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ECONOMIC DESIGN OF X-BAR CONTROL CHART WITH VARIABLE SAMPLE SIZE AND SAMPLING INTERVAL UNDER NON-NORMALITY ASSUMPTION: A GENETIC ALGORITHM

Abstract. *While the main assumption of an economic model of a variable sample size and sampling interval (VSSI) X-Bar control chart is normality, some process data may not follow a normal distribution. In this paper, a model for an economic design of the VSSI X-Bar chart under non-normality of the process data is first developed. Then a parameter-tuned genetic algorithm is proposed to solve the model and compare its performances in terms of the expected loss per hour to the ones of a fixed sample size and sampling interval X-Bar control charts that works under the normality assumption. Finally, a numerical example is given to illustrate the applications of the proposed methodology and to perform a sensitivity analysis on the model input parameters (i.e. the cost and the process parameter).*

Keywords: *Economic design, Non-normality, Variable sample size and sampling interval, Markov chain; Genetic algorithm.*

JEL Classification: C06

1. Introduction and Literature Review

Statistical quality control is an efficient tool to improve product quality while saving production cost. Since Shewhart presented the first control chart to monitor the

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process mean, several control chart techniques have been developed and widely employed as a primary tool in statistical process control environments. The main application of a control chart is to detect assignable causes in order to take necessary corrective actions ahead of manufacturing a large number of nonconforming products.

In the designing process of a control chart, three parameters are involved; sample size, sampling interval, and control limit coefficient. Furthermore, economic and/or statistical designs are the two common practices in this regard. In a statistical design, the design parameters are determined based on the statistical performances of the chart, which are measured either in terms of type-I and II errors or in terms of average run lengths (ARL) and average time to signal (ATS). Besides, in an economic design, the design parameters are selected based on minimizing a cost model or a loss function.

The $\overline{X}$ control charts with variable sample size (VSS), variable sampling interval (VSI), variable sample size and sampling interval (VSSI), and variable parameters (VP) have all been shown faster than the basic form of the Shewhart $\overline{X}$ chart in detecting process changes. The interested reader is referred to Cui and Reynolds [15], Park and Reynolds [23], Reynolds et al. [26], Runger and Pignatiello [29], Prabhu et al. [24], Reynolds [27,28], Costa [9,10,11,12,13], Zimmer et al. [33,34], and Bai and Lee [2].

Montgomery [22] and Woodall [31] proposed economic designs of $\overline{X}$ control chart; both of which are based on the Duncan's cost model [18]. In the first design, the expected cost is minimized, and in the second approach, it is assumed the process continues while a search for an assignable cause is carried on. Chiu [6] proposed another popular model in which the process is stopped during the search for assignable cause. Costa & Rahim [14] considered the economic design of VP $\overline{X}$ chart using a Markov chain approach. Besides, Chen [5] proposed the economic design of VSI T^2 chart using a hybrid Markov chain and genetic algorithm approach.

All the mentioned researches are based on the assumption that the process data come from a normal distribution. However, this may not be true in many manufacturing processes. If the measurements (process data) follow a non-normal distribution, the statistic used in the $\overline{X}$ chart follows a normal distribution, only when the sample size is large enough (based on the central limit theorem). Nonetheless, there are many situations in practice where the chart is employed when sample sizes are small, restricting the use of the chart.

Burr [3] introduced the Burr distribution that has been used to model various non-normal distributions since then. Zimmer & Burr [35] offered the statistical design of VSI $\overline{X}$ chart under the non-normality assumption. Yourstone and Zimmer [32] used the Burr distribution to represent different non-normal distributions. Recently Lin and Chou [20,21] have developed a statistical design of the VSSI $\overline{X}$ chart and VP $\overline{X}$ chart based on non-normality assumption, respectively. Moreover, for an economic design of the VSI $\overline{X}$ chart, Bai and Lee [1] used Chiu's [6] model. Further, Chen [4] applied the developed model of Bai and Lee [1] for economic design of VSI $\overline{X}$ chart under the non-normality assumption of the process data using the Burr distribution.

This research is an extension of the researches performed by Costa & Rahim [14] and Chen [4,5], in which an economic design of VSSI $\overline{X}$ chart is proposed assuming non-normal data. We employ the Burr distribution to estimate the distribution of the sample means. A cost model is derived using the Markov chain method and a parameter-tuned (using Taguchi method) genetic algorithm is used to determine the optimal design parameters.

The organization of the rest of the paper is as follows. Sections 2 and 3 contain brief backgrounds on the VSSI $\overline{X}$ control chart and a specific application of the Burr distribution to model non-normality, respectively. The assumptions with the development of the economic model are given in section 4. A solution method based on genetic algorithm is proposed in section 5. A numerical example and a sensitivity analysis of the solution procedure for VSSI $\overline{X}$ chart are presented in section 6, and finally the comparison results are discussed in section 7.

2. A brief background on the VSSI $\overline{X}$ control chart

The variable sample size and sampling interval $\overline{X}$ control charts with symmetric and asymmetric control limits are described in the following two subsections.

2.1 The VSSI $\overline{X}$ charts with symmetric limits

The basic idea of the variable sample size and sampling interval Shewhart control charts in monitoring the process mean was presented by Prabhu et al. [24,25], Costa [10], and Das et al. [16]. In applying the procedure to a normal process with an in-control mean and standard deviation of μ_0 and σ, the VSSI $\overline{X}$ chart chooses two sample size variables $(n_1 \text{ and } n_2)$ and two fixed sampling intervals $(h_1 \text{ and } h_2)$. In this method, $n_1 < n_0 < n_2$, $h_1 > h_0 > h_2$, n_0 is the average sample size; usually set 4 or 5, and h_0 is the fixed sampling interval; regularly set 1. The VSSI $\overline{X}$ chart applies the action and the warning limits to divide the chart into three regions; central, warning, and action.

