Seyed Habib A.RAHMATI, M.Sc. Student
Department of Industrial and Mechanical Engineering
Islamic Azad University
Qazvin Branch, Qazvin, Iran
E-mail: sd_rahmaty@yahoo.com
Seyed Hamid Reza PASANDIDEH, PhD
Department of Industrial Engineering, Faculty of Engineering
Tarbiat Moallem University,Tehran, Iran
E-mail: shr_pasandideh@tmu.ac.ir

A FUZZY APPROACH TO QUEUING SYSTEM'S MEASURES AND STEADINESS

Abstract. *Queuing systems are naturally stochastic. Besides, these systems have many measures which are calculated by taking samples. Mostly, these samples are consisted of imprecise data especially in those systems which don't have historical data and are unknown for experts of the system. Therefore, the measures which are calculated by such data are expected to be imprecise. To face with these uncertainty and impreciseness, this paper, based on Buckley's estimation method, proposes a fuzzy method to evaluate queue's measures. This method by considering both of the uncertainty and impreciseness concepts introduces triangular shape fuzzy numbers for different measures. Another important consideration in queue's studies is to determine whether they are steady or transient. In this paper, to do such important determination, a new fuzzy method, which is consistent with special form of fuzzy numbers of the paper, is also introduced. To illustrate the methods more explicitly, numerical example are also given.*

Keywords: *Queue theory, Buckley's* estimation approach, Fuzzy measures, Fuzzy steadiness tasting.

JEL Classification: **C15, D81, A12, C02**

1-Introduction

Surly, one of the most important usages of stochastic processes and probability theory is in queuing system modeling. Queuing systems have been applied in various practical terms like; industrial systems, computerized systems, transportation systems, different types of service systems, financial systems and many other aspects of real world systems that have similar features with queuing systems. In all mentioned systems, wasting of sources (like time, energy, capital etc) due to existence of queues, force owners of the systems to recognize their systems much better by studying queue's features. The recognition of these systems (like any other system) is due to identifying input, output, and other component of the systems. Most of the important features of queuing systems are introduced by their different measures (like utilization factor). It means that precise recognition of each queuing systems is due to the preciseness of their measures. This precise recognition helps us to control our queuing system more efficiently and reduce or eliminate the wasting of sources.

Since in practical systems different type of measurements are usually estimated by taking sample, validation of those measurements and analyzes obtained from measurements are severely relied on the quality of obtained data. Usually, in a system like queuing system obtained data have two special features;1) uncertainty and 2) vagueness. The first features force us to use different statistical concepts, like calculating expected value of measures; estimating point and interval estimate of measures and other similar concept to prevent bad effect of uncertainty on the systems. On the other hand, the second features force us to prevent another type of bad effects on our systems which are usually due to; 1) having no data or 2) having imprecise data. To cope with second feature usually a concept which was developed by Zadeh (1978) is applied. Therefore, in this paper because of importance of fuzzy queues rather than commonly used crisp queues (see Li and Lee (1989)) fuzzy assessment of queuing systems is developed.

Fuzzy studying of queuing systems has been done in different studies. Negi and Lee (1992) to analysis fuzzy queues, proposed the α-cut and two-variable simulation. Li and Lee (1989) using a general approach based on Zadeh's extension principle (Zadeh, (1978) and Yager, (1986)), the possibility concept and fuzzy Markov chains (Stanford, R.E.,1982), developed the analytical results for two typical fuzzy queues (called M/F/1/1 and FM/1/1, where F represents fuzzy time and FM represents fuzzy exponential distributions).

Buckley (1990) on the basis of possibility theory discussed elementary multiple-server queuing systems with finite or infinite capacity and source population. Buckley et al. (2001) to determine optimal number of servers extended the results of Buckley (1990) to a fuzzy queuing decision problem.

Buckley has also introduced (Buckley and Eslami, (2003a, b), Buckley, (2005a, b)) a method to create fuzzy numbers for parameters of different distributions (discrete and continuous), in which both of the uncertainty and vagueness features are considered. In his method, $(1-\beta)100\%$ confidence intervals for a parameter are used as a family of $\alpha - cuts$ of a triangular shaped fuzzy number. In this paper, developing his approach some measures of queuing systems is estimated. Besides, he has also developed (Buckley, J.J., 2005a) a method to compare fuzzy statistic with fuzzy critical value in a hypothesis testing. In this paper, his idea of fuzzy hypothesis testing is used to create a new steadiness testing of queuing systems. Of course, it should be mentioned that his approach is modified to be used much simpler in our specific study.

