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VALUE AT RISK ESTIMATION BY COMBINING SEMI-PARAMETRIC DENSITY ESTIMATION WITH HISTORICAL SIMULATION

Abstract. *The Value at Risk (VaR) has become a standard risk measure for financial assets. In this paper we propose an alternative way to implement the historical simulation approach to VaR estimation, utilizing a semi-parametric density estimation approach with multiplicative adjustment. The semi-parametric density estimator has the very same asymptotic variance as the standard non-parametric method, while there is substantial room for reducing the bias if the chosen parametric initial function belongs to a wide neighborhood around the true density function. We derive an expression for the pdf of any order statistic of the return distribution utilizing the semi-parametric method. The mean of the estimated pdf is the VaR estimate, and the standard deviation of the estimated pdf can be used to construct a confidence interval around the estimate. We apply this approach to four financial returns series.*

Key words: *Value at Risk, historical simulation, semi-parametric density estimation, multiplicative adjustment, kernel estimation.*

JEL Classification: C14, G10.

1. Introduction

The Value at Risk (VaR) has become a standard measure of financial risks for financial assets, which is used by financial institutions and nonfinancial firms. VaR is a single number which statistically measures the maximum likely loss over a specified time horizon at a particular probability level. More information on VaR is available in Duffie and Pan (1997) and Jorion (2001).

From a statistical prospective, the major challenge of estimating VaR is to precisely predict the tail probability of financial returns series. The assumption that financial asset returns are normally distributed is not supported by empirical evidence. The existing literatures showed that distribution of stock returns exhibits negative skewness and heavy tails (Fama, 1965; Gray and French, 1990; Bekaert et al., 1998). This property is very important in risk measure. The existing estimation methods of VaR include parametric and non-parametric approaches.

The parametric approach is based on parametric model for the return distribution, e.g., Gaussian or t-distribution. Bali and Theodossiou (2007) proposed a conditional technique for estimating the VaR on the basis of the skew generalized t (SGT) distribution in the S&P 500 index returns. Another class of parametric approaches is based on generalized autoregressive conditional heteroskedastic (GARCH) models, which can be able to resemble to certain degrees the fat-tail phenomenon of financial returns (So, Yu, 2006, Predescu and Stancu, 2011).

The parametric approach is attractive for a number of reasons. First of all, the parameters of a model often have important interpretations to a subject matter specialist. Another attractive aspect of parametric approach is its statistical simplicity, i.e., estimation of the entire function boils down to inferring a few parameter values. The third reason is that it can provide an excellent estimator if the class of parametric functions happens to be correctly chosen. However, the parametric approach is model dependent and is subject to errors of model misspecification.

The non-parametric approach, has in general a slower rate of convergence, but has attractive flexibility that can be used without the structural assumption that underlying structure is controlled by a finite dimensional parameter. So, the non-parametric estimator has the advantage of being free of distributional assumptions on returns, while being able to capture fat-tail and asymmetry distribution of returns automatically.

Model-free non-parametric estimation of VaR has been proposed by Dowd (2001) based on the sample quantile, which is commonly called the historical simulation method. Gourieroux, Laurent and Scaillet (2000) introduced non-parametric kernel VaR estimators, and Chen and Tang (2005) investigated their statistical properties. Butler and Schachter (1998) proposed a method to implement the historical simulation approach, employing a non-parametric kernel quantile estimator of the probability density function (pdf) of the returns on a portfolio.

In recent years, there have been increasing interests and activities in the general area of semi-parametric approaches. Among others, a semi-parametric approach with multiplicative adjustment has been used to improve the density estimation. The approach can be viewed as semi-parametric in such a case that it combines parametric and non-parametric methods. In the proposed approach, a parametric estimator is used as a crude guess of true density function. This initial parametric approximation is adjusted via multiplication by a non-parametric factor. It is shown that the semi-parametric estimator has the very same asymptotic variance as the standard non-parametric method, while there is substantial room for reducing the bias if the chosen parametric initial function belongs to a wide neighborhood around the true density function. Hjort and Glad (1995) proposed a density estimator based on the naive estimator of the non-parametric factor. Hjort and Jones (1996) suggested and investigated two versions of multiplicative density estimator. Naito (2004) proposed a local L_2-fitting criterion with index α. Wang and Lin (2008) showed that the multiplicative adjustment method can be applied to density estimation for time series. Similar ideas have been used to improve the regression estimation (Glad, 1998, Wang et al., 2009) and time series conditional variance estimation (Mishra et al., 2010).

