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VALUE-AT-RISK ESTIMATION COMPARATIVE APPROACH WITH APPLICATIONS TO OPTIMIZATION PROBLEMS

Abstract. *Mean-risk models have received much attention from researchers and practitioners in the last years. The classical mean-risk models are based on variance and other variance-related risk measures. Recently, risk measures concerned with the left tails of the distributions, that evaluate the extremely unfavourable outcomes, are used. The most important in this class of risk measure is Value-at-Risk (VaR). In this paper we develop a new integrated VaR estimation with risk optimization model. Three methods for estimating VaR are presented and used to estimate the risk corresponding to a set of assets from Bucharest Stock Exchange. We analyze the forecasting performances of these methods based on the computational results provided. We present the mean-risk models and propose a new class of optimization problems, which can be solved using linear programming techniques . In order to illustrate the behavior and the advantages of this approach we will apply our results to build a minimal risk portfolio at Bucharest Stock Exchange.*

Keywords: *risk measure, Value-at-Risk, estimation, portfolio, asset, optimization.*

JEL Classification: C02, C44, C61, C63, G11.
Math. Subj. Classification 2010: 90B50, 90C29, 91G10.

1. INTRODUCTION

Financial portfolio optimization is an important field which has developed from the mean-variance theory (Markowitz, 1952) and from the expected utility theory. The mean-variance theory has some limitations, like in the case when the random outcome of the assets follows a non-normal distribution. The financial literature that contradicts the normality assumption for random outcomes of financial portfolios is a strong argument for introducing new risk measures, like quantile-based risk measures. The most important in this class is Value-at-risk (VaR), which evaluates the maximal loss of a portfolio over a specified time horizon for a certain probability level. VaR risk measure is used for setting the capital adequacy limits for banks and other financial institutions and plays an important role in investment, risk management and regulatory control of financial institutions. In 1993 the Bank of International Settlements members amended the Basel Accord to require Banks and other financial institutions to hold in reserve enough capital to cover 10 days of potential losses based on the 95% 10-day VaR. Furthermore, financial institutions were required to report their overall risk exposure on this basis. Since in most cases the distribution of the loss random variable is not known, a method for evaluating or approximating VaR is required. There are typically three approaches to estimating VaR: the parametric or analytic method, the historical simulation or empirical method and the Monte-Carlo simulation method. VaR is one of the risk measures used to build mean-risk models, the most widely used techniques in solving portfolio optimization problems. In the recent literature, many research papers are devoted to the topic of portfolio optimization using different risk measures, see for example Armeanu and Balu (2009), Fulga (2009a, 2009b), Fulga et al. (2009), Fulga and Dedu (2009), Fulga and Pop (2007, 2008), Marinescu and Marin (2009), Ştefanescu et al. (2008, 2010), Topaloglu et al. (2008). Mean-risk models consist in evaluating and comparing return distributions using two statistics: the expected value of the return and the value of a risk measure which evaluates the loss. Thus, mean-risk models have a ready interpretation of the results and in most cases are convenient from a computational point of view. The risk measure used plays an important role in decision making. Variance was the first risk measure used in mean-risk models in Markowitz (1952). In spite of criticism and many proposals of new risk measures as in Fishburn (1977), Konno and Yamazaki (1991), Ogryczak and Ruszczynski (1999, 2001), Rockafellar and Uryasev (2000, 2002), variance is still used in the practice of portfolio selection. For regulatory and reporting purposes, risk measures concerned with the left tails of distributions, which evaluate the extremely unfavourable outcomes, are used. The most widely used risk measure for such purposes is VaR. In spite of a considerable amount of research, optimizing VaR is still an open problem, like in Larsen et al. (2002), Pang and Leyffer (2005).

The goal of our work consists in developing an integrated VaR estimation with risk optimization model, which can be applied to asset allocation. We will use this approach to build a portfolio with minimal aggregate risk. The rest of this paper is structured as follows.

In section 2 we introduce VaR risk measure and present three methods for estimating VaR of a random variable: the analytic method, the historical simulation and the Monte-Carlo simulation method. We use them to estimate the risk corresponding to a set of assets from Bucharest Stock Exchange and analyze the forecasting performances of the three methods, based on the computational results provided.

In Section 3 the portfolio selection problem is introduced and different mean-risk models are presented in order to complete the theoretical framework.