Jau-Chuan et al (2006) to describe the family of crisp queues with a vacationing server, by means of the membership functions of the system characteristics developed a set of parametric nonlinear programs. Shih-Pin (2007) to construct the membership function of the minimal expected time of their fuzzy objective value, calculate the lower and upper bounds of the minimal expected total cost per unit time by the means of formulating three pairs of mixed integer nonlinear programs (MINLP) parameterized by the possibility level α. Pardo and Fuente, (2009) consider a model of design and control of queues in which the parameters that describe the distributions of the service time are fuzzy numbers.

The organization of this paper is as follows. Section2 reviews the classical concepts of queuing systems and their measures. Section3 discusses Buckley's approach and presents the $\alpha - cuts$ of fuzzy estimation for different measures. Section4 using Buckley's approach, proposes a new algorithm for fuzzy estimation of measures based on predefined $\alpha - cuts$ and illustrates it by numerical example. Section5 presents our new fuzzy steadiness testing method and explain it by example. Section6 concludes the paper.

2- Queue measures' definitions and formulations

There are different types of queuing systems. However, measures that should be calculated in each systems (or we would better to say in each model) are usually the same. By adding new features to each model like the limited capacity of servers the calculation of measures gradually getting hard and sometimes new measures are created. But like any other developed model queues' model has some specific forms (for some specific condition) that are calculated easier. In this paper, measures of one specific model are considered to become fuzzy. This model is M/M/m queue model. Of course, two of its specific forms which are M/M/1 and M/M/2 are studied in detail. It should be mentioned that by adapting the method for these two specified model the method can be developed for many various queues' model (like M/M/1/K) easily and this paper just for more explicitly and convenience in the explanations of the method choose these two specified form.

A preliminary assessment before doing any queue's calculations is to determine whether it is steady or it is transient. To do so, we should compare the utilization factor (ρ) of that model with number 1. If ρ is less than 1 ($\rho < 1$) the system is steady otherwise, ($\rho > 1$) it is transient (Gross & Harris 1984). Usually, queuing systems studies are done in steady environments (especially in practical studies). In this paper the model is studied under the condition of steadiness too. The paper also introduced a new steadiness testing which is compatible with special form of created fuzzy numbers of the paper. The rest of the section discusses measures of the queues in two mentioned models.

2.2. M/M/1

Before discussing the measures, following mathematical notation are introduced:

: Number of the servers m

: Number of the system's customers n

: Arrival rate of the customers λ

: Service rate μ

: Average number of the customers in system (in steady condition) L

: Average number of the customers which are waiting in queue (in steady condition) L_q

: Average waiting time of the customers in system (in steady condition) W

: Average waiting time of the customers in queue (in steady condition) W_q

: The probability of existence n customers in the system (in steady condition) π_n
: Amount of the customers in the system (in steady condition) $N(t)$

$$\pi_0 = 1 - \rho \quad (2)$$

$$\pi_n = \rho^n (1 - \rho) \quad (3)$$

$$L = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho} \quad (4)$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho} \quad (5)$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1 - \rho)} \quad (6)$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu(1 - \rho)} \quad (7)$$

$$P(N(t) > n) = \rho^{n+1} \quad (8)$$

2.3. M/M/2

$$\pi_0 = \frac{1 - \rho}{1 + \rho} \quad (9)$$

$$L_q = \frac{2\rho^3}{1 - \rho^3} \quad (10)$$

$$W_q = \frac{L_q}{\lambda} \quad (11)$$

3. Buckley's estimation approach

In what follows, with modification, fuzzy estimation based on Buckley's approach (Buckley and Eslami, (2003a, b)) is presented. First some notations are introduced. A triangular shaped fuzzy number N is a fuzzy subset of the real numbers R satisfying:

(1) Normality: N(x) =1 for exactly one x $\in$ R

(2) Convexity: one way for creating a convex shape for a fuzzy number is to create an L-R (Left-Right) number. A L-R number is a number which its left side is created by an increasing function and its right side is created by a decreasing number. L-R function in this paper is created as follows:
For $\alpha \in (0,1]$, the $\alpha - cut$ of N is a closed and bounded interval, which denoted by $N_\alpha = [n_1(\alpha), n_2(\alpha)]$, where $n_1(\alpha)$ is increasing and $n_2(\alpha)$ is decreasing continuous functions.