In this paper we propose an alternative way to implement the historical simulation approach to VaR estimation, utilizing a semi-parametric quantile estimator of the pdf of the financial returns. We derive an expression for the pdf of any order statistic of the return distribution utilizing the semi-parametric method with multiplicative adjustment. The mean of the estimated pdf is the VaR estimate, and the standard deviation of the estimated pdf can be used to construct a confidence interval around the estimate. We apply this approach to four financial returns series.

The rest of this paper is structured as follows: Section 2 introduces VaR risk measure and outlines two non-parametric VaR estimation methods: historical simulation method and combination of kernel estimation with historical simulation. Section 3 presents the proposed semi-parametric estimator. Empirical analyses of four financial returns series are carried out in Section 4. Section 5 gives a conclusion.

2. Non-parametric Value at Risk estimation

Let $\{X_t\}_{t=1}^{T}$ be the market value of an asset over T periods of a time unit, and let $Y_t = 100 \cdot [\log(X_t) - \log(X_{t-1})]$ be the log-returns with pdf f and cumulative distribution function (cdf) F. Given a positive value p close to zero, the $1-p$ level VaR is

$$\nu_p = \inf \{u : F(u) \geq p\},$$

which specifies the smallest amount of loss such that the probability of the loss in market value being larger than ν_p is less than p.

2.1 Historical simulation method

Historical simulation method assumes that the historically observed financial returns used in the VaR calculation are taken from independent and identical distributions which are the same as the distribution applicable to the forecast. If we have a sample of T observations, we can regard each observation as giving an estimate of VaR at an implied probability level. Let $F_T(x) = T^{-1} \sum_{t=1}^{T} I(Y_t \leq x)$ be the empirical distribution function of the return series $\{Y_t\}$, where $I(\cdot)$ is the indicator function. The historical simulation VaR proposed by Dowd (2001) is $\hat{\nu}_p = Y_{([Tp]+1)}$, where $Y_{(r)}$ is the r th order statistic. For example, if $T = 100$, we can take the 95% VaR as the negative of the sixth-smallest return observation, the 99% VaR as the negative of the second-smallest, and so on. The advantage of this method is that it is non-parametric and the main shortcoming of it is the potential for imprecise estimation of VaR because the VaR is an extreme quantile situated in the tail region of the distribution where the amount of data is small.

2.2 Combination of kernel estimation with historical simulation

Butler and Schachter (1998) proposed a two-step estimation method to implement the historical simulation approach, employing a non-parametric kernel quantile estimator of the pdf of the returns on a portfolio.

The first step is to estimate the pdf and cdf of financial returns using kernel density estimation. The kernel density estimation (Silverman, 1986) is a way of generalizing a histogram constructed with the sample data. The kernel estimator $\tilde{f}(x)$ of the pdf of financial returns, is defined by

$$\tilde{f}(x) = T^{-1} \sum_{t=1}^{T} K_h(X_t - x),$$

where $K_h(z) = h^{-1}K(h^{-1}z)$ and $K(z)$ is a kernel function, which is taken to be a symmetric probability density, and h is the bandwidth. The cdf $\tilde{F}$ of the portfolio return distribution is computed directly from this estimate by linear interpolation.

The second step is to estimate the distribution of any order statistic. According to the theory of order statistic which is well established in the statistical literature (Stuart and ord, 1987, Reiss, 1989), the distribution of the rth order statistic is derived as follows. Let the order statistic be called x, with pdf $g_r(x)$ and cdf $G_r(x)$. Then the probability that exactly r of the data are less than or equal to x is

$$\frac{T!}{r!(T-r)!} F(x)^r \left[1 - F(x)\right]^{T-r},$$

so that the probability that at least r of the data are less than or equal to x is

$$G_r(x) = \sum_{k=r}^{T} \frac{T!}{k!(T-k)!} F(x)^k \left[1 - F(x)\right]^{T-k}. \qquad (1)$$

If at least r of the data are less than or equal to x, then the rth order statistic is less than or equal to x. Thus, equation (1) defines the cdf of the rth order statistic. The pdf that follows from this by differentiation with respect to x is, after some manipulation

$$g_r(x) = \frac{T!}{(r-1)!(T-r)!} f(x) F(x)^{r-1} \left[1 - F(x)\right]^{T-r}. \quad (2)$$

Equation (2) states that $r-1$ of the data must be less than or equal to x, one must

equal to x, and the rest must be greater than or equal to x.
Using pdf estimated with the kernel density estimator above, we can derive the pdf of the r th order statistic and calculate its mean and variance. The pdf is not analytic, but its moments can be calculated by numerical methods. The mean implied by that pdf is the estimate of VaR. From the standard deviation of the estimate we can calculate confidence intervals.