In Section 4 we will apply our risk estimation and minimization model to solve an optimization problem. In order to illustrate the behavior and the advantages of this approach we will apply our results to build a minimal risk portfolio. We provide computational results based on real data drawn from Bucharest Stock Exchange.

Section 5 summarizes the conclusions and reveals the advantages of our original approach.

2. VALUE-AT-RISK ESTIMATION

We consider the case of a single period portfolio consisting in s assets. We will evaluate the outcome of the assets using the log-return function, since it is widely used in financial analysis. For each j, $j \in \overline{1,s}$, let $S_j(t)$ be the closing price of the asset j at the moment t. The log-return of the asset j corresponding to the time horizon $[t, t+k]$ is defined as follows:

$$R_j(t,k) = \ln S_j(t+k) - \ln S_j(t), \quad j \in \overline{1,s}, \ t > 0, \ k > 0. \tag{1}$$

For $k = 1$ we will use the notation $R_j(t) = R_j(t,1), \ j \in \overline{1,s}, \ t > 0.$

Similarly, we define the loss random variable $L_j(t,k)$ of the asset j corresponding to the time horizon $[t, t+k]$ in the following manner:

$$L_j(t,k) = -R_j(t,k) = \ln S_j(t) - \ln S_j(t+k), \quad j \in \overline{1,s}, \ t > 0, \ k > 0. \tag{2}$$

For $k = 1$ we denote:

$$L_j(t) = L_j(t,1) = \ln S_j(t) - \ln S_j(t+1), \quad j \in \overline{1,s}, \ t > 0. \tag{3}$$

Definition 1. The Value-at-Risk of the loss random variable $L_j(t,k)$ corresponding to the asset j for the time horizon $[t, t+k]$ with probability level $\alpha \in (0,1)$ is defined as follows:

$$VaR_\alpha\left(L_j(t,k)\right) = \min\left\{z \in \mathbf{R} \,\middle|\, P(L_j(t,k) \le z) \ge \alpha\right\}. \tag{4}$$

Since in most cases the distribution of the loss random variable is not known, a method for evaluating or approximating VaR is required. There are typically three approaches to estimating VaR: the parametric or analytic method, the historical simulation or empirical method and the Monte-Carlo simulation method. In chosing one of these methods, it must be taken into account the accuracy and speed of each model. Parametric method is simple, but it is based on the assumption that the distribution the loss random variable is known. Historical method is easy to implement, but it does not accurately capture the risk of future events, since it predicts the future development based on past data, which could lead to inaccurate forecasts if the trend of the past no longer complies, or if the portfolio changes. Monte Carlo simulation approach is more general and thus it can be applied to a wide range of risk models. This method requires powerful computational tools and more time than the others.

2.1. THE PARAMETRIC METHOD

The parametric or analytic method requires an assumption to be made about the statistical distribution from which data are drawn. The attraction of parametric VaR is that relatively little information is needed to compute it. But its main weakness is that the distribution chosen may not accurately reflect all possible states of the market and may under or overestimate the risk. This problem is particularly acute when using value at risk to assess the risk of asymmetric distributions such as portfolios containing options and hedge funds. In such cases the higher statistical moments of skewness and kurtosis which contribute to more extreme losses (fat tails) need to be taken into account.

We will study the analytic method in the case when the random variables considered can be well approximated by a normal distribution. Consider a set of s assets, with asset j giving the return R_j, $j \in \overline{1,s}$, at the end of the investment period. We model the return R_j using a random variable, since the future price of the asset is not known. Let L_j be the loss random variable corresponding to the asset j, $j \in \overline{1,s}$. We will derive the analytical form of VaR risk measure of the loss random variable L_j in the case of normal distribution. For each j, $j \in \overline{1,s}$, let the loss random variable L_j be normal distributed, with $E(L_j) = m_j$ and $Var(L_j) = \sigma_j^2$. Following the approach in Fulga, Dedu (2009), we will derive the analytical expression of VaR risk measure of the portfolio corresponding to the probability level α, as stated in the next proposition.

Proposition 2. Let the loss random variable L_j be normal distributed, with $E(L_j) = m_j$ and $Var(L_j) = \sigma_j^2$. Then the VaR risk measure of the portfolio corresponding to the probability level α is:

$$VaR_\alpha\left(L_j\right) = m_j + \sigma_j \cdot \Phi_j^{-1}(\alpha), \tag{5}$$

where Φ_j denotes the cumulative distribution function corresponding to the normal distribution with parameters m_j and σ_j.