In the design process of the VSSI $\overline{X}$ chart one first needs to determine the action and the warning limit coefficients k and w , respectively. Then, the lower and the upper action and warning limits are set using equations (1) and (2), correspondingly.

$$UCL_i = \mu_0 + k\frac{\sigma}{\sqrt{n_i}} \quad \text{and} \quad LCL_i = \mu_0 - k\frac{\sigma}{\sqrt{n_i}} \quad ; \quad i = 1,2 \tag{1}$$

$$UWL_i = \mu_0 + w\frac{\sigma}{\sqrt{n_i}} \quad \text{and} \quad LWL_i = \mu_0 - w\frac{\sigma}{\sqrt{n_i}} \quad ; \quad i = 1,2 \tag{2}$$

Where UCL_i and LCL_i are defined as the upper and lower action limits and UWL_i and LWL_i are the upper and the lower warning limits, respectively.

In the application of the VSSI $\overline{X}$ control charts with symmetric limits, if the sample point is located in the central region, the pair $(n_1$ and $h_1)$ is used for the next sampling. If the sample point is plotted in the warning region, the $(n_2$ and $h_2)$ pair will be employed. The chart signals when a sample point falls in the action region.

2.2 The VSSI $\overline{X}$ charts with asymmetric limits

Yourstone and Zimmer [32] offered a procedure for designing VSSI $\overline{X}$ control charts with asymmetric limits, where the distribution of the sample means is not normal. The asymmetric action and warning limits are given in equations (3) and (4), respectively.

$$UCL_i = \mu_0 + k_{i1}\frac{\sigma}{\sqrt{n_i}} \quad \text{and} \quad LCL_i = \mu_0 - k_{i2}\frac{\sigma}{\sqrt{n_i}} \quad ; \quad i = 1,2 \tag{3}$$

$$UWL_i = \mu_0 + w_{i1}\frac{\sigma}{\sqrt{n_i}} \quad \text{and} \quad LWL_i = \mu_0 - w_{i2}\frac{\sigma}{\sqrt{n_i}} \quad ; \quad i = 1,2 \tag{4}$$

Where $0 < w_{i1} < k_{i1}$ and $0 < w_{i2} < k_{i2}$. Hence, the symmetric VSSI $\overline{X}$ chart is a special case of the asymmetric in which $k_{i2} = k_{i1} = k$, and $w_{i2} = w_{i1} = w \quad , \; i = 1,2$.

3. The Burr distribution

Burr [3] presented a simple distribution function that was capable of modeling various types of continuous distributions. The probability density function of the two-parameter Burr distribution is as follows.

$$f(y) = \frac{cky^{c-1}}{\left(1+y^c\right)^k} \quad ; \quad y > 0 \tag{5}$$

With a cumulative distribution function of

$$F(y)=\begin{cases}1-\dfrac{1}{\left(1+y^{c}\right)^{k}} & y\geq 0\\ 0 & y<0\end{cases} \tag{6}$$

or simply $F(y)=1-\dfrac{1}{\left(1+Max\,\{0,y\}^{c}\right)^{k}}$, where c and k are greater than one.

Considering different combinations of c and k , this distribution can cover an extensive range of skewness and kurtosis coefficients of variform probability density functions (e.g., Normal, Gamma, Beta, etc.). For instance for c=4.8621 and k=6.3412 the Burr distribution approximates the normal distribution [32]. Further, the first four moments of the empirical distribution are used to determine c and k .

Burr [3] tabulated the first two moments and the coefficients of skewness and kurtosis for the family of the Burr distribution. These tables allow users to establish a standardized transformation between the variable of the Burr distribution (Y) and any random variable (X) when they have similar coefficients of skewness and kurtosis. The standardized transformation between Y and X is defined as follows.

$$\frac{X-\overline{X}}{S_X}=\frac{Y-M}{S} \tag{7}$$

where S and M are the sample mean and standard deviation of the Burr distribution, and $\overline{X}$ and S_X are the sample mean and standard deviation of the data set (random variable). Furthermore, according to Dodge and Rousson [17], the coefficients of skewness $(\alpha_3(\overline{X}))$ and kurtosis $(\alpha_4(\overline{X}))$ for $\overline{X}$ are:

$$\alpha_3\left(\overline{X}\right)=\frac{\alpha_3}{\sqrt{n}} \quad \text{and} \quad \alpha_4\left(\overline{X}\right)=\frac{\alpha_4-3}{n}+3 \tag{8}$$

Where α_3 and α_4 are the coefficients of skewness and kurtosis estimated by population. Using the value of $\alpha_3\left(\overline{X}\right)$ and $\alpha_4\left(\overline{X}\right)$ and the tables available in Burr [3], it is possible to calculate k , c , S , and M for the distribution with the value near to $\alpha_3\left(\overline{X}\right)$ and $\alpha_4\left(\overline{X}\right)$ by interpolation.

4. Developing the cost model

The optimal design parameters of the proposed VSSI $\overline{X}$ control chart under non-normality of the process data are determined by minimizing a cost model or a loss function. However, before developing the cost model, the assumptions are first defined.

4.1. Model assumptions

The assumptions involved in the development process of the proposed VSSI $\overline{X}$ chart are:

1. The sampling interval h and the sample size n range between their maximum and minimum values. Moreover, $0 < n_1 \le n_2$ and $h_1 \ge h_2 > 0.01$.
2. The process starts in an in-control state with an on-target mean of $\mu = \mu_0$ and goes to an out-of-control state to an off-target mean of $\mu_1 = \mu_0 + \delta\sigma,\ \delta > 0$, at some random time.
3. Since the exponential distribution is often used for modeling certain intervals of time, it is assumed the time until the first assignable cause follows an exponential distribution with parameter $\lambda > 0$. This assumption is essential to develop the Markov chain cost model in section 4.2..
4. After the shift, the process mean will remain off-target until the assignable cause is eliminated.
5. During the search for an assignable cause, the process remains shut-down.

4.2. Developing a Markov chain cost model

One of the statistical properties of a control chart is measured in terms of the speed of detecting mean shifts. When the interval between samples is constant, this speed can be measured by ARL. However, when the interval is variable, it has to be measured by an adjusted average time to signal $(AATS)$ that is defined as the mean of the time required from the actual process mean shift to the time the chart signals. Defining the average time cycle (ATC) to be the mean of the time from the beginning of the production to the first signal after the process shift, we have:

$$AATS = ATC - 1/\lambda \tag{9}$$

Because of the memoryless property of the exponential distribution, one may estimate ATC by a Markov chain model as follows.