In this paper, triangular shaped fuzzy number is used and created for parameter estimation and following definition are considered:
X**:** a random variable with p. d .f. (p .m .f) $f(x;\theta)$ for single parameter θ
θ**:** an unknown variable that must be estimated from a random sample $X_1, X_2, X_3, ..., X_n$
θ**:** represent a statistic which is used to estimate y=u($X_1, X_2, X_3, ..., X_n$)
Given the values of our random variable, e.g. $X_i = x_i,\ 1 \le i \le n$, one can obtain a point estimate. $\hat{\theta} = y = u(x_1, x_2, x_3, ..., x_n)$ for θ. In most of statistical studies, since we never expect this point estimate be exactly equal to θ, we also compute a $(1-\alpha)100\%$ confidence interval for θ.

In this paper a $(1-\alpha)100\%$ confidence interval for θ denotes by $[\theta_1(\alpha), \theta_2(\alpha)]$ (for $0 < \alpha < 1$). According to mentioned notation, the interval $\theta_1 = [\hat{\theta}, \hat{\theta}]$ shows 0% confidence interval for θ and $\theta_0 = \ddot{\theta}$ shows a 100% confidence interval for θ, where $\ddot{\theta}$ is the whole parameter space .Then for $0 \le \alpha \le 1$ a family of $(1-\alpha)100\%$ confidence intervals for θ is obtained. Placing these confidence intervals, one on top of the other, a triangular shaped fuzzy number θ whose $\alpha - cuts$ are the following confidence intervals is obtained:

$$\theta_\alpha = [\theta_1(\alpha), \theta_2(\alpha)] \text{ for } 0 < \alpha < 1 : \theta_0 = \ddot{\theta} \text{ and } \theta_1 = [\hat{\theta}, \hat{\theta}].$$

Hence by using this method we have more information about θ rather than a point estimate, or just a single interval estimate. It is easy to generalize Buckley's method in the case where θ is a vector of parameters (Buckley, Eslami, (2003a, b), Buckley, (2005a)).

It should be mentioned that in this paper α represent both significant level (therefore as usual our confidence intervals show as $(1-\alpha)100\%$) and $\alpha - cuts$ of fuzzy numbers. The rest of the section computes $(1-\alpha)100\%$ confidence intervals for ρ , L, L_q, W, W_q, π_0, π_n in M/M/1 and for ρ, π_0, L_q, W_q in M/M/2 . In section 4 interval of each factor will use as $\alpha - cuts$ of its fuzzy estimator.

3.1. M/M/1

According to normal approximation to the Poisson, we can obtain following confidence intervals for parameter of Poisson distribution (λ) for a determined level of α :

$$\{\lambda - Z_{\frac{\alpha}{2}}\sqrt{\frac{\lambda}{n}}, dpo + Z_{\frac{\alpha}{2}}\sqrt{\frac{\lambda}{n}}\} \text{ for } 0<\alpha<1 . \quad (12)$$

Since most of the time λ is unknown and we should estimate it by sampling and also since according to MLE approach the point estimator of λ is $\overline{X}$ (Rohatgi and Ehsanes Saleh (2001)), we would better to use $\overline{X}$ instead of λ as follow:

$$\hat{\lambda}_{\alpha} = \{\overline{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{X}}{n}}, \overline{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{X}}{n}}\}\ 0<\alpha<1 \quad (13)$$

In M/M/1 both of the service and demand rate have Poisson distribution therefore, Eq. (13) indicates their approximated confidence intervals. It should be mentioned that in this paper $\overline{X}$ is used to denote the mean of samples which taken to estimate λ and similarly $\overline{Y}$ is used for μ and have following confidence interval:

$$\hat{\mu}_{\alpha} = \{\overline{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{Y}}{n}}, \overline{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{Y}}{n}}\}\ 0<\alpha<1 \quad (14)$$

Notice that in what follows, notation n in the dominator of the L-R functions denotes amount of samples and the n on power in different part of the functions denote amount of people in the system.