2.2. HISTORICAL SIMULATION METHOD

The historical simulation or empirical method is useful in the case when empirical evidence indicates that the random variable considered cannot be well approximated by normal distribution or in the case when we are not able to make distributional assumptions. Historical simulation method calculates the hypothetical value of a change in the current portfolio depending on historical variations of the risk factors. The great advantage of this method is that it makes no assumption regarding the distribution of probability, using the empirical distribution obtained from analysis of past data, while being a relatively simple calculation. Because it is not dependent on assumptions regarding the parameters of the markets evolution, this methodology can be adapted to leptokurtic, asymmetric and other abnormal distributions. The disadvantage of the historical simulation method lies in the fact that it predicts the future development based on past data, which could lead to inaccurate forecasts if the trend of the past no longer complies, or if the portfolio changes. Let L_j be the loss random variable corresponding to asset j, $j \in \overline{1,s}$. Let $L_j^1, L_j^2, ..., L_j^n$ be n independent and identically distributed random observations of L_j and let $\hat{F}_n^j$ be the empirical cumulative distribution function of L_j. Then we have:

$$\hat{F}_n^j(z)=\frac{1}{n}\sum_{i=1}^{n} I_{\{L_j \le z\}}, \tag{6}$$

where I_A represents the indicator function of the set A. Historical estimation of $VaR_\alpha(L_j)$ involves generating n independent and identically distributed random observations of L_j, denoted as $L_j^1, L_j^2, \ldots, L_j^n$ and estimating $VaR_\alpha(L_j)$ by:

$$\hat{v}_n^j(L_j)=(\hat{F}_n^j)^{-1}(\alpha)=\min\left\{z\in \mathbf{R} \mid \hat{F}_n^j(z)\ge \alpha\right\}.$$

Proposition 3. Let L_j be the loss random variable corresponding to asset $j,\ j\in\overline{1,s}$ and $L_j^1, L_j^2, \ldots, L_j^n$ be n independent and identically distributed random observations of L_j. If $\hat{F}_n^j$ is the empirical cumulative distribution function of L_j, then $\hat{v}_n^j$ is an unbiased and consistent estimator for $VaR_\alpha(L_j)$.

Using (4) and (6), we obtain

$$\hat{v}_n^j(L_j)=\min\left\{ z\in \mathbf{R} \mid \frac{1}{n}\sum_{i=1}^{n} I_{\{L_j^i \le z\}} \ge \alpha \right\}. \tag{7}$$

2.3. MONTE CARLO SIMULATION METHOD

Monte Carlo method is most helpful when the assets in the portfolio are not amenable to analytical treatment. The Monte Carlo method for VaR estimation is based on the statistical simulation of the joint behaviour of all relevant market variables and uses this simulation to generate future possible values of the portfolio This method is uses in the first step scenario generation techniques, that means producing a large number of future price scenarios. The next step, the portfolio valuation, consists in computing a portfolio value for each scenario. In the final step, the summary, we report the results of the simulation, either as a portfolio distribution or as a particular risk measure. The VaR of the loss random variable corresponding to an asset is estimated by creating a hypothetical time series of returns on that asset, obtained by running the asset through actual historical data and computing the changes that would have occurred in each period. Historical simulation represents the simplest way of estimating VaR for many portfolios. However, the Monte Carlo simulation approach is often time-consuming. In risk management, since the probability level is typically close to 1, it results that and a large number of replications are needed to obtain accurate estimation of the tail behavior.

In the next section we introduce a class of the most important problems whose solving process requires risk estimation as an essential phase.

3. THE PORTFOLIO SELECTION PROBLEM

The problem of portfolio selection with one investment period belongs to the general problem of deciding between random variables when larger outcomes are preferred. Decisions are required on the proportion of capital to be invested in each of a number of available assets such that at the end of the investment period the return is as high as possible.

