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NEW ROBUST TESTS FOR THE DETECTION OF ARCH EFFECT

Abstract. *ARCH model has become very popular because it enables the econometricians to estimate the variance of a series at a particular point of time. Although a number of tests are available in the literature, there is evidence that they fail to detect ARCH effect especially when outliers are present in the data. In this article, we proposed two new tests for the detection of ARCH effect of innovation based on simple Chi-square (* χ^2 *) and Student's -t tests and call them the Arch Test-Chi-square based (Arch* χ^2 *) and Arch Test- Student's –t based (Arch Student's –t), respectively. The performance of the proposed tests is examined by real life data as well as Monte Carlo simulations.*

Keywords: *ARCH, Innovation, Outlier, Chi-square, Student's -t, Simulation.*

JEL Classification: C12, C22, C52, C63

1. INTRODUCTION

It is conventional to note that the problem of heteroscedasticity is often associated with cross-sectional data; whereas econometrics or time series are usually studied in the context of homoscedastic processes. On the other hand, autocorrelation is a common feature of time series data. In analyses of macroeconomic data, Engel (1982, 1983) found the evidence that for some kinds of data, the innovation variances in time series models were less stable than usually assumed. To handle this situation, Engel's results suggested that in models of inflation, large and small forecast innovations appeared to occur in clusters, suggesting a form of heteroscedasticity in which the variance of the forecast innovation depends on the size of the previous innovation. He suggested the autoregressive, conditionally heteroscedastic, or ARCH, model as an alternative to the usual time series process. More recent studies of financial markets suggest that the phenomenon is quite common. Different aspects of ARCH models are studied by many authors in the literatures. The ARCH model has proven to be useful in studying the volatility of inflation (Coulson and Robins 1985), the term structure of interest rates (Engle et al. 1985), the volatility of stock market returns (Engle et al. 1987) and the behaviour of foreign exchange markets (Domowitz & Hakkio 1985; Bollerslev & Ghysels 1996). A large body of literature (Campbell et al. 1997; Johnston & Dinardo 1997; Enders 2008; Greene 2008; Gujarati 2010) are now available to make valid inferential statements in the presence of ARCH. In detecting an ARCH effect in time series data, it is conventional to use the Lagrange Multiplier (LM) test mainly because of the tradition and ease of computation. In identifying an ARCH effect by LM test we often observe that this test is affected by unusual observations. Imon et al. 2007 mentioned that the existence of a set of observations, called *ARCH-influential*, whose presence or absence may cause a huge interpretative problem in understanding the ARCH effect of the data. They presented several examples to demonstrate the fact that outliers are the prime source of ARCH-influential observations in time series data. Moreover, they suggested Modified Breusch Pagan Goldfrey (MBPG) test for the detection of ARCH by robustifying the conventional Breusch Pagan Goldfrey test. As expected, this test is more robust than the existing ones. The main limitation of this test is based on the auxiliary variables which make the test procedure impractical in many cases. It motivates us to develop new alternative tests for the detection of ARCH effect in the presence of outliers. Most of the ARCH detection tests are based on the mean squared error. Evidence shows that the mean squared error is more sensitive to outliers than mean absolute deviation of error (Maronna et al. 2006). In this respect, we proposed the tests based on the mean absolute deviation of error are likely to be less sensitive to outliers.

The key idea of ARCH is that the variance of u at time t $(=\sigma_t^2)$ depends on the size of the square innovation term at time $(t-1)$, that is, on u_{t-1}^2. To be more specific, let us consider the regression model:

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t \tag{1}$$

and assume that conditional on the information available at time $(t-1)$, the innovation term is distributed as

$$u_t \sim N\left[0, \left(\alpha_0 + \alpha_1 u_{t-1}^2\right)\right] \tag{2}$$

Since in (2) the variance of u_t depends on the square innovation term in the previous time period, it is called an ARCH(1) process. But the generalize form of ARCH(*p*) process can be written as

$$\text{var}(u_t) = \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \cdots + \alpha_p u_{t-p}^2 \tag{3}$$

Assuming an Auto Regressive (AR) process with Gaussian noise in equation (3), Fox (1972) at first consider outliers within time series. There are two categories of outliers are defined broadly: "Additive Outliers (AO)" where a single observation is affected and "Innovation Outliers (IO)" where an unusual observation in the generating process affects all later observations and the subsequent series. In this article, we suggest that innovations based on the ordinary least square (OLS) fit of the model should be more convincing and reliable in a diagnostic test for detection of the ARCH effect of the innovation in time series data.