To develop the Markov chain model, consider the pictorial representation of a typical production cycle that is depicted in Figure (1) [4]. Moreover, at each sampling point, one of the four following transition states can happen:

1. The process is in-control and the sample is small.
2. The process is in-control and the sample is large.
3. The process is out-of-control and the sample is small.
4. The process is out-of-control and the sample is large.

Table (1) shows different positions of the i^{th} sample point, the state of the process in $i+1^{st}$ sampling point, and the corresponding states of the Markov chain model.

When the sample point is plotted in the action region, the control chart signals. In this case, if the new state is either 1 or 2, the signal will be a false alarm. Otherwise, if the new state is either 3 or 4, the signal will be a correct alarm. The absorbing state, the fifth state, will be obtained when the correct alarm is signaled.

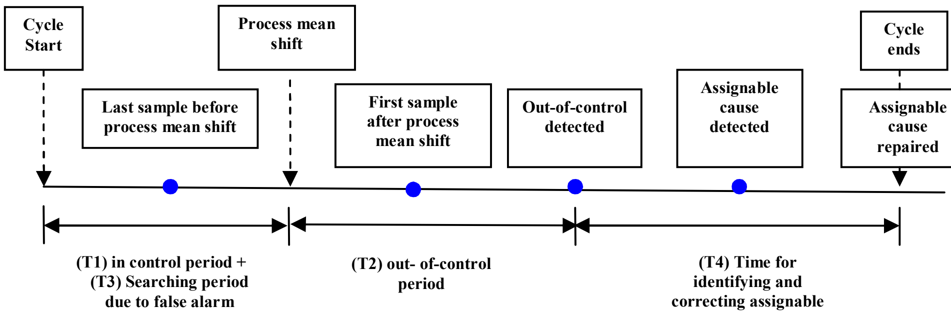

Figure (1): A typical production cycle

Table (1): The states of the Markov chain

i^{th} sample	$(i+1)$ st sample	
Sample point position	Process status	State of the Markov chain
Warning	On-target	2
Warning	Off-target	4
Central	On-target	1
Central	Off-target	3

✶On-target (in-control) mean $\mu = \mu_0$

✶Off-target (out-of-control) mean $\mu = \mu_0 + \delta\sigma$

The transition probability matrix of the Markov chain model is defined as:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & 0 \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Where P_{ij} denotes the transition probability that i is the prior state and j is the new state. Then, based on the initial properties of Markov chains we have [8]:

$$ATC = b'(I - Q)^{-1} t \quad (10)$$

Where $b' = (p_{11}, p_{12}, p_{13}, p_{14})$ is the vector of initial probabilities, I is the identity matrix of order 4, Q is the transition matrix with eliminated elements corresponding to the absorbing state, and $t' = (h_1, h_2, h_1, h_2)$ is the vector of the sampling intervals. The derivations of the transition probabilities for an asymmetrical limits VSSI $\overline{X}$ chart are given in the appendix.

4.3. The loss function

A process cycle involves in-control, out-of-control, detecting the assignable cause, and repair periods [14]. As a result, the average length of a production cycle is obtained as follows.

$$E(T) = ATC + t_0 E(FA) + t_1 \tag{11}$$

Where t_0 is the average amount of time to search for the assignable cause when the process is in-control and t_1 is the average time of detecting and correcting the assignable cause. Moreover, $E(FA)$ is the average number of false alarms at each production cycle that is obtained by:

$$E(FA) = b'(I - Q)^{-1}\alpha \tag{12}$$

Where $\alpha' = (\alpha_1, \alpha_2, 0, 0)$; $\alpha_i = \Pr[Z > K_{i1}] + \Pr[Z < K_{i2}] \quad \forall \quad i = 1,2$, in which Z follows a standard normal distribution.

For the average net profit of the production cycle, we have:

$$E(C) = V_0\left(\frac{1}{\lambda}\right) + V_1\left(ATC - \frac{1}{\lambda}\right) - C_0 E(FA) - C_1 - sE(M) \tag{13}$$

Where:

V_0 is the profit earned per hour when the process is in-control

V_1 is the profit earned per hour when the process is out-of-control

C_0 is the average cost based on false warnings

C_1 is the average cost of detecting and removing the assignable cause

s is the inspection cost of an item

$E(M)$ is the average number of inspected items per cycle and is obtained by:

$$E(M) = b'(I - Q)^{-1} m, \quad \text{where} \quad m' = (n_1, n_2, n_1, n_2) \tag{14}$$

Hence, using equations (11) and (13), the loss function $E(L)$ is derived as follows.

$$E(L) = V_0 - \frac{E(C)}{E(T)} \tag{15}$$

The loss function given in equation (15) is a function of the process parameters $\left(t_0,t_1,\lambda,\delta\right)$, the cost parameters $\left(s,C_0,C_1,V_0,V_1\right)$, and the design parameters $\left(n_1,h_1,w_{11},w_{12},k_{11},k_{12}\right)$, $\left(n_2,h_2,w_{21},w_{22},k_{21},k_{22}\right)$. The economic design of the VSSI $\overline{X}$ control chart is obtained by minimizing $E(L)$ over the design parameters given specific values of the other parameters. In other words, one needs to solve the following mixed-integer non-linear minimization model:

$$
\begin{aligned}
&Min\ E\left(L\right)=V_0-\frac{E(C)}{E(T)}\\
&s.t.:\\
&n_1,n_2\in Z^{+}\\
&0<n_1\le n_2\\
&h_1\ge h_2\ge 0.01\\
&0\le w_{i1}\le k_{i1}\ and\ \ 0\le w_{i2}\le k_{i2}\qquad \forall\ i=1,2
\end{aligned}
\tag{16}
$$

In the next section, a solution procedure along with an example is given to demonstrate the application of the proposed methodology.

5. An example and a solution procedure

The model in (16) is a mixed-integer non-linear optimization model and an exact solution procedure is an inefficient and time-consuming method to solve it. Therefore, a heuristic search algorithm is required to solve the model. Historically, among the search algorithms, the genetic algorithm (GA) has been successful in solving similar models [7].