Now according to what define in Eq. (13) and Eq. (14) we have the following confidence interval for ρ :

$$\hat{\rho}_{\alpha} = \{\frac{\overline{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{X}}{n}}}{\overline{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{Y}}{n}}}, \frac{\overline{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{X}}{n}}}{\overline{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{Y}}{n}}}\}\ 0<\alpha<1 \quad (15)$$

And confidence intervals for other factors are as follows:

$$\hat{\pi}_{0_\alpha} = \{1 - \frac{\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}}, 1 - \frac{\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}}\} \quad \text{for } 0<\alpha<1 \tag{16}$$

$$\hat{\pi}_{n_\alpha} = \{(1 - \frac{\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}}) * (\frac{\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}})^n, (1 - \frac{\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}}) * ((\frac{\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}})^n\} \quad 0<\alpha<1 \tag{17}$$

$$\hat{W}_{q_\alpha} = \{\frac{\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{(\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}) * (\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}, \frac{\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{(\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}) * (\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}\} \quad 0<\alpha<1 \tag{18}$$

$$\hat{W}_{\alpha} = \{\frac{1}{(\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}, \frac{1}{(\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}\} \quad 0<\alpha<1 \tag{19}$$

$$\hat{L}_{\alpha} = \{\frac{\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}})}, \frac{\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}})}\} \quad 0<\alpha<1 \tag{20}$$

$$\hat{L}_{q_\alpha} = \{\frac{(\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}})^2}{(\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}) * (\bar{Y} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}, \frac{(\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}})^2}{(\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}) * (\bar{Y} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}} - (\bar{X} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}))}\} \quad 0<\alpha<1 \tag{21}$$

$$\hat{P}_{\alpha}(N(t)>n)=\{(\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}})^{n+1},(\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}}})^{n+1}\}\ 0<\alpha<1 \quad (22)$$

3.2. M/M/m

According to what was obtained in 3.1, similar results for M/M/m are as follow:

$$\hat{\rho}_{\alpha}=\{\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})},\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})}\}\ 0<\alpha<1 \quad (23)$$

3.2.1. M/M/m=2

$$\hat{\pi}_{0_{\alpha}}=\{\frac{1-\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})}}{1+\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})}},\frac{1-\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})}}{1+\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})}}\}\ \ 0<\alpha<1 \quad (24)$$

$$\hat{\pi}_{n_{\alpha}}=\{\frac{2*(\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})})^{3}}{1-(\frac{\bar{X}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}-Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})})^{3}},\frac{2*(\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})})^{3}}{1-(\frac{\bar{X}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}}{m(\bar{Y}+Z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{Y}}{n}})})^{3}}\}\qquad 0<\alpha<1 \quad (25)$$

4. A new algorithm to estimate fuzzy factors

Let θ_α^i for $\alpha \in (0,1)$ and i=1, 2, 3, 4 be as in section 3. In the next part Buckley's approach is extended to create a new algorithm to find fuzzy estimates for ρ , π_n , W_q of M/M/1 and also ρ of M/M/m=5 (of course, it should be noticed that, all other factors that were mentioned can be estimated and plotted like these four factors, consequently there is no need to demonstrate the method for all of them).

FEQF (Fuzzy Estimates of Queuing theory's Factors) Algorithm:

FEQF is a three steps algorithm. The first step by taking samples creates point estimates that are used in different confidence intervals of step two. The second step estimates confidence intervals that are used as $\alpha - cuts$ of step three. Finally, step three using confidence intervals of different parameters as their $\alpha - cuts$ create their L-R fuzzy numbers.

(1)Taking sample and estimating λ and μ by $\overline{X}$ and $\overline{Y}$ respectively.
(2)Let $\theta_0^i = R^+$, $\theta_1^i = [\hat{\theta}^i, \hat{\theta}^i]$ for i=1,2,3 $\hat{\theta}^1 = \rho$, $\hat{\theta}^2 = \pi_n$, $\hat{\theta}^3 = W_q$ of M/M/1 and for i=4, $\hat{\theta}^4 = \rho$ of M/M/m and R^+ is the set of all positive real numbers.
(3)Place θ_α^i ; $0 \le \alpha \le 1$, one on top of the other , to produce a triangular shaped fuzzy number $\hat{\theta}^i$ for i=1,2,3,4 ;where $\hat{\theta}^1 = \rho$, $\hat{\theta}^2 = \pi_n$, $\hat{\theta}^3 = W_q$, $\hat{\theta}^4 = \rho$.

Uncertainty due to sampling variability is unavoidable because all these factors are generally unknown and must be estimates from observations. Therefore this paper introduces FEQF algorithm to guard against uncertainty in order to get close to the real value of the measures. In what follows performance of FEQF algorithm is illustrated by numerical examples.