Consider a set of s assets, with asset j giving a return R_j at the end of the investment period, $j \in \overline{1,s}$. We model the return R_j using a random variable, since the future price of the asset is not known. Let w be the total amount of capital to be invested and w_j the capital to be invested in asset j, $j \in \overline{1,s}$. Then the proportion of capital invested in asset j is $x_j = \dfrac{w_j}{w}$, $j \in \overline{1,s}$. Let $\boldsymbol{x} = (x_1,...,x_s)^T \in \mathbf{R}^s$ represent the decision vector or the portfolio resulting from choice. The portfolio return is the random variable $R_{\boldsymbol{x}} = \sum_{j=1}^{s} x_j R_j$. A feasible set A of decision vectors consists of the weights $(x_1,...,x_s)^T$ that must satisfy a set of constraints. The simplest way to define a feasible set is by the requirement that the weights must sum to 1 and short selling is not allowed. For this basic version of the problem, the set of feasible decision vectors is

$$A = \left\{ (x_1,...,x_s)^T \in \mathbf{R}^s \;\middle|\; \sum_{j=1}^{s} x_j = 1,\ x_j \geq 0,\ j = \overline{1,s} \right\}.$$

Consider a different portfolio defined by the decision vector $\boldsymbol{y} = (y_1,...,y_s)^T \in A$, where y_j represents the proportion of capital invested in asset j, $j \in \overline{1,s}$. The return of this portfolio is given by the random variable $R_{\boldsymbol{y}} = \sum_{j=1}^{s} y_j R_j$.

The problem of choosing between portfolios $\boldsymbol{x}$ and $\boldsymbol{y}$ becomes the problem of choosing between random variables $R_{\boldsymbol{x}}$ and $R_{\boldsymbol{y}}$. The criteria used to compare two random variables need to be specified and models for choosing between random variables (models for preference) are required. The purpose of such models is firstly to define a preference relation among random variables and secondly to identify random variables that are non-dominated with respect to that preference relation. Our specific problem requires to decide between two random variables on the basis of two criteria, mean and risk. Therefore we recall that generally, for a multi-objective problem:

$$\max_{\boldsymbol{x} \in A} \left\{ f(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}),..., f_p(\boldsymbol{x}) \right) \right\}, \tag{8}$$

the Pareto preference relation is defined as follows.

Definition 4. A feasible solution $\boldsymbol{x}^1 \in A$ Pareto dominates another feasible solution $\boldsymbol{x}^2 \in A$ if $f_i(\boldsymbol{x}^1) \geq f_i(\boldsymbol{x}^2), \forall i = \overline{1, p}$, with at least one strict inequality.

Definition 5. $\boldsymbol{x}^0 \in A$ is a Pareto efficient (non-dominated) solution of (8) if and only if there does not exist a feasible solution $\boldsymbol{x}$ such that $\boldsymbol{x}$ Pareto dominates $\boldsymbol{x}^0$.
In other words, a Pareto efficient solution is a feasible solution such that, in order to improve upon one objective function, at least one other objective function must assume a worse value.

3.1. THE GENERAL MEAN-RISK MODEL FOR PORTFOLIO OPTIMIZATION

Mean-risk models were developed in the early 1950's in order to solve the portfolio selection problem. In mean-risk models, two scalars are attached to each random variable: the expected value and the value of a risk measure corresponding to the loss function. Preference is defined using a trade-off between the mean, where a larger value is desirable and risk, where a smaller value is desirable.

Markowitz (1952) proposed variance as a risk measure. Since then, many alternative risk measures have been proposed. The question of which risk measure is most appropriate is still the subject of much debate. In the mean-risk approach with the risk measure denoted by ρ, the preference relation is defined as follows.

Definition 3. The random variable R_x dominates (is preferred to) random variable R_y if and only if:

$$E(R_x) \geq E(R_y)$$

and

$$\rho(R_x) \geq \rho(R_y),$$

with at least one strict inequality.
Alternatively, we can say that portfolio $\boldsymbol{x}$ dominates portfolio $\boldsymbol{y}$.

Definition 6. The choice $\boldsymbol{x}$ (or the random variable R_x) is efficient (non-dominated) if and only if there is no other choice $\boldsymbol{y}$ such that R_y has higher expected value and less risk than R_x.