3. Semi-parametric VaR estimation with multiplicative adjustment

In the semi-parametric approach with multiplicative adjustment, a parametric density estimator is utilized, but it is seen as a crude guess of the true density f. This initial parametric approximation is adjusted via multiplication by an adjustment factor ξ which can be determined by non-parametric approaches using some criteria.
Suppose $g(x,\theta)$ be a given parametric family of densities, where the possibly multidimensional parameter $\theta = (\theta_1, \cdots \theta_p)'$ belongs to some open and connected region in p -space. Let the parametric-start estimate be $g(x,\hat{\theta})$, where $\hat{\theta}$ is an estimator of the least false value θ_0 according to a certain distance measure between f and $g(\cdot,\theta)$. For concreteness we here chose $\hat{\theta}$ as the maximum likelihood estimator and define θ_0 as the minimizer of the kullback-Leibler distance on θ.
The next problem is the determination of the adjustment factor ξ. Hjort and Glad (1995) proposed a density estimator based on the naive estimator of ξ. Hjort and Jones (1996) suggested and investigated two versions of multiplicative density estimator. Naito (2004) proposed a local L_2 -fitting criterion with index α,

including the above estimators proposed by Hjort and Glad (1995) and Hjort and Jones (1996) as special cases. However, the focus of all of the above mentioned papers is in i.i.d. observations. Wang and Lin (2008) showed that the multiplicative adjustment method can be extended to density estimation for time series.

3.1 The local L_2-fitting criterion with index α

Following Naito (2004), the adjustment factor ξ is determined by minimization of the empirical version of the function

$$Q(x,\xi \mid \alpha) = \int K_h(u-x)\frac{[f(u)-g(u,\hat{\theta})\xi]^2}{g(u,\hat{\theta})^{\alpha}}du \qquad (3)$$

for a fixed target point x, where α is a real number called the index. This method is called the local L_2-fitting criterion. After omitting the irrelevant term, the empirical version of equation (3) can be expressed as

$$Q_T(x,\xi \mid \alpha) = \xi^2 \int K_h(u-x)g(u,\hat{\theta})^{2-\alpha}du - \frac{2\xi}{T}\sum_{t=1}^{T}K_h(X_t-x)g(X_t,\hat{\theta})^{1-\alpha} .$$

The minimizer can be easily determined as

$$\hat{\xi}(x) = \arg\min_{\xi} Q_T(x,\xi \mid \alpha) = \frac{T^{-1}\sum_{t=1}^{T}K_h(X_t-x)g(X_t,\hat{\theta})^{1-\alpha}}{\int K_h(u-x)g(u,\hat{\theta})^{2-\alpha}du} .$$

Using this $\hat{\xi}$, a class of semi-parametric density estimators is obtained by

$$\hat{f}_{\alpha}(x) = g(x,\hat{\theta})\hat{\xi} = g(x,\hat{\theta})\frac{T^{-1}\sum_{t=1}^{T}K_h(X_t-x)g(X_t,\hat{\theta})^{1-\alpha}}{\int K_h(u-x)g(u,\hat{\theta})^{2-\alpha}du}. \qquad (4)$$

3.2 Advantages of the semi-parametric density estimation approach with multiplicative adjustment

According to Wang and Lin (2008), under some mixing and smoothness conditions, let $g_0(x) = g(x,\theta_0)$, with θ_0 be the best parametric approximation to f, the asymptotic bias and variance of $\hat{f}_\alpha(x)$ are respectively,

$$Bias\{\hat{f}_\alpha(x)\} = \frac{h^2}{2}\sigma_K^2\left[\frac{(g_0(x)^{1-\alpha}f(x))''}{g_0(x)^{1-\alpha}} - \frac{f(x)(g_0(x)^{2-\alpha})''}{g_0(x)^{2-\alpha}}\right] + O\left(\frac{h^2}{T} + h^4 + T^{-2}\right),$$

$$Var\{\hat{f}_\alpha(x)\} = \frac{1}{Th}R(K)f(x) + o\left(\frac{1}{Th}\right),$$

where $\sigma_K^2 = \int z^2 K(z)dz$ and $R(K) = \int K(z)^2 dz$.