Example1.1 M/M/1

Consider in a service system λ and μ are estimated by observations gained by samples (each consisting of 30 observations) and have following point estimates:

$\hat{\lambda} = \overline{X} = 10$, $\hat{\mu} = \overline{Y} = 14$ where n=30

So in classical calculation (crisp) ρ , π_n , W_q compute as follows:

$$\hat{\rho} = \frac{10}{14} \approx 0.71 \tag{26}$$

$$\hat{\pi}_3 \approx 0.104 \tag{27}$$

$$\hat{W}_q \approx 0.178 \tag{28}$$

now, according to what was defined in Eq. (15), Eq. (17)and Eq. (18)we can compute $\alpha - cuts$ of $\hat{\theta}^i$ for i=1,2,3 as follows:

$$\hat{\rho}_\alpha = \{\frac{10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}}, \frac{10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}}\} \tag{29}$$

$$\hat{\pi}_{n_\alpha} = \{(1 - \frac{10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}}) * (\frac{10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}})^n, (1 - \frac{10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}}) * ((\frac{10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}})^n\} \tag{30}$$

$$\hat{W}_{q\alpha} = \{\frac{10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{(14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}) * (14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}} - (10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}))}, \frac{10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{(14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}}) * (14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}} - (10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}))}\} \tag{31}$$

In the first step of FEQF algorithm, we can obtain θ_0^i and θ_1^i for $i = 1, 2, 3$ (for each i separately) . During the second step, by placing $\theta_0^i, \theta_\alpha^i$ for $\alpha \in (0,1)$, and θ_1^i for i=1,2,3(for each i separately),which are calculated by the first step and Eq. (15), Eq. (17) and Eq. (18) respectively ,one on top of the other, we can obtain fuzzy estimates for ρ, π_n ,W_q respectively. The graphs of their membership functions are shown in Fig. 1, by Minitab software. Note that in classical method, as it was shown in Eq. (1), Eq. (3) and Eq. (7), one can find estimates $\hat{\rho} \approx 0.71$, $\hat{\pi}_3 \approx 0.104$ and $\hat{W}_q \approx 0.178$ .We would never expect these precise point estimates to be exactly equal to the parameter value, so we often compute $(1-\alpha)100\%$ confidence intervals for our parameters. The L-R fuzzy estimate obtained by the FEQF algorithm contains more information than a point or interval estimate, in the sense that the fuzzy estimate contains point estimates and $(1-\alpha)100\%$ confidence intervals for all at once for $\beta \in [0,1)$, which is very useful for a practitioner.

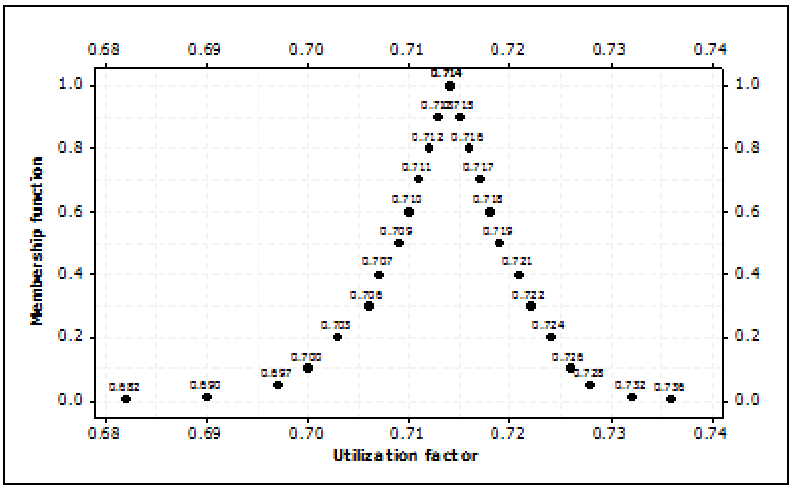

Figure 1. Fuzzy utilization factor (ρ)