2. PROPOSED ARCH TESTS

If X be an $N\left(\mu, \sigma^2\right)$ variable, then the mean absolute deviation d is given by

$$d = \frac{\sum_{t=1}^{T} | X_t - \mu |}{T} \tag{4}$$

Stuart & Odd 1987 showed that

$$E(d) = \sigma\sqrt{\frac{2}{\pi}} = \sqrt{\frac{2\sigma^2}{\pi}} \tag{5}$$

and $Var(d) = \dfrac{\sigma^2}{T^2}\left(1 - \dfrac{2}{\pi}\right)$ (6)

Using (5) they also showed that

$$E(d) = E\left(| x_t - \bar{x} |\right) = \sqrt{\frac{2\sigma^2 (T-1)}{\pi T}} \text{ as } T \to \infty \tag{7}$$

Let us consider the multiple linear regression model

$$Y = X\beta + U \tag{8}$$

where, Y is a $(T \times 1)$ vector of the responses, X is a $(T \times p)$ matrix of the levels of the regressor variables, β is a $(p \times 1)$ vector of the regression coefficients, and U is a $(T \times 1)$ vector of the random innovations. We also assume that $U \sim NID(0, \sigma^2 I)$. Now we consider a mean deviation about mean of the innovation which is given by

$$d = \frac{\sum_{t=1}^{T} |u_t|}{T} \tag{9}$$

Here we ignore $\bar{u}$ as $E(u) = 0$. In linear regression model, (7) can be written as

$$E(d) = \sqrt{\frac{2\sigma^2 (T-p)}{\pi T}} \tag{10}$$

From (9), it is easy to show that $\sum_{t=1}^{T} |u_t| = Td$

Taking mathematical expectation on both sides and using (10), we obtain

$$E\left(\sum_{t=1}^{T} |u_t|\right) = E(Td) = TE(d) = T\sqrt{\frac{2\sigma^2 (T-p)}{\pi T}} = \sqrt{\frac{2\sigma^2 T^2 (T-p)}{\pi T}} \tag{11}$$

Again, using (6), we obtain

$$Var\left(\sum_{t=1}^{T} |u_t|\right) = Var(Td) = T^2 Var(d) = T^2 \frac{\sigma^2}{T^2}\left(1 - \frac{2}{\pi}\right) \; \sigma^2\left(1 - \frac{2}{\pi}\right) \tag{12}$$

From the identity $Var\left(\sum_{t=1}^{T} |u_t|\right) = E\left(\sum_{t=1}^{T} |u_t|\right)^2 - \left(E\left(\sum_{t=1}^{T} |u_t|\right)\right)^2$

we obtain

$$E\left(\sum_{t=1}^{T} |u_t|\right)^2 = \sigma^2\left(1 - \frac{2}{\pi}\right) + \left(\sqrt{\frac{2\sigma^2 T(T-p)}{\pi}}\right)^2 = \sigma^2\left(1 - \frac{2}{\pi}\right) + \left(\frac{2\sigma^2 T(T-p)}{\pi}\right)$$

$$= \frac{\sigma^2(\pi - 2 + 2T(T-p))}{\pi}$$

$$\Rightarrow E\left(\frac{\left(\sum_{t=1}^{T} |u_t|\right)^2 \pi}{\pi - 2 + 2T(T-p)}\right) = \sigma^2 \tag{13}$$

as $T \to \infty$.

Now, we want to test the hypothesis:

H_0 : There is no ARCH effect

H_1 : H_0 is not true

Testing of hypothesis in general linear regression model, we use

$$\frac{(T-p)MSE}{\sigma^2} \overset{a}{\sim} \chi^2_{T-p} \tag{14}$$

and

$$\frac{MSE-\sigma^2}{\sqrt{\frac{2\sigma^4}{T-p}}} \overset{a}{\sim} t_{T-p} \tag{15}$$

where $MSE = \frac{\sum_{t=1}^{T} u_t^2}{T-p}$.

Using equation (13), we obtain from (14)

$$\frac{(T-p)MSE}{\sigma^2} = \frac{(T-p)MSE}{\frac{\left(\sum_{t=1}^{T}|u_t|\right)^2 \pi}{\pi-2+2T(T-p)}} \overset{a}{\sim} \chi^2_{T-p} \tag{16}$$

which is our required propose χ^2 test statistic for testing the ARCH effect.

Again using equation (13), our required statistic for the propose *t* test obtained from (15) is

$$\frac{MSE-\sigma^2}{\sqrt{\frac{2\sigma^4}{T-p}}} = \frac{\sqrt{T-p}\left[MSE - \frac{\left(\sum_{t=1}^{T}|u_t|\right)^2 \pi}{\pi-2+2T(T-p)}\right]}{2\left(\frac{\left(\sum_{t=1}^{T}|u_t|\right)^2 \pi}{\pi-2+2T(T-p)}\right)} \overset{a}{\sim} t_{T-p} \tag{17}$$

3. EXAMPLE

Here, we consider a real life data. The data set consists of the monthly U.S. air passenger miles (in millions) for the period January 1960 to December 1977 (T=216), which is taken from Upender 2008. At first, we check outliers by the robust LTS method. This method can detect 9 outliers (case 128, 140, 152, 164, 176, 188, 200, 205 and 212). Next we check the order of ARCH and compute Lagrange multiplier and modified BPG tests together with our proposed χ^2-based and t-based tests. The results are shown in Table 1. Values in parenthesis for each test in the tables indicate the number of lag.

