underline{\gamma}\left[2S'(\underline{q}) - 2C_q(\underline{q},\underline{\theta})\right] = 0 \tag{10}$$

$$\frac{\partial H}{\partial \tilde{q}_2} = 0 \qquad \text{or} \qquad \tilde{\gamma}\left[S'(\tilde{q}_2) - C_q(\tilde{q}_2,\underline{\theta})\right] = 0 \tag{11}$$

$$\frac{\partial H}{\partial \tilde{q}_1} = 0 \qquad \text{or} \qquad -\underline{\gamma} f'(\tilde{q}_1) + \tilde{\gamma}\left[S'(\tilde{q}_1) - C_q(\tilde{q}_1,\overline{\theta})\right] = 0$$

$$\text{or } S'(\tilde{q}_1) = C_q(\tilde{q}_1,\overline{\theta}) + \frac{\underline{\gamma}}{\tilde{\gamma}} f'(\tilde{q}_1) \tag{12}$$

$$\frac{\partial H}{\partial \overline{q}} = 0 \qquad \text{or} \qquad -\underline{\gamma} f'(\overline{q}) - \tilde{\gamma} f'(\overline{q}) + 2\overline{\gamma}\left[S'(\overline{q}) - C_q(\overline{q},\overline{\theta})\right] = 0$$

or
$$S'(\overline{q}) = \frac{\underline{\gamma} + \tilde{\gamma}}{2\overline{\gamma}} f'(\overline{q}) + C_q(\overline{q},\overline{\theta}) \tag{13}$$

Using (10) and (11) we get:

$\underline{q}^{SB} = \tilde{q}_2^{SB}$ (the second best solution)

The second best quantity $\tilde{q}_1^{SB}$ is given by (12), and from (13) we can derive $\overline{q}^{SB}$.

We must check now if the solution given by the equations (12) and (13) satisfies the initial condition $\tilde{q}_1 \geq \overline{q}$. This latter condition is equivalent to $S'(\overline{q}) \geq S'(\tilde{q}_1)$ or:

$$\frac{\underline{\gamma} + \tilde{\gamma}}{2\overline{\gamma}} f'(\overline{q}) + C_q(\overline{q},\overline{\theta}) \geq C_q(\tilde{q}_1,\overline{\theta}) + \frac{\underline{\gamma}}{\tilde{\gamma}} f'(\tilde{q}_1)$$

or

$$\frac{\underline{\gamma} + \tilde{\gamma}}{2\overline{\gamma}} f'(\overline{q}) - \frac{\underline{\gamma}}{\tilde{\gamma}} f'(\tilde{q}_1) \geq C_q(\tilde{q}_1,\overline{\theta}) - C_q(\overline{q},\overline{\theta}) \geq 0$$

We have then:

$$\frac{\underline{\gamma} + \tilde{\gamma}}{2\overline{\gamma}} f'(\overline{q}) \geq \frac{\underline{\gamma}}{\tilde{\gamma}} f'(\tilde{q}_1) \geq f'(\overline{q}) \frac{\underline{\gamma}}{\tilde{\gamma}}$$

And this yields to:

$$\frac{\underline{\gamma}}{\tilde{\gamma}} \leq \frac{\underline{\gamma}+\tilde{\gamma}}{2\overline{\gamma}} \tag{14}$$

Remarks:

1. Using the expression of the correlation coefficient $\rho = \underline{\gamma}\overline{\gamma} - \frac{\tilde{\gamma}^2}{4}$, the above condition can be rewritten as: $\rho \leq \frac{\tilde{\gamma}}{4}\left(2\underline{\gamma}+\tilde{\gamma}\right)$. This latter condition is obviously satisfied when θ_1 and θ_2 are independent variables, since their correlation is then zero.
2. We consider this case as a case with *weak correlation* between the adverse selection parameters θ_1 and θ_2.