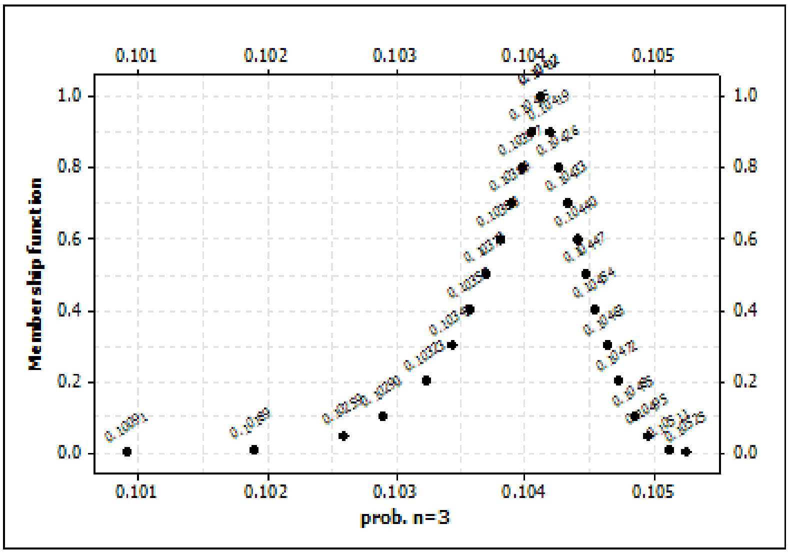

Figure 2. Fuzzy estimated prob. for n=3 (or π_3)

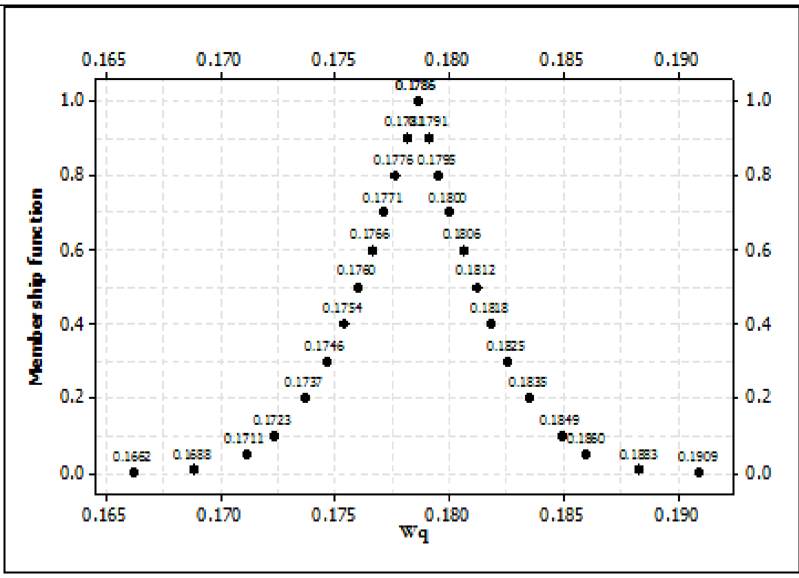

Figure 3. Fuzzy estimated Wq

Example 1.2 Now suppose with same assumption of Example 1.1 we have 5 servers. Therefore, we have the following utilization factor:

$$\hat{\rho}_\alpha = \{\frac{10 - Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{5*(14 - Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}})}, \frac{10 + Z_{\frac{\alpha}{2}}\sqrt{\frac{10}{30}}}{5*(14 + Z_{\frac{\alpha}{2}}\sqrt{\frac{14}{30}})}\}\ 0 < \alpha < 1 \qquad (32)$$

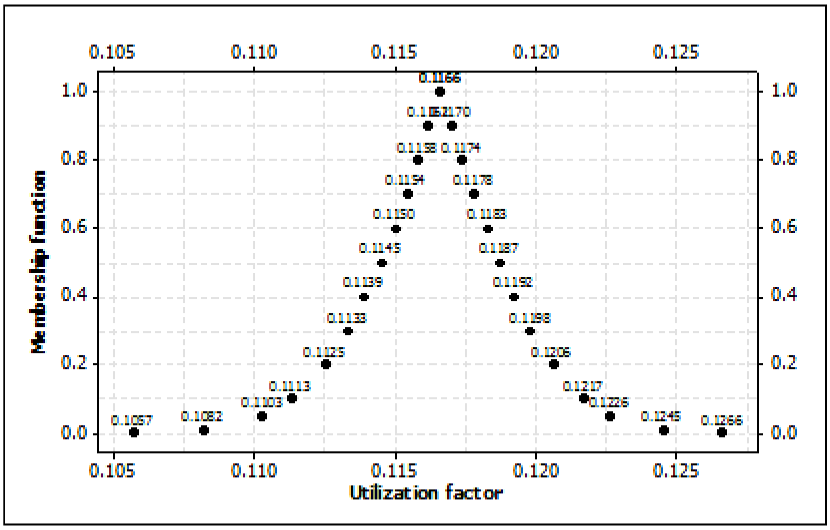

Figure 4. Fuzzy estimated utilization factor of M/M/5

From the fuzzy estimate, one can conclude that the classical estimate $\hat{\rho} \approx 0.714$ belongs to the fuzzy estimate ρ with grade of membership equal to 1. It is obvious that fuzzy set of ρ contains more elements other than "0.714" with corresponding grades of membership. For example, one can says that $\hat{\rho} \approx 0.71$ belongs to the ρ with grade of membership $\hat{\rho}(0.71) = 0.6$ (Fig. 1). Similarly, we can see that $\hat{W}_q \approx 0.1791$ belongs to the W_q with grades of membership $\hat{W}_q(0.1791) = 0.9$ (Fig. 3).