Then the asymptotic MISE (AMISE) of $\hat{f}_\alpha(x)$ is

$$AMISE\{\hat{f}_\alpha(x)\} = \frac{h^4}{4}(\sigma_K^2)^2\Re(\hat{f}_\alpha) + \frac{R(K)}{Th}, \qquad (5)$$

where $\Re\{\hat{f}_\alpha\} = \int\left[\frac{(g_0(x)^{1-\alpha}f(x))''}{g_0(x)^{1-\alpha}} - \frac{f(x)(g_0(x)^{2-\alpha})''}{g_0(x)^{2-\alpha}}\right]^2 dx$.

The AMISE of kernel density estimator $\tilde{f}(x)$ of f is (Fan and Yao, 2003, p.206)

$$AMISE\{\tilde{f}(x)\} = \frac{h^4}{4}(\sigma_K^2)^2\Re(\tilde{f}) + \frac{R(K)}{Th}, \qquad (6)$$

where $\Re\{\tilde{f}\} = \int [f''(x)]^2 dx$.

From (5) and (6), the semi-parametric approach is better than the traditonal kernel density estimator in all cases where $\Re\{\hat{f}_\alpha\}$ is smaller in size than $\Re\{\tilde{f}\}$. So the semi-parametric approach presents several potential improvements over both pure parametric and non-parametric estimators.

Firstly, in the case where the parametric model is misspecified so that the parametric estimator for the true density is usually inconsistent, the semi-parametric estimator can still be consistent with the density.

Secondly, in comparison with the kernel density estimator, the semi-parametric estimator can result in bias reduction as long as the initial parametric model can capture some roughness feature of the true density function, whereas the two estimators have the same asymptotic variance.

3.3 The semi-parametric VaR estimator

Similar to the kernel estimation method in section 2.2, the semi-parametric approach also includes the following two steps.

The first step is to estimate the pdf and cdf of financial returns using semi-parametric density estimator with multiplicative adjustment. The semi-parametric estimator of the pdf of financial returns is given by (4). The cdf of the return distribution is computed directly from this estimate by linear interpolation.

The second step is to estimate the distribution of a percentile or order statistic. Using pdf and cdf estimated with the semi-parametric density estimator, we can derive the pdf of the r th order statistic by equation (2) and calculate its mean and variance by numerical methods. The mean implied by that pdf is the estimate of VaR. From the standard deviation of the estimate we can calculate confidence intervals.

4. Empirical results

4.1 Data and descriptive statistics

Here we examine the performance of the semi-parametric estimator with four daily closing stock market indexes from different countries, namely USA (S&P 500), Japan (Nikkei 225), Germany (DAX) and China (SSEC). Logarithmic daily returns are calculated by $100 \cdot \ln(p_t / p_{t-1})$ for alternative sample periods of three years (June 30, 2008 – June 30, 2011) and five years (June 30, 2006 – June 30, 2011), where p_t denotes closing price index at time t. The summary information about the empirical distributions of stock returns, together with Kolmogorov-Smirnov (K-S) and Jarque-Bera (J-B) normality test statistics are presented in Table 1 and Table 2.

Table 1

Descriptive statistics for sample period of three years

	S&P 500	Nikkei 225	DAX	SSEC
No. of observations	757	734	772	731
Min	-9.4695	-12.1110	-7.3355	-6.9827
Max	10.9572	13.2346	10.7975	9.0343
Mean	0.0041	-0.0432	0.0180	0.0013
Standard Deviation	1.8609	2.0721	1.7653	1.8594
Skewness	-0.2245	-0.5203	0.3169	-0.0941
Kurtosis	9.5492	10.7415	9.4043	5.3692
Kolmogorov-Smirnov	0.0768	0.1034	0.0828	0.1084
	p-value	p-value	p-value	p-value
	0.0000	0.0000	0.0000	0.0000
Jarque-Bera	1.3592e+003	1.8660e+003	1.3322e+003	172.0421
	p-value	p-value	p-value	p-value
	0.0001	0.0010	0.0010	0.0010