This means that, for a given level of minimum expected return, R_x has the lowest possible risk, and, for a given level of risk, it has the highest possible expected return. Plotting the efficient portfolios in a mean-risk space gives the efficient frontier. Thus, the efficient solutions in a mean-risk model are Pareto efficient solutions of a multi-objective problem, in which the expected return is maximized and the risk is minimized:

$$\max_{\mathbf{x} \in A} \left\{ E(R_{\mathrm{x}}), -\rho(R_{\mathrm{x}}) \right\}.$$

In order to find an efficient portfolio, we solve an optimization problem with decision variable $\boldsymbol{x} = (x_1, ..., x_n)^T$:

$$\min_{\boldsymbol{x} \in A} \rho(R_{\boldsymbol{x}})$$
$$E(R_{\mathbf{x}}) \geq \mu_0,$$

where μ_0 represents the desired level of expected return for the portfolio. Varying μ_0 and repeatedly solving the corresponding optimization problem identifies the minimum risk portfolio for each value of μ_0. These are the efficient portfolios that compose the efficient set. By plotting the corresponding values of the objective function and of the expected return respectively in a return-risk space, we trace out the efficient frontier.

Equivalently, we may consider the model:

$$\min_{\mathbf{x} \in A} E(R_{\mathbf{x}})$$
$$\rho(R_{\mathbf{x}}) \leq \rho_0,$$

where ρ_0 represents the maximum accepted level of risk for the portfolio. As before, varying ρ_0 and plotting the corresponding values we trace out the efficient frontier.

An alternative formulation, which explicitly trades risk against return in the objective function, is

$$\max_{\boldsymbol{x} \in A} E(R_{\boldsymbol{x}}) - \tau \rho(R_{\boldsymbol{x}})$$
$$\tau \geq 0.$$

Varying the trade-off coefficient τ and repeatedly solving the corresponding optimization problems traces out the efficient frontier.

3.2. THE MEAN-VARIANCE MODEL

The mean-variance approach is the earliest method used to solve the portfolio selection problem and it was designed by Markowitz (1952, 1959). Consider n assets with rates of return $R_j,\ j \in \overline{1,n}$. Let $\mu_j = E(R_j),\ j \in \overline{1,n}$ the means of the rates of return and $\sigma_{kj} = \operatorname{cov}(R_k, R_j)$ the covariance between returns of asset k and asset j, with $k,\ j = \overline{1,n}$.

The variance of the portfolio resulting from choice of $\boldsymbol{x} = (x_1, ..., x_n)^T$ can be expressed as:

$$\sigma^2(R_{\mathbf{x}}) = \sum_{k=1}^{n} \sum_{j=1}^{n} x_k x_j \sigma_{kj},$$

which is a quadratic function of $x_1, ..., x_n$.

Two models can be defined for the mean-variance principle. The first one requires that for a given lower bound μ_0 for the mean of the portfolio return, select a portfolio $\boldsymbol{x}$, such that its variance $\sigma^2(R_{\boldsymbol{x}})$ is minimum:

$$\min_{\boldsymbol{x}\in A} \sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j \sigma_{kj}$$

$$\sum_{j=1}^{n} \mu_j x_j \geq \mu_0,$$

where μ_0 is the desired expected value of the portfolio return.

The second model states that for a given upper bound σ_0^2 for the variance of the portfolio, select a maximal return portfolio x, as follows:

$$\max_{\boldsymbol{x}\in A} \sum_{j=1}^{n} \mu_j x_j$$

$$\sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j \sigma_{kj} \leq \sigma_0^2.$$

The theory of mean_variance efficient portfolios was first given in Markowitz (1959) and has also been subject to a lot of criticism. One of the most important reasons for its disadvantages is the computational difficulty associated with solving a large-scale quadratic programming problem.

3.3. THE MEAN-VaR MODEL

The *VaR* risk measure is extensively used in the practice of risk management as a criterion for choosing between portfolios in the following sense. Given a loss function $l(\boldsymbol{x},\boldsymbol{p})$ and a probability level $\alpha \in (0,1)$, we can define two models for the optimization portfolio problem. The first one minimizes the Value-at-Risk of the portfolio, provided that its expected return has a lower bound μ_0:

$$\begin{aligned} &\min_{\boldsymbol{x}\in A} VaR_\alpha\big(l(\boldsymbol{x},\boldsymbol{p})\big) \\ &\sum_{j=1}^{n} \mu_j x_j \geq \mu_0, \end{aligned} \tag{9}$$

The second model searches for a portfolio $\boldsymbol{x}$ that maximizes the expected return, for a given upper bound v_0 of the Value-at-Risk of the loss random variable:

$$\begin{aligned} &\max_{\boldsymbol{x}\in A} \sum_{j=1}^{n} \mu_j x_j \\ &VaR_\alpha\big(l(\boldsymbol{x},\boldsymbol{p})\big) \leq v_0, \end{aligned} \tag{10}$$

where μ_0, v_0 are the parameters of the models.