5-New method for testing system's steadiness

In queue model it is important to determine whether our model is steady or not. Mostly, this determination is done by comparing fuzzy utilization ρ factor with number one. Then, if ρ is smaller than one our system is steady; otherwise it is unsteady (Gross, D. & C.M. Harris. (1984)).Since in this paper the utilization factor is a fuzzy number we would better to compare it with a fuzzy one. Therefore, we have the following fuzzy number to create a fuzzy one. We now have a fuzzy number for utilization factor (Fig.1 and Eq. (15)) and a fuzzy number for approximating one (Fig.6 and Eq. (33)). Our final decision will depends on the relationship between utilization factor and fuzzy one. This can be best explained through studying Fig. 6. Figure 5 illustrates our final decision rule: steady or not steady. In fact, the fuzzy number for utilization factor should be triangular shaped fuzzy number, like in Fig. 1, instead of triangular fuzzy numbers, but in order to simplify the calculations, this paper uses estimated fuzzy utilization factor like the one you see in Fig. 4 with membership function as Eq. (33) and create a fuzzy comparing method. In linear estimation of convex shape (Fig. 4 and Eq. (33)), the number (utilization factor which is) in the most left (0.69) with the middle number (0.714) are used to create left function, and the number (utilization factor which is) in the most right (0.736) with the middle number (0.714) are used to create right function. Similar linear estimating can be used in utilization of this method.

$$\begin{cases} (x-0.690) / (0.714-0.690), & 0.690<x<0.714 \\ 1, & x=0.714 \\ (0.736-x) / (0.736-0.714), & 0.714<x<0.736 \end{cases} \quad (33)$$

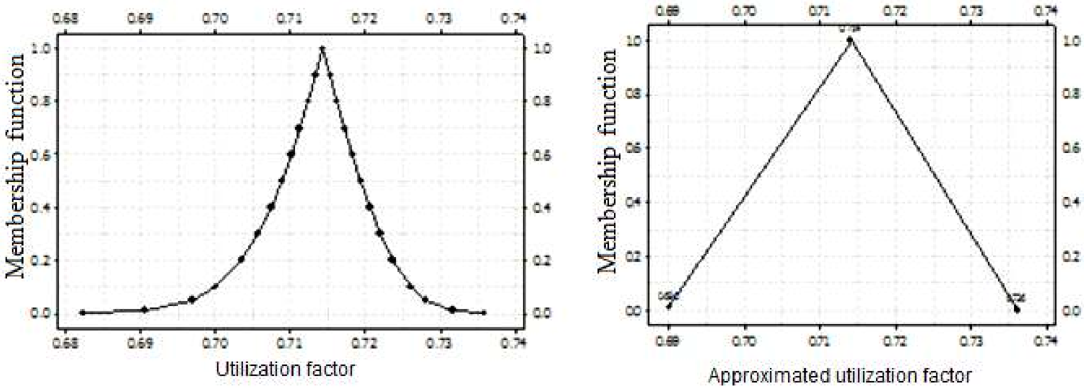

Figure 5. Fuzzy utilization factor vs. its approximated form

As you can see in Fig.5 the estimation is pessimistic and cause more fuzziness. Therefore the new method which this section is introduced is strict. The next paragraph explains the new strict method to compare fuzzy utilization factor with fuzzy one.

Fig.6 shows that the vertex of fuzzy utilization factor is at $x = d$ and the vertex of fuzzy one is at $x = c$. The total area under the graph of fuzzy utilization factor is represented by A.U. and the area under the graph of fuzzy utilization factor, but to the right of the vertical line through $x = c$ is represented by A.R. We choose a value of $\varphi \in (0,1)$ and our decision rule is: if $\frac{A.R.}{A.D.} \geq \phi$, then the process is not steady, otherwise it is steady. Lets in this paper choose $\phi = 0.4$. Surely, for $\phi \geq 0.5$ the process is not steady. Notice that in Fig. 6 we get $\frac{A.R.}{A.D.} \geq 0.5$ when $x = d$ lies to the right of $x = c$.