To illustrate the solution procedure, a numerical example is given in section 5.1 for which a GA is proposed in section 5.2 to search for the optimal solution of the economic design of the VSSI $\overline{X}$ control chart. Besides the chromosomes that contain different values of the chart design parameters, i.e., $\left(n_1,h_1,w_{11},w_{12},k_{11},k_{12}\right)$ and $\left(n_2,h_2,w_{21},w_{22},k_{21},k_{22}\right)$, four GA parameters have to be first specified. They are the crossover rate (CR), the mutation rate (MR), number of generation (GN), and the population size (PS) that usually the quality of the solution acquired from GAs depends on these four parameters [5]. Orthogonal-array experiments described in section 5.3 are used to determine the tuned values of these parameters.

5.1. The numerical example

To explain the proposed GA solution procedure, let $s=5$, $C_0=500$, $C_1=500$, $V_0=500$, $V_1=50$, $t_0=5$, $t_1=1$, and $\lambda=0.01$. Further, suppose a data set is collected from the process with the sample statistics as sample mean $\overline{X} = 50.42$, sample standard deviation $S_x = 5.68$, sample skewness coefficient $\alpha_3 = 1.4322$, and sample kurtosis coefficient $\alpha_4 = 4.3558$. We use α_3 and α_4 to fit the Burr distribution with $c = 2$ and $k = 4$. From the Table in Burr [3], the mean and the standard deviation of the Burr random variable Y with $c = 2$ and $k = 4$ are $M = 0.4909$ and $S = 0.3039$, respectively.

5.2. The proposed GA

Genetic algorithms are global search optimization techniques that have been inspired by the processes of natural selections in biological system [19]. The basic characteristics of the genetic algorithms that cause them to be different from other search algorithms are:

1. They consider many points, rather than a single point, in the search space concurrently. This will reduce the chance of trapping in the local optimum point.
2. They work directly with strings of characters representing the solution set, not the solutions themselves.
3. They apply probabilistic rules instead of deterministic rules to guide their search.

The initial setting of a GA involves encoding, fitness function, selection mechanism, crossover operation, mutation, and culling. Accordingly, the steps involved in the proposed GA that are partially taken from Chen [4] are:

1. Randomly generate an initial solution set (population) of PS individuals and assess each solution (individual) by a fitness function. An individual is shown as a numerical string.
2. If the termination condition is not satisfied, repeatedly do the following:
    a. Select parents from the population for crossover.
    b. Generate offspring.
    c. Mutate some of the members.
    d. Merge mutants and offspring into population.
    e. Cull some members of the population.
3. Stop and return the best fitting solution.

In what follows, the details of the proposed GA are briefly described.

5.2.1 Initial population

A chromosome of the proposed GA involves 12 genes, each gene a representative of a decision variable of the model. Figure (2) shows a typical chromosome.

Generally, evolutionary algorithms generate the initial population at random. Accordingly, in the initial population of the proposed GA, the number of feasible chromosomes that are generated using Uniform pseudo random numbers is equal to PS .

n_1	n_2	h_1	h_2	w_{11}	k_{11}	w_{21}	k_{21}	w_{12}	k_{12}	w_{22}	k_{22}

Figure (2): A typical chromosome

5.2.2 Chromosome evaluation

In the one hand, the loss function given in equation (15) is used to evaluate the performance of a chromosome in terms of its loss value. In a minimization problem, the fittest chromosomes are the ones with the minimum loss values The loss is a function of the process parameters, $(t_0, t_1, \lambda, \delta)$, the cost parameters, (s, C_0, C_1, V_0, V_1), and the design parameters, $(n_1, h_1, w_{11}, w_{12}, k_{11}, k_{12})$ and $(n_2, h_2, w_{21}, w_{22}, k_{21}, k_{22})$. On the other hand, the fitness function transforms the chromosomes' loss values to their fitness values using a transformation function. Some of the transformation functions that are usually employed in the literature are rank, top, and proportional functions. In the third transformation function that is employed in this research and is the most applied method, the fitness of a chromosome is proportional to its loss divided by the total losses of the chromosomes in the population. In other words, denoting $F(x_i)$ the fitness value of chromosome i , $f(x_i)$ its loss value, N the number of chromosomes in a population, and $\sum_{i=1}^{N} f(x_i)$ the total loss of all chromosomes, the fitness function is given by $F(x_i) \propto f(x_i) \Big/ \sum_{i=1}^{N} f(x_i)$. This relation guarantees that every chromosome of a population has a probability (that is proportional to its fitness) of being selected as a parent in the new generated population. However, sometimes when the differences between chromosomes are considerable, this function leads to fast convergence. We will restrict the number of children of parents to one to avoid fast convergence.

5.2.3 Chromosome selection

After evaluation, the chromosomes are sorted based on their fitness value, from lowest to highest. The first $copy - size$ highest ranked chromosomes are then directly copied to the next generation (elitism).

The selection operator determines the parents to set up the next generation. The chromosomes with the highest fitness have high a higher chance of being selected. The tournament selection strategy is adopted in this research. For a parent in this strategy, n players are first selected randomly and then the best individual out of the

n players is selected to be the second parent. The tournament size (n or sub-group) must be at least two. This size influences the selective pressure, i.e. more individuals in the sub-groups increase the chance of selecting better individuals.

5.2.4 Crossover

Several crossover operators such as partial-mapped crossover (PMX), order crossover (OX), position based crossover (PBX), and order-based crossover (OBX) have been proposed in the literature. These operators can be viewed as an extension of two-point or multipoint crossovers of binary strings representation. In the proposed GA, we employed the tow-point crossover operator as well. In other words, based on the roulette- wheel selection procedure, a number of $2(CR)(PS)$ parents for construction of the next generation are first selected. In this regard, the fittest chromosomes have a better chance of being selected. Then, for each parent two cross-points are chosen randomly and the section between them is exchanged between the two parents. Figure (3) depicts an example of this operation.