Table 2

Descriptive statistics for sample period of five years

	S&P 500	Nikkei 225	DAX	SSEC
No. of observations	1262	1229	1271	1218
Min	-9.4695	-12.1110	-7.4335	-9.2562
Max	10.9572	13.2346	10.7975	9.0343
Mean	0.0031	-0.0372	0.0205	0.0412
Standard deviation	1.5764	1.8215	1.5492	2.0318
Skewness	-0.2462	-0.5423	0.1895	-0.4142
Kurtosis	11.4530	11.3485	10.5787	5.3400
Kolmogorov-Smirnov	0.0726	0.0825	0.0577	0.1384
	p-value	p-value	p-value	p-value
	0.0000	0.0000	0.0003	0.0000
Jarque-Bera	3.7700e+003	3.6293e+003	3.0493e+003	312.7127
	p-value	p-value	p-value	p-value
	0.0010	0.0010	0.0010	0.0010

As is commonly found for stock index returns, the assumption of normality is rejected for all returns series. Based on the skewness and kurtosis estimates, we may argue that all the distributions of returns are skewed and leptokurtic, thus exhibiting heavy tails (and high peaks), and both the Kolmogorov-Smirnov test and Jarque-Bera test rejecting the null hypothesis that the returns are normally distributed.

4.2 Estimation results

Using kernel and semi-parametric estimation approaches respectively, we compute the values of VaR corresponding to each indexes for five probability levels ($p = 0.05, 0.025, 0.02, 0.015, 0.01$) and two time horizons (3 years and 5 years). The results of the VaR estimation are shown in Table 3 – Table 6.

For the semi-parametric approach, we utilize normal distribution as an initial parametric approximation of the true density, i.e.,

$$g(x, \hat{\theta}) = \phi_{\hat{\sigma}}(x - \hat{\mu}),$$

where $(\hat{\mu}, \hat{\sigma}^2)$ is the MLE of (μ, σ^2). For the adjustment factor, the Gaussian kernel was used, the index α and bandwidth h were selected by data-based method (Naito, 2004). For the kernel method, the bandwidth was choosed by the cross-validation criterion. Figure 1 displays the estimated pdf for three-yearly S&P 500 index returns using semi-parametric and kernel method, together with the histogram of the data. Other returns and periods would produce similar results. The econometric methods and techniques used in the paper are implemented in Matlab.

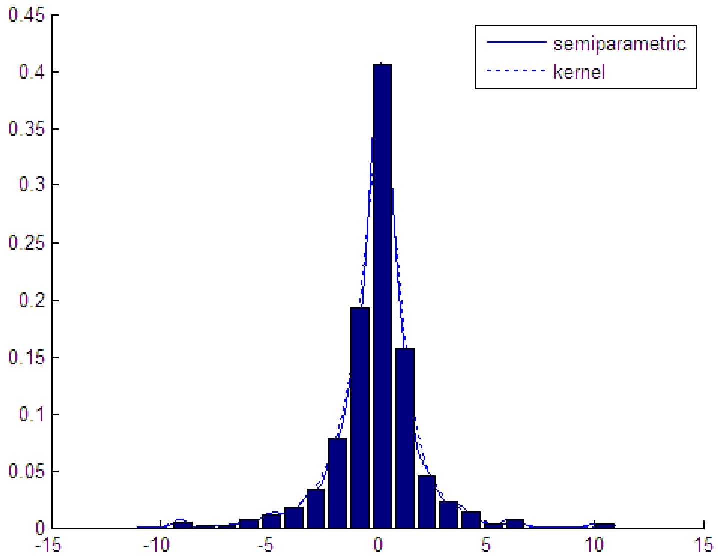

Figure 1. Kernel and semi-parametric density estimates for three-yearly S&P 500 returns, together with the histogram of the data.