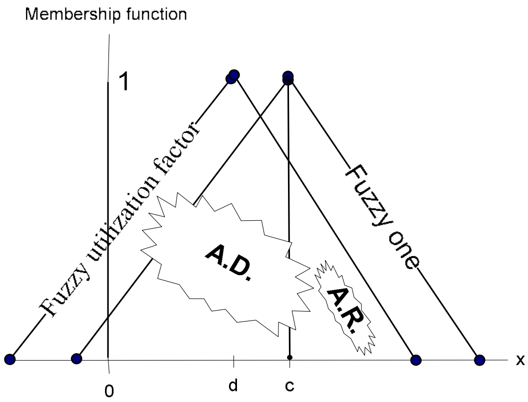

Figure 6. Fuzzy estimated utilization factor in comparison with fuzzy one in decision making

Example2. Let all assumptions be as the same as Example1 but the model is M/M/1. Consider approximate one is a fuzzy number with shape and membership function like (Fig.6) and (Equation 34) respectively.

$$
\begin{cases}
(x-0.8) / (1-0.8), & 0.8<x<1 \\
1, & x=1 \\
(1.2-x) / (1.2-1), & 1<x<1.2
\end{cases}
\qquad (34)
$$

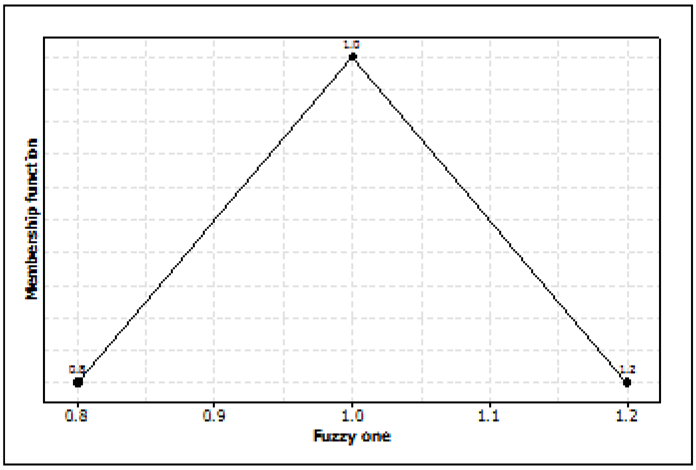

Figure 7. Membership function of the approximate one

So Figure 6 in Example 2 shows as Fig.8.

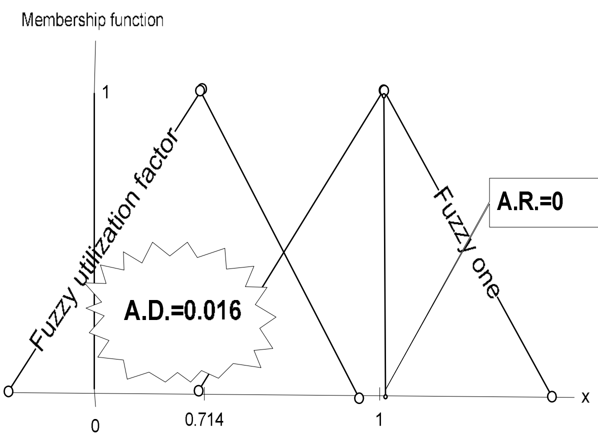

Figure 8. Illustrates Fig.6.in Example 1

Now after doing some simple calculations decision rule leads to acceptance of the process as follows:

$$\frac{A.R.}{A.D.} = \frac{0}{0.016} = 0 \leq 0.4 \tag{35}$$

Since the result of the example show the process is not steady and also since proposed estimation is pessimistic, therefore the crisp utilization factor must be smaller than crisp one. So in exact decision making we should accept the steadiness of queuing system.

6. Conclusion

Since different factors of queuing theory are estimated using sample data, it is of interest to obtain confidence intervals rather than simple point estimates to estimate them. To find fuzzy estimates for these factors (especially utilization factor), a new algorithm based on Buckley's approach named FEQM is introduced. The final results of the proposed algorithm not only contain point estimates but also interval estimates and hence provide more information for us. A new method to test system's steadiness (thorough comparing fuzzy estimated utilization factor with fuzzy one) is also introduced and numerical examples are given to illustrate the performance of the new algorithm and new fuzzy system's steadiness test.

Future research may use these methods to create different fuzzy queue models or a more efficient fuzzy logic control.

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