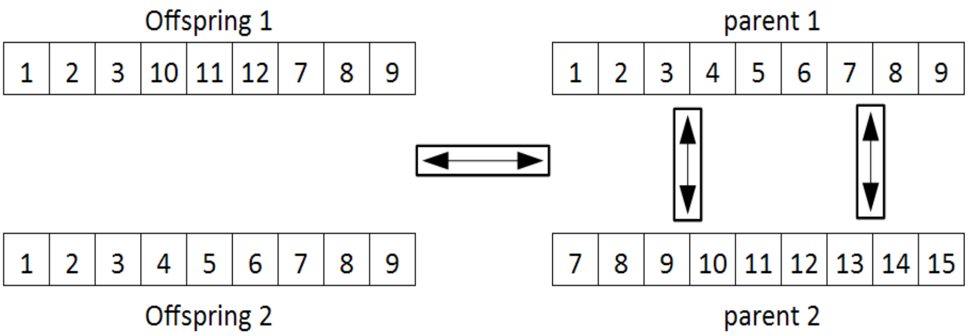

Figure (3): An example of the crossover operation

5.2.5 Mutation operation

After the crossover operation, a number equal to $(MR)(PS)$ parents are mutated using the most popular (a uniform mutation) operator. This operator acts in two stages. In the first stage, the operator determines the position of the mutation and in the second stage it replaces the value of the determined position with another value at random.

5.2.6 Stopping

When the GA algorithm repeats GN times, the chromosome with the lowest fitness value is reported as the solution of the problem.

It should be mentioned that, the proposed genetic algorithm of this research is programmed in Matlab 7.0.4 environment and is executed on a Pentium 4 2.8 MHz personal computer. Moreover, the basic structure of the proposed genetic algorithm is presented in Figure (4).

5.3. Determining the values of the GA parameters

The quality of the final solution gained by a GA algorithm depends on four parameters: the population size (PS), the crossover rate (CR), the mutation rate (MR), and the number of generation (GN). Orthogonal array experiments are performed in this section to tune these parameters [7]. In these experiments, three levels of each parameter are defined as shown in Table (2).

Table (2): The GA parameter levels

Parameters	Level 1	Level 2	Level 3
PS	50	75	100
CR	0.5	0.6	0.7
MR	0.1	0.15	0.25
GN	20,000	40,000	60,000

The L9 orthogonal array was used to assign the four control parameters [30]. In the experiment of the L9 orthogonal array, there are entirely nine arrays (or 9 different level combinations of the four parameters). For each array, three cost values $(E(L))$ denoted by y_1, y_2, and y_3, are obtained by different runs of the proposed GA. The responses are given in Table (3). Since the "smaller-the-better" of the expected loss is desired, the suitable signal-to-noise ratio (SN) for assessment of the experimental results is given by [30]:

$$SN = -10.\log\left(\frac{1}{no}\sum_{i=1}^{no} y_i^2\right) \tag{17}$$

Where no is the total number of $E(L)$ evaluations in each array. Moreover, the signal-to-noise ratios of the three levels of the control parameters are given in Table (4). The results in Table (4) show the best combination of the parameter levels are $PS = 100$, $CR = 0.5$, $MR = 0.1$, and $GN = 60,000$.

In the next section, the performance of the proposed methodology in designing a VSSI $\overline{X}$ control chart is compared to the one of a fixed sample size and sampling interval (FSSI) $\overline{X}$ scheme in terms of the expected loss function.