Table 3

Kernel and semi-parametric VaR estimates corresponding to S&P 500 for five probability levels and two time horizons

		Kernel estimation		Semi-parametric estimation	
p	Time horizon	Mean	s.d.	Mean	s.d.
$p = 0.05$	3 years	-3.1024	0.2622	-3.2162	0.2983
	5 years	-2.5478	0.1560	-2.5723	0.1603
$p = 0.025$	3 years	-4.3672	0.4280	-4.6245	0.4510
	5 years	-3.4400	0.2470	-3.5837	0.3189
$p = 0.02$	3 years	-4.6965	0.4526	-4.9744	0.5096
	5 years	-3.7638	0.3308	-3.9899	0.3912
$p = 0.015$	3 years	-5.2233	0.5154	-5.6258	0.7096
	5 years	-4.3464	0.4296	-4.6214	0.4435
$p = 0.01$	3 years	-6.0241	0.7591	-6.7516	0.9908
	5 years	-5.0396	0.4787	-5.4007	0.6184

Table 4

Kernel and semi-parametric VaR estimates corresponding to Nikkei 225 for five probability levels and two time horizons

		Kernel estimation		Semi-parametric estimation	
p	Time horizon	Mean	s.d.	Mean	s.d.
$p = 0.05$	3 years	-3.1576	0.2427	-3.1977	0.3160
	5 years	-2.8287	0.1548	-2.8308	0.1752
$p = 0.025$	3 years	-4.5300	0.5937	-5.0646	0.7366
	5 years	-3.8783	0.3276	-4.1228	0.4215
$p = 0.02$	3 years	-5.1834	0.7130	-5.8427	0.8621
	5 years	-4.3045	0.4119	-4.6612	0.5083
$p = 0.015$	3 years	-5.8170	0.8013	-6.6444	1.1543
	5 years	-4.9087	0.5119	-5.3903	0.6101
$p = 0.01$	3 years	-7.0518	1.2041	-8.3184	1.4153
	5 years	-5.7896	0.6372	-6.5214	1.0315

Table 5

Kernel and semi-parametric VaR estimates corresponding to DAX for five probability levels and two time horizons

		Kernel estimation		Semi-parametric estimation	
p	Time horizon	Mean	s.d.	Mean	s.d.
$p = 0.05$	3 years	-2.8381	0.2111	-2.8354	0.2326
	5 years	-2.3918	0.1275	-2.4209	0.1439
$p = 0.025$	3 years	-3.9107	0.4161	-4.0836	0.4078
	5 years	-3.2751	0.2731	-3.4991	0.3468
$p = 0.02$	3 years	-4.3405	0.4641	-4.4806	0.4139
	5 years	-3.6306	0.3601	-3.9211	0.3867
$p = 0.015$	3 years	-4.8558	0.4727	-4.9462	0.4399
	5 years	-4.1521	0.4456	-4.4164	0.3886
$p = 0.01$	3 years	-5.4982	0.5261	-5.5497	0.4737
	5 years	-4.9333	0.4345	-5.1497	0.4610

Table 6

Kernel and semi-parametric VaR estimates corresponding to SSEC for five probability levels and two time horizons

		Kernel estimation		Semi-parametric estimation	
p	Time horizon	Mean	s.d.	Mean	s.d.
$p = 0.05$	3 years	-3.3659	0.2329	-3.7555	0.2517
	5 years	-3.7477	0.2156	-4.1296	0.2059
$p = 0.025$	3 years	-4.3459	0.3112	-4.4371	0.2616
	5 years	-4.8043	0.2407	-5.2033	0.2949
$p = 0.02$	3 years	-4.6589	0.3168	-4.8941	0.2671
	5 years	-5.0892	0.2536	-5.5741	0.3433
$p = 0.015$	3 years	-5.0237	0.3256	-5.2113	0.2873
	5 years	-5.4411	0.2919	-6.0500	0.3799
$p = 0.01$	3 years	-5.3622	0.3524	-5.5049	0.2954
	5 years	-5.9613	0.3948	-6.6519	0.4006

From Table 3 - Table 6, all semi-parametric VaRs are more conservative than kernel ones. For semi-parametric approach, the normal start is also the easiest initial parametric approximation. Thus, it seems to be a good choice for VaR estimation.

5. Conclusions

In this paper, we propose a new VaR estimator that uses a semi-parametric density estimator with multiplicative adjustment to estimate the moments of any order statistic of the distribution of the stock index returns. The semi-parametric density estimator has the very same asymptotic variance as the standard non-parametric kernel method, while there is substantial room for reducing the bias if the chosen parametric initial function belongs to a wide neighborhood around the true density function. Just like kernel method, this estimator is also in the class of historical simulation estimators of VaR and produces a standard deviation, which can be used to construct confidence intervals to help in making ex ante risk management decisions. We illustrate the application of the estimator with data from four different stock index returns. The empirical results illustrate the performance of our approach.

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