4. CASE STUDY. RISK ESTIMATION AND OPTIMIZATION AT BUCHAREST STOCK EXCHANGE

We consider the case of 10 assets from Bucharest Stock Exchange: ALBZ, ATB, BIO, BRK, IPRU, OLT, SIF5, SNP, TBM, TLV, which will be used to build a portfolio with minimal aggregated risk measure. First it is necessary to estimate the VaR of the loss function for each asset, based on the values of the closing price in 31 consecutive days. We use real data drawn from the daily transaction reports between June 8, 2010 and July 20, 2010, available on www.bvb.ro site. We first compute the values of the loss function $L_j(t),\ t=\overline{1,30}.$ corresponding to each asset $j \in \overline{1,10}$, using (2).

4.1. PARAMETRICAL METHOD

We will use this method in the case when the random variables considered can be well approximated by a normal distribution. First we estimate the parameters m_j and σ_j, $j=\overline{1,10}$, of the normal distribution which models the loss random variable corresponding to each asset. Table 1 provides the computational results.

Asset	ALBZ	ATB	BIO	BRK	IPRU	OLT	SIF5	SNP	TBM	TLV
m_j	0.002	-0.002	-0.002	-0.006	0.004	0.002	-0.005	-0.002	0.000	-0.001
σ_j	0.031	0.030	0.023	0.039	0.027	0.031	0.034	0.024	0.037	0.027

Table 1. The estimated parameters of the normal distribution for each asset.

Figure 1 illustrates the graph of the normal density corresponding to the estimated parameters, compared with the histogram of the time series obtained using real data in the case of one asset (BIO).

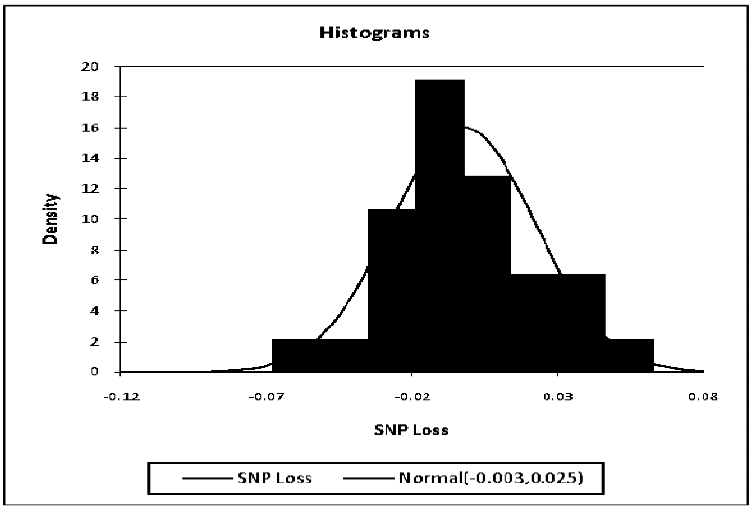

Figure 1. The graph of the normal density which models the loss random variable compared with the histogram of the time series obtained using real data

Using (5) and the estimated values of the parameters m_j and σ_j, $j=\overline{1,10}$, from Table 1, we compute the values of VaR corresponding to each asset for three probability levels and 30 days time horizon and provide a summary of the results in the next table.

α	ALBZ	ATB	BIO	BRK	IPRU	OLT	SIF5	SNP	TBM	TLV
0.99	0.1060	0.0951	0.0719	0.1175	0.0987	0.1080	0.103	0.0769	0.1221	0.0864
0.95	0.0850	0.0748	0.0562	0.0909	0.0800	0.0866	0.0798	0.0602	0.0971	0.0683
0.90	0.0739	0.0640	0.0478	0.0768	0.0700	0.0751	0.0674	0.0513	0.0837	0.0587

Table 2. The VaR of the loss function corresponding to each asset for three probability levels and 30 days time horizon, estimated using the parametric method

4.2. HISTORICAL SIMULATION METHOD

Let L_j be the loss random variable corresponding to asset j, $j \in \overline{1,10}$. Let $L_j^1, L_j^2, ..., L_j^n$ be n independent and identically distributed random observations of L_j and let $\hat{F}_n$ be the empirical cumulative distribution function of L_j. We compute the loss function corresponding to each