Figure (4): Structure of the algorithmic process

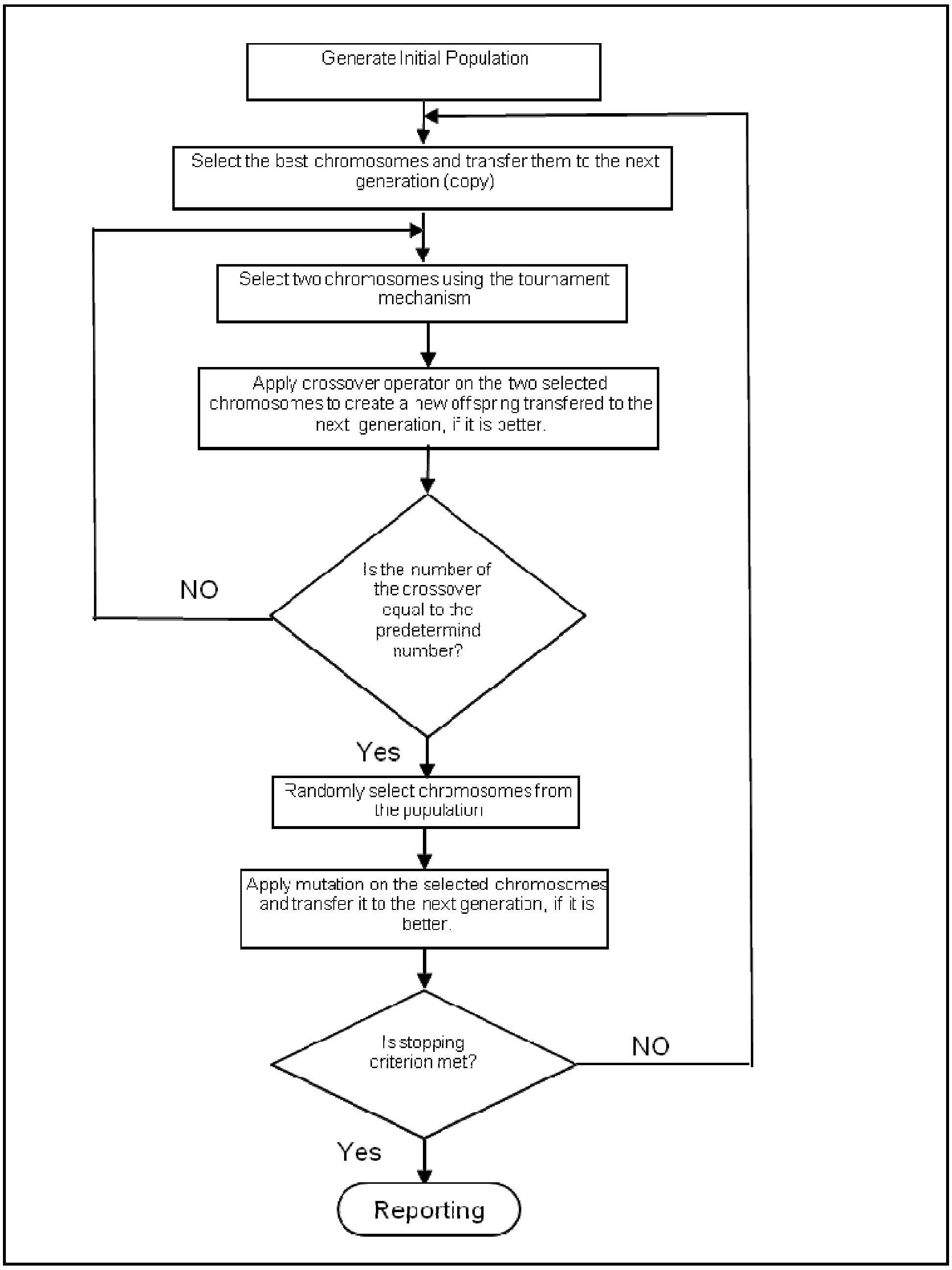

Table (3): Experimental layout of the L9 orthogonal array and the results

Array	*PS*	*CR*	*MR*	*GN*	y_1	y_2	y_3	*SN*
1	1	1	1	1	31.7875	32.7846	32.0538	-30.1602
2	1	2	2	2	33.2731	32.7040	32.0091	-30.2820
3	1	3	3	3	32.6601	32.5078	32.7626	-30.2760
4	2	1	2	3	31.4807	30.9842	31.8487	-29.9496
5	2	2	3	1	35.9639	36.1853	34.5044	-31.0190
6	2	3	1	2	32.3651	32.7712	32.6166	-30.2603
7	3	1	3	2	32.1808	33.4928	32.8641	-30.3308
8	3	2	1	3	30.9674	31.8311	31.5630	-29.9540
9	3	3	2	1	32.9486	33.1535	33.5145	-30.4244

Table (4): S/N of the control parameters

Levels	*PS*	*CR*	*MR*	*GN*
Level 1	**-90.718131**	**-90.440582**	**-90.374517**	**-91.603611**
Level 2	**-91.228892**	**-91.255001**	**-90.656000**	**-90.873023**
Level 3	**-90.709248**	**-90.960688**	**-91.625755**	**-90.179637**

6. A comparison study

The expected loss function of the FSSI $\overline{X}$ scheme can be easily obtained by setting $h_1 = h_2$, $n_1 = n_2$, and $w = 0$; implying the transition probabilities of $p_{11} = p_{21} = 0$ and $p_{13} = p_{23} = p_{33} = p_{43} = 0$. Thirteen different sets of cost and process parameters are taken from Chen [5] and are given in Table (5). As it was mentioned in 5.1 section, for a non-normal process the parameters of the Burr distribution are chosen $c = 2$ and $k = 4$. In order to calculate P_{ij} and ultimately computing $AATS$, $E(FA)$, and $E(L)$, we first find the skewness and kurtosis coefficient ($\alpha_{3\hat{X}}, \alpha_{4\hat{X}}$) corresponding to $c = 2$ and $k = 4$ from the Burr [3] Table II. Then, we compute $\alpha_{3\bar{X}}$ and $\alpha_{4\bar{X}}$ using equation (8). Next, we obtain c, k, M, and S corresponding to $\alpha_{3\bar{X}}$, $\alpha_{4\bar{X}}$ using the Burr Table II and III and interpolation. Using these values in each of the probability states of the Markov chain, we calculate P_{ij} s. Finally, inserting P_{ij} s in equations (9), (12), and (15), the values of $AATS$, $E(L)$, and $E(FA)$ are calculated, respectively.

Table (5): The cost and the process parameter sets

Set #	s	C_0	C_1	V_0	V_1	t_0	t_1	λ	δ
1	5	500	500	500	50	5	1	0.01	1
2	**10**	500	500	500	50	5	1	0.01	1
3	5	**250**	500	500	50	5	1	0.01	1
4	5	500	**50**	500	50	5	1	0.01	1
5	5	500	500	**250**	50	5	1	0.01	1
6	5	500	500	500	**100**	5	1	0.01	1
7	5	500	500	500	**0**	5	1	0.01	1
8	5	500	500	500	50	**2.5**	1	0.01	1
9	5	500	500	500	50	5	**10**	0.01	1
10	5	500	500	500	50	5	1	**0.05**	1
11	5	500	500	500	50	5	1	0.01	**1.5**
12	5	500	500	500	50	5	1	0.01	**0.5**
13	5	500	500	500	50	5	1	0.01	**2**

For the thirteen sets of the cost and the process parameters, the optimum designs of the FSSI $\overline{X}$ and VSSI $\overline{X}$ schemes along with their corresponding $E(L)$, $E(FA)$, and $AATS$ are given in Table (6). The percent reductions of the expected loss obtained by the VSSI scheme rather than using the FSSI scheme are given in the last column of this table as well.

A careful investigation of the results in Table (6) reveals the following:

Table (6): The optimum design of FSSI and VSSI X-Bar charts under non-normality (Burr distribution ($c=2$, $k=4$))