TABLE 1

ARCH test results for the monthly U.S. air passenger miles (in millions) data with and without Outliers

Test	Value of statistic		Critical value (5%)		p-value	
	With Outlier	Without Outlier	With Outlier	Without Outlier	With Outlier	Without Outlier
LM (5)	0.0992	32.7747	3.8415	11.0705	0.753	0.000
BPG (5)	483.443	318.405	11.0705	11.0705	0.000	0.000
χ^2-based (5)	450.799	262.918	240.484	228.579	0.000	0.001
t-based (5)	11.9594	3.0418	1.6524	1.6527	0.000	0.001

From the above table we observe that when outliers are not present in the data all tests suggest rejecting homoscedastic innovation variance at the 5% level of significance, but in the presence of outliers the Lagrange Multiplier (LM) test fails to detect the ARCH effect. The rest of the tests (Modified Breusch Pagan Goldfery (MBPG), Chi-Square-based and Student's –*t* based tests) suggest that we should accept that the innovation is conditionally heteroscedastic even when outliers are present in the data.

4. SIMULATION RESULTS

From the above example we get the impression that the conventional ARCH detection tests may be unsuccessful in the presence of outliers but our proposed test can perform better irrespective of the existence of outliers. Now from our experience with an individual data set we want to confirm it by reporting the results of a Monte Carlo simulation experiment. We first simulate homoscedastic time series data from uniform distribution and take one period lag on this sample to get the independent variable y_{t-1}. Next we generate innovation u_t from Gaussian distribution with mean 0 and constant variance. Next we compute the dependent variable y_t from the equation

$$y_t = \beta_1 + \beta_2 y_{t-1} + u_t \qquad (18)$$

for given values of β_1 and β_2. We run this simulation experiment 10,000 times for four different sample of sizes T = 50,100, 150 and 200 each at 5% level of significance and the results of the experiment are given in Table 2.

Table 2

Simulation power of ARCH tests at α =0.05, under homoscedasticity

Test Name	T = 50	T = 100	T = 150	T = 200
LM (1)	5.73	5.91	7.21	8.06
MBPG(1)	4.25	4.68	4.74	4.61
χ^2 -based (1)	1.09	0.78	0.65	0.012
t-based (1)	1.29	0.89	0.69	0.17

We observe from Table 2 that in the presence of outliers, the LM test is very poor, followed by the MBPG test. The power of LM test for rejecting homoscedasticity when in fact it is true, is higher than other tests. But it is interesting to note that both the Chi-square (χ^2)-based and t-based tests have lower powers which indicate that the tests successfully confirm the homoscedasticity even though outliers are present in the data.

Next we simulate ARCH data of order 3. We generate samples from uniform distribution and take one period lag on this sample to obtain the independent variables. We generate innovation from Gaussian distribution with mean 0 and variance $\alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \alpha_3 u_{t-3}^2$ for the arbitrary chosen constants $\alpha_0, \alpha_1, \alpha_2$ and α_3. In our simulation experiment, we consider four different sample sizes *T* = 50, 100, 150 and 200 each at 5% level. Each of the result based on 10,000 replications are presented in Table 3.

From Table 3, we observe that the power of the Chi-square (χ^2)-based and *t*-based tests are very high than the rest of the tests. The Modification of Breusch Pagan Goldfery (MBPG) test is the next choice but Lagrange Multiplier (LM) test perform poorly throughout.

Table 3

Simulation power of ARCH tests at α =0.05, under heteroscedasticity

Test	T = 50	T = 100	T = 150	T = 200
LM (1)	8.87	11.21	16.05	35.98
LM (2)	10.22	16.54	27.58	46.50
LM (3)	9.94	18.85	21.87	40.24
MBPG(1)	94.01	94.85	94.98	94.94
MBPG(2)	95.65	93.25	93.39	94.29
MBPG(3)	97.01	95.58	96.54	95.68
χ^2 -based (1)	98.09	99.98	100	100
χ^2 -based (2)	98.87	99.99	100	100
χ^2 -based (3)	98.45	99.97	100	100
t –based (1)	99.54	99.99	100	100
t –based (2)	99.59	99.98	100	100
t –based (3)	99.86	99.99	100	100

5. CONCLUSION

In this article, we developed two new tests for the detection of ARCH based on the mean absolute deviation of errors. Empirical results show that these two tests are very successful in the identification of ARCH in the presence of outliers whereas the most popular Lagrange Multiplier (LM) test for conditional heteroscedasticity fails to do so. Simulation results show that these tests possess very good power under a variety of situations.

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