Proposition 5. If the condition $\frac{\underline{\gamma}}{\tilde{\gamma}} \leq \frac{\underline{\gamma}+\tilde{\gamma}}{2\overline{\gamma}}$ holds, then all the neglected participation and incentive compatibility constraints (the local and global downward incentive constraints) are satisfied.

Proof

Remember that the optimal informational rents are given by:

$$\underline{U} = f(\overline{q}) + f(\tilde{q}_1),\ \tilde{U} = f(\overline{q}) \text{ and } \overline{U} = 0$$

The incentive compatibility constraint $\tilde{U} \geq \underline{U} - f(\underline{q})$ becomes:

$$f(\overline{q}) \geq f(\overline{q}) + f(\tilde{q}_1) - f(\underline{q}) \text{ or } \underline{q} \geq \tilde{q}_1,$$

and this is true from the assumption made on the quantities.

In a similar way, the constraint $\overline{U} \geq \tilde{U} - f(\tilde{q}_2)$ becomes:

$$f(\tilde{q}_2) \geq f(\overline{q}) \text{ or } \tilde{q}_2 \geq \overline{q}$$

and this corresponds to the implementability condition.

The global incentive compatibility constraint can be written as:

$$0 \geq f(\overline{q}) + f(\tilde{q}_1) - 2f(\underline{q})$$

or

$$\left[f(\underline{q}) - f(\overline{q})\right] + \left[f(\underline{q}) - f(\tilde{q}_1)\right] \geq 0$$

and this condition is simply implied by the implementability condition.

Case II

Suppose now that $\max\left\{2f(\overline{q}), f(\overline{q}) + f(\tilde{q}_1)\right\} = 2f(\overline{q})$ with $2f(\overline{q}) > f(\overline{q}) + f(\tilde{q}_1)$ or $\overline{q} > \tilde{q}_1$.

This case shows that the inefficient production designed for the type $\left(\bar{\theta},\bar{\theta}\right)$ is higher than the inefficient production designed for the type ($\underline{\theta},\bar{\theta}$).

The Principal's objective function becomes:

$$
\begin{aligned}
H(\cdot) &= \underline{\gamma}\left[2S\left(\underline{q}\right)-2C\left(\underline{q},\underline{\theta}\right)-2f\left(\bar{q}\right)\right]+ \\
&+\tilde{\gamma}\left[S\left(\tilde{q}_2\right)+S\left(\tilde{q}_1\right)-C\left(\tilde{q}_2,\underline{\theta}\right)-C\left(\tilde{q}_1,\bar{\theta}\right)-f\left(\bar{q}\right)\right]+ \\
&+\bar{\gamma}\left[2S\left(\bar{q}\right)-2C\left(\bar{q},\bar{\theta}\right)\right]
\end{aligned}
$$

Optimizing with respect to the outputs, we get the following first order conditions:

$$\frac{\partial H}{\partial \underline{q}}=0 \quad \text{or} \quad \underline{\gamma}\left[2S'\left(\underline{q}\right)-2C_q\left(\underline{q},\underline{\theta}\right)\right]=0 \tag{15}$$

$$\frac{\partial H}{\partial \tilde{q}_2}=0 \quad \text{or} \quad \tilde{\gamma}\left[S'\left(\tilde{q}_2\right)-C_q\left(\tilde{q}_2,\underline{\theta}\right)\right]=0 \tag{16}$$

$$\frac{\partial H}{\partial \tilde{q}_1}=0 \quad \text{or} \quad \tilde{\gamma}\left[S'\left(\tilde{q}_1\right)-C_q\left(\tilde{q}_1,\bar{\theta}\right)\right]=0 \tag{17}$$

$$\frac{\partial H}{\partial \bar{q}}=0 \quad \text{or} \quad -2\underline{\gamma}f'\left(\bar{q}\right)-\tilde{\gamma}f'\left(\bar{q}\right)+\bar{\gamma}\left[2S'\left(\bar{q}\right)-2C_q\left(\bar{q},\bar{\theta}\right)\right]=0$$

or

$$S'\left(\bar{q}\right)=C_q\left(\bar{q},\bar{\theta}\right)+\frac{2\underline{\gamma}+\tilde{\gamma}}{2\bar{\gamma}}f'\left(\bar{q}\right) \tag{18}$$

Using the equations (15)-(18) we obtain the second best solution:

$\underline{q}^{SB}=\tilde{q}_2^{SB}$, $\tilde{q}_1^{SB}$ and $\bar{q}^{SB}$.

We must check if the above solution satisfies the assumption made on quantities $\bar{q}>\tilde{q}_1$. But this implies that $S'\left(\bar{q}\right)<S'\left(\tilde{q}_1\right)$ or:

$$C_q\left(\bar{q},\bar{\theta}\right)+\frac{2\underline{\gamma}+\tilde{\gamma}}{2\bar{\gamma}}f'\left(\bar{q}\right)<C_q\left(\tilde{q}_1,\bar{\theta}\right)$$

or

$$\frac{2\underline{\gamma}+\tilde{\gamma}}{2\bar{\gamma}}f'\left(\bar{q}\right)<C_q\left(\tilde{q}_1,\bar{\theta}\right)-C_q\left(\bar{q},\bar{\theta}\right)<0,$$

And this latter inequality is impossible.

Therefore, the Case II is impossible.

Case III

This case is a particular case. We suppose that $2\overline{q} = \overline{q} + \tilde{q}_1$ or $\overline{q} = \tilde{q}_1$. In this case, the Principal's objective function becomes:

$$\begin{aligned} H(\cdot) = \underline{\gamma}\left[2S(\underline{q}) - 2C(\underline{q},\underline{\theta}) - 2f(\overline{q})\right] + \\ + \tilde{\gamma}\left[S(\tilde{q}_2) + S(\overline{q}) - C(\tilde{q}_2,\underline{\theta}) - C(\overline{q},\overline{\theta}) - f(\overline{q})\right] + \\ + \overline{\gamma}\left[2S(\overline{q}) - 2C(\overline{q},\overline{\theta})\right] \end{aligned}$$

Optimizing with respect to the outputs, the second best solution is given by the following relations:

$$S'(\underline{q}) = C_q(\underline{q},\underline{\theta}) \tag{19}$$

$$S'(\tilde{q}_2) = C_q(\tilde{q}_2,\underline{\theta}) \tag{20}$$

$$S'(\overline{q}) = C_q(\overline{q},\overline{\theta}) + \frac{2\underline{\gamma} + \tilde{\gamma}}{2\overline{\gamma} + \tilde{\gamma}} f'(\overline{q}) \tag{21}$$

with $\overline{q}^{SB} = \tilde{q}_1^{SB} = q^G$.

This case corresponds to a *bunching of inefficient types*.

Remark: We consider this case as a case with *strong correlation* and *bunching of types*.

5. Conclusions

We studied a symmetric two-dimensional adverse selection model where the Agent has a generalized cost function, with two marginal costs parameters representing his private information. We assumed that these parameters are correlated variables and we derived the features of the optimal contracts in both situations: weak and strong correlation between types. In the first case (weak correlation), the second best solution has almost the same features as in the one-dimensional model with two types: the production of the Agent with low marginal costs on both activities is efficient and for each Agent with mixed type the production with low marginal cost is also efficient (such that $\underline{q}^{SB}$ and $\tilde{q}_2^{SB}$ correspond to the first best solution); the production for each activity with high marginal cost is distorted downward with respect to the first best (such that $\tilde{q}_1^{SB}$ and $\overline{q}^{SB}$ correspond are second best quantities). In the second case (strong correlation) the optimal contract entails some bunching of the inefficient types such that the productions $\underline{q}^{SB}$ and $\tilde{q}_2^{SB}$ are still efficient and now we have $\overline{q}^{SB} = \tilde{q}_1^{SB} = q^G$.

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Appendix

Positive correlation between types holds if $\rho = \underline{\gamma}\overline{\gamma} - \dfrac{\tilde{\gamma}^2}{4} > 0$.

The random variable vector $\theta = (\theta_1, \theta_2)$ has the following distribution:

$$\theta_1 / \theta_2 : \begin{pmatrix} \underline{\theta} & \overline{\theta} \\ \underline{\gamma} + \dfrac{\tilde{\gamma}}{2} & \overline{\gamma} + \dfrac{\tilde{\gamma}}{2} \end{pmatrix} \text{ and } \theta_2 / \theta_1 : \begin{pmatrix} \underline{\theta} & \overline{\theta} \\ \underline{\gamma} + \dfrac{\tilde{\gamma}}{2} & \overline{\gamma} + \dfrac{\tilde{\gamma}}{2} \end{pmatrix}.$$

We can easily derive the expected value of each variable:

$$M(\theta_1) = M(\theta_2) = \underline{\theta}\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}\left(\frac{\tilde{\gamma}}{2} + \overline{\gamma}\right)$$

And the dispersion of each variable is:

$$D^2(\theta_1) = D^2(\theta_2) = M\left(\theta_1^{\,2}\right) - M^2(\theta_1) = M\left(\theta_2^{\,2}\right) - M^2(\theta_2) =$$

$$= \underline{\theta}^2\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}^2\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right) - \left[\underline{\theta}\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right]^2$$

The correlation coefficient between θ_1 and θ_2 is written then as:

$$\rho(\theta_1,\theta_2) = \frac{M(\theta_1\theta_2) - M(\theta_1)M(\theta_2)}{\sqrt{D^2(\theta_1)D^2(\theta_2)}} =$$

$$= \frac{M(\theta_1\theta_2) - \left[\underline{\theta}\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right]^2}{\underline{\theta}^2\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}^2\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right) - \left[\underline{\theta}\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right]^2} \tag{A.1}$$

Proposition A.1. If the cost parameters θ_1 and θ_2 are positive correlated, then the following inequality holds:

$$\rho = \underline{\gamma}\overline{\gamma} - \frac{\tilde{\gamma}^2}{4} > 0$$

Proof

The coefficient satisfies $\rho(\theta_1,\theta_2) > 0$ if and only if $M(\theta_1\theta_2) - M(\theta_1)\cdot M(\theta_2) > 0$
or $M(\theta_1\theta_2) > M(\theta_1)\cdot M(\theta_2) \geq 0$.

Therefore, we can write the implicit condition:

$$M(\theta_1\theta_2) > 0 \quad \text{(A.2)}$$

(and this is because we have $M(\theta_2)\cdot M(\theta_2) = M^2(\theta_1) = M^2(\theta_2)$).

The above condition (A.2) can be rewritten as:

$$M(\theta_1\theta_2) = \underline{\theta}^2\underline{\gamma} + \overline{\theta}^2\overline{\gamma} + 2\underline{\theta}\overline{\theta}\frac{\tilde{\gamma}}{2} > 0$$

Let be $g(\underline{\theta}) = \underline{\theta}^2\underline{\gamma} + \overline{\theta}^2\overline{\gamma} + \underline{\theta}\overline{\theta}\tilde{\gamma}$.

It is obvious that $g\left(\underline{\theta}\right) \geq g_{\min}$ and this simply implies that:

$$g\left(\underline{\theta}\right) \geq g_{\min} > 0 \qquad \text{(A.3)}$$

It is easy to derive the minimum of the function $g\left(\cdot\right)$:

$$g_{\min} = \frac{\overline{\theta}^2}{4\underline{\gamma}}\left[\tilde{\gamma}^2 + 4\underline{\gamma}\overline{\gamma} - 2\tilde{\gamma}^2\right]$$

Inserting this value into (A.3) we get:

$$\rho = \underline{\gamma}\overline{\gamma} - \frac{\tilde{\gamma}^2}{4} > 0 .$$

Remarks:

The correlation coefficient $\rho\left(\theta_1\theta_2\right)$ is nonnegative (in this case), meaning that:

$$M\left(\theta_1\theta_2\right) - M\left(\theta_1\right)\cdot M\left(\theta_2\right) \geq 0$$

Evaluating the above expression we also have:

$$M\left(\theta_1\theta_2\right) - M\left(\theta_1\right)M\left(\theta_2\right) = \underline{\theta}^2\underline{\gamma} + \overline{\theta}^2\overline{\gamma} + \overline{\theta}\,\underline{\theta}\tilde{\gamma} - \left[\underline{\theta}\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right]^2 =$$

$$= \underline{\theta}^2\left[\underline{\gamma} - \left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right)^2\right] + 2\underline{\theta}\overline{\theta}\left[\frac{\tilde{\gamma}}{2} - \left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right] + \overline{\theta}^2\left[\overline{\gamma} - \left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)^2\right]$$

(A.4)

If we consider the above expression as a quadratic function with respect to the variable $\underline{\theta}$, the discriminant of this function is:

$$\Delta' = \overline{\theta}^2\left[\frac{\tilde{\gamma}}{2} - \left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\right]^2 - \overline{\theta}^2\left[\underline{\gamma} - \left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right)^2\right]\left[\overline{\gamma} - \left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)^2\right] =$$

$$= \overline{\theta}^2\left[\frac{\tilde{\gamma}^2}{4}\left(1 - 2\overline{\gamma} - 2\underline{\gamma} - \tilde{\gamma} + \underline{\gamma} + \overline{\gamma}\right) - \underline{\gamma}\overline{\gamma}\left(1 - \overline{\gamma} - \underline{\gamma} - \tilde{\gamma}\right)\right]$$

We get at the end:

$$\Delta' = 0$$

Hence $\rho\left(\theta_1, \theta_2\right)$ is nonnegative.

Perfect correlation condition (or strongly positive correlated variables)

Proposition A.2. The cost parameters θ_1 and θ_2 are perfectly correlated if $\tilde{\gamma} = 0$.

Proof

Using the above expression (A.1) for $\rho(\theta_1, \theta_2)$, it follows that $\rho(\theta_1, \theta_2) = 1$ if:

$$M(\theta_1, \theta_2) = \underline{\theta}^2\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right) + \overline{\theta}^2\left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)$$

or

$$\underline{\theta}^2\underline{\gamma} + \overline{\theta}^2\overline{\gamma} + \underline{\theta}\overline{\theta}\tilde{\gamma} = \underline{\theta}^2\underline{\gamma} + \overline{\theta}^2\overline{\gamma} + \frac{\tilde{\gamma}}{2}\left(\underline{\theta}^2 + \overline{\theta}^2\right)$$

or

$$\frac{\tilde{\gamma}}{2}\left(\overline{\theta} - \underline{\theta}\right)^2 = 0$$

Hence, we obtain $\tilde{\gamma} = 0$.

Proposition A.3. If the variables θ_1 and θ_2 are independently distributed, then $\rho = 0$.

Proof

If θ_1 and θ_2 are independently distributed, it must be that:

$$\Pr\left(\theta_1 = \overline{\theta}, \theta_2 = \underline{\theta}\right) = \Pr\left(\theta_1 = \overline{\theta}\right) \cdot \Pr\left(\theta_2 = \underline{\theta}\right)$$

or

$$\frac{\tilde{\gamma}}{2} = \left(\overline{\gamma} + \frac{\tilde{\gamma}}{2}\right)\left(\underline{\gamma} + \frac{\tilde{\gamma}}{2}\right)$$

or

$$\overline{\gamma}\underline{\gamma} - \frac{\tilde{\gamma}^2}{2} + \frac{\tilde{\gamma}^2}{4} = 0$$

And this yields to:

$$\rho = \overline{\gamma}\underline{\gamma} - \frac{\tilde{\gamma}^2}{4} = 0.$$