NO.	FSSI									VSSI															
	n	h	k_{11}	k_{21}	k_{12}	k_{22}	$E(L)$	$E(FA)$	$AATS$	n_1	n_2	h_1	h_2	w_{11}	k_{11}	w_{21}	k_{21}	w_{12}	k_{12}	w_{22}	k_{22}	$E(L)$	$E(FA)$	$AATS$	%
1	16	6.190	2.961	2.961	4.629	4.629	42.101	0.111	3.920	4	12	3.60	0.02	1.01	4.98	1.45	3.52	3.19	4.33	2.873	3.221	31.655	0.0356	2.3724	24.811
2	13	8.039	2.645	2.645	4.575	4.575	52.169	0.174	5.256	4	9	6.51	0.05	0.94	4.96	1.07	3.07	4.16	4.8	3.658	4.723	40.348	0.0433	4.2082	22.660
3	16	6.193	2.941	2.941	4.600	4.600	41.830	0.117	3.882	5	10	3.95	0.03	1.29	4.88	1.08	3.38	4.86	4.99	2.849	2.872	31.773	0.0338	2.6657	24.044
4	17	6.367	3.001	3.001	3.814	3.814	37.884	0.099	3.900	5	14	3.87	0.04	1.19	4.92	1.63	3.48	4.68	4.77	3.389	4.487	27.773	0.0275	2.4904	26.690
5	14	8.948	2.728	2.728	3.635	3.635	27.464	0.129	5.728	4	11	6.01	0.02	1.06	4.86	1.31	3.16	4.48	4.98	3.718	3.771	21.672	0.0356	4.024	21.089
6	17	6.819	2.999	2.999	3.591	3.591	40.199	0.092	4.176	4	9	5.22	0.03	0.83	4.99	0.96	3.41	3.36	3.63	4.645	4.678	30.654	0.0363	3.1952	23.743
7	17	6.056	2.999	2.999	4.210	4.210	43.921	0.104	3.705	4	9	3.51	0.03	1.1	4.99	0.96	3.43	4.43	4.89	3.841	4.855	32.670	0.0437	2.445	25.616
8	15	6.027	2.78	2.78	4.181	4.181	40.597	0.172	3.726	5	11	4.54	0.02	1.03	4.99	1.24	3.27	4.59	4.89	3.528	4.965	31.403	0.0359	2.7082	22.648
9	17	6.709	2.98	2.98	4.908	4.908	78.108	0.098	4.081	5	13	4.54	0.02	1.15	4.99	1.48	3.41	2.44	2.5	3.303	4.215	69.291	0.0256	2.8351	11.288
10	16	3.037	2.906	2.906	3.807	3.807	111.47	0.049	1.901	5	12	1.79	0.02	1.27	4.99	1.45	3.18	3.31	4.27	3.593	4.018	94.383	0.0163	1.2057	15.330
11	10	4.919	3.452	3.452	4.481	4.481	33.556	0.059	2.829	4	8	3.27	0.02	1.7	5	1.93	3.83	4.78	4.95	4.541	4.557	26.295	0.0248	1.847	21.639
12	41	10.16	2.312	2.312	3.561	3.561	63.828	0.238	6.873	10	27	5.32	0.01	0.93	4.99	0.93	2.79	3.48	4.54	3.446	4.562	49.991	0.0624	4.527	21.679
13	9	4.127	3.847	3.847	3.105	3.105	28.990	0.038	2.2611	3	7	3.2	0.02	2.01	4.99	3.44	4.39	3.38	4.31	3.805	3.872	24.050	0.0302	1.7173	17.038

1. In all trials, the expected loss values of the VSSI control schemes are consistently smaller than that of the FSSI control scheme.
2. Compared to the FSSI scheme, the VSSI scheme requires a smaller sample size, a larger upper control limit, and a more frequent sampling.
3. The optimal value of h_2 in all cases is close to zero. This implies that if the sampling point is plotted in the warning zone, a more frequent sampling will be required.
4. Not only smaller $AATS$ of the VSSI scheme indicates a faster detection of the process mean shifts, but also lower $E(FA)$ implies the VSSI control schemes suggests better protection against false alarms than the FSSI scheme.
5. The minimum expected loss is achieved for situations in which the obtained profit per hour during an in-control period is high or the time the process remains in-control is short.
6. A longer time spent on detection and elimination of an assignable cause leads to a higher expected hourly loss.
7. For cases in which δ, λ, t_1, C_1 are small, V_0 is large compared to V_1, or s is small, the percent reduction becomes larger.

7. Conclusions

In this paper, a methodology was proposed for economic design of asymmetric $\overline{X}$ control charts with variable sample size and sampling interval under the non-normality of the process data. In this methodology, the Burr distribution was first employed to model non-normal sample means. Second, a Markov chain approach was used to determine the chart's adjusted average time to signal (AATS). Third, the mixed-integer non-linear loss function of the problem was derived. Finally, a parameter-tuned (using Taguchi approach) genetic algorithm was utilized to solve the model and determine the chart's optimal design parameters.

A numerical example was given to demonstrate the application of the proposed methodology and to compare its performances with the ones of a fixed sample size and sampling interval (FSSI) $\overline{X}$ control charts under different cost and process settings. The results of the comparison study showed that in all parameter settings the expected loss values of the proposed scheme is consistently lower than the one of the FSSI chart for non-normal data. Further, the adjusted average time to signal and the expected number of false alarms of the proposed scheme were lower than the ones of the FSSI scheme.

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Appendix: **Transition probabilities for asymmetrical control limits**

Let q_{0i} ; $\forall\ i = 1,2$ be the conditional probability that a sample mean $\overline{X}$ falls outside the control limits given that $\mu = \mu_0$; $\forall\ n_i$ i $= 1,2$. Further, let P_{ij} $\forall\ i = 1,2$ & $j = 1,2,3,4$ be the conditional probability that $\overline{X}$ falls within central (safe) or warning regions, given that $\overline{X}$ falls inside the two controls limits when $\mu = \mu_0$, and finally let P_{ij} $\forall\ i = 3,4$ & $j = 3,4$ be the conditional probability that $\overline{X}$ falls within central (safe) or warning regions when $\mu = \mu_0 + \delta\sigma$. Then, for non-normal process data that is modeled by a 2-parameter Burr distribution we have:

$$q_{0i} = \Pr\left(\frac{Y - M_i}{S_i} > k_{i1} \middle| \mu = \mu_0\right) + \Pr\left(\frac{Y - M_i}{S_i} < -k_{i2} \middle| \mu = \mu_0\right)$$

$$= 1 + \frac{1}{\left[1 + Max\{0, M_i + S_i K_{i1}\}^{c_i}\right]^{k_i}} - \frac{1}{\left[1 + Max\{0, M_i - S_i K_{i2}\}^{c_i}\right]^{k_i}}, \forall\ i = 1,2$$

Hence, the transition probabilities are obtained as

$$P_{11} = \Pr\left(\mu_0 - W_{12}\frac{\sigma}{\sqrt{n_1}} < \overline{X} < \mu_0 + W_{11}\frac{\sigma}{\sqrt{n_1}} \middle| \mu = \mu_0\ ,\ LCL < \overline{X} < UCL\right)$$

$$= \frac{e^{-\lambda h_1}}{1 - q_{01}}\left(\frac{1}{\left[1 + Max\ \{0, M_1 - S_1 W_{12}\}^{c_1}\right]^{k_1}} - \frac{1}{\left[1 + Max\ \{0, M_1 + S_1 W_{11}\}^{c_1}\right]^{k_1}}\right)$$

$$P_{12} = \Pr\left(\mu_0 + W_{11}\frac{\sigma}{\sqrt{n_1}} < \overline{X} < \mu_0 + K_{11}\frac{\sigma}{\sqrt{n_1}} \middle| \mu = \mu_0\ ,\ LCL < \overline{X} < UCL\right)$$

$$+ \Pr\left(\mu_0 - K_{12}\frac{\sigma}{\sqrt{n_1}} < \overline{X} < \mu_0 - W_{12}\frac{\sigma}{\sqrt{n_1}} \middle| \mu = \mu_0\ ,\ LCL < \overline{X} < UCL\right)$$

$$=\frac{e^{-\lambda h_2}}{1-q_{01}}\left(\frac{1}{\left[1+Max\left\{0,M_1+S_1W_{11}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1+S_1K_{11}\right\}^{c_1}\right]^{k_1}}\right.$$

$$\left.+\frac{1}{\left[1+Max\left\{0,M_1-S_1K_{12}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1-S_1W_{12}\right\}^{c_1}\right]^{k_1}}\right)$$

$$P_{21}=\Pr\left(\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$=\frac{e^{-\lambda h_1}}{1-q_{02}}\left(\frac{1}{\left[1+Max\ \left\{0,M_2-S_2W_{22}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\ \left\{0,M_2+S_2W_{21}\right\}^{c_2}\right]^{k_2}}\right)$$

$$P_{22}=\Pr\left(\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+K_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$+\Pr\left(\mu_0-K_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$=\frac{e^{-\lambda h_2}}{1-q_{02}}\left(\frac{1}{\left[1+Max\left\{0,M_2+S_2W_{21}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2+S_2K_{21}\right\}^{c_2}\right]^{k_2}}\right.$$

$$\left.+\frac{1}{\left[1+Max\left\{0,M_2-S_2K_{22}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2-S_2W_{22}\right\}^{c_2}\right]^{k_2}}\right)$$

$$P_{13}=\Pr\left(\mu_0-W_{12}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0+W_{11}\frac{\sigma}{\sqrt{n_1}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$=\frac{e^{-\lambda h_1}}{1-q_{01}}\left(\frac{1}{\left[1+Max\ \left\{0,M_1-S_1W_{12}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\ \left\{0,M_1+S_1W_{11}\right\}^{c_1}\right]^{k_1}}\right)$$

$$P_{23}=\Pr\left(\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$= \frac{e^{-\lambda h_1}}{1-q_{02}}\left(\frac{1}{\left[1+Max\left\{0,M_2-S_2W_{22}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2+S_2W_{21}\right\}^{c_2}\right]^{k_2}}\right)$$

$$P_{14}=\Pr\left(\mu_0+W_{11}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0+K_{11}\frac{\sigma}{\sqrt{n_1}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$+\Pr\left(\mu_0-K_{12}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0-W_{12}\frac{\sigma}{\sqrt{n_1}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$=\frac{e^{-\lambda h_2}}{1-q_{01}}\left(\frac{1}{\left[1+Max\left\{0,M_1+S_1W_{11}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1+S_1K_{11}\right\}^{c_1}\right]^{k_1}}\right.$$

$$\left.+\frac{1}{\left[1+Max\left\{0,M_1-S_1K_{12}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1-S_1W_{12}\right\}^{c_1}\right]^{k_1}}\right)$$

$$P_{24}=\Pr\left(\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+K_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$+\Pr\left(\mu_0-K_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0\ ,\ LCL<\overline{X}<UCL\right)$$

$$=\frac{e^{-\lambda h_2}}{1-q_{02}}\left(\frac{1}{\left[1+Max\ \left\{0,M_2+S_2W_{21}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\ \left\{0,M_2+S_2K_{21}\right\}^{c_2}\right]^{k_2}}\right.$$

$$\left.+\frac{1}{\left[1+Max\left\{0,M_2-S_2K_{22}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2-S_2W_{22}\right\}^{c_2}\right]^{k_2}}\right)$$

$$P_{33}=\Pr\left(\mu_0-W_{12}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0+W_{11}\frac{\sigma}{\sqrt{n_1}}\middle|\ \mu=\mu_0+\delta\sigma\right)$$

$$=\frac{1}{\left[1+Max\left\{0,M_1-S_1W_{12}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1+S_1W_{11}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}$$

$$P_{34}=\Pr\left(\mu_0+W_{11}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0+K_{11}\frac{\sigma}{\sqrt{n_1}}\middle|\mu=\mu_0+\delta\sigma\right)$$

$$+\Pr\left(\mu_0-K_{12}\frac{\sigma}{\sqrt{n_1}}<\overline{X}<\mu_0-W_{12}\frac{\sigma}{\sqrt{n_1}}\middle|\mu=\mu_0+\delta\sigma\right)$$

$$=\frac{1}{\left[1+Max\left\{0,M_1+S_1W_{11}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1+S_1K_{11}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}$$

$$+\frac{1}{\left[1+Max\left\{0,M_1-S_1K_{12}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}-\frac{1}{\left[1+Max\left\{0,M_1-S_1W_{12}-S_1\delta\sqrt{n_1}\right\}^{c_1}\right]^{k_1}}$$

$$P_{43}=\Pr\left(\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0+\delta\sigma\right)$$

$$=\frac{1}{\left[1+Max\left\{0,M_2-S_2W_{22}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2+S_2W_{21}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}$$

$$P_{44}=\Pr\left(\mu_0+W_{21}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0+K_{21}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0+\delta\sigma\right)$$

$$+\Pr\left(\mu_0-K_{22}\frac{\sigma}{\sqrt{n_2}}<\overline{X}<\mu_0-W_{22}\frac{\sigma}{\sqrt{n_2}}\middle|\mu=\mu_0+\delta\sigma\right)$$

$$=\frac{1}{\left[1+Max\left\{0,M_2+S_2W_{21}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2+S_2K_{21}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}$$

$$+\frac{1}{\left[1+Max\left\{0,M_2-S_2K_{22}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}-\frac{1}{\left[1+Max\left\{0,M_2-S_2W_{22}-S_2\delta\sqrt{n_2}\right\}^{c_2}\right]^{k_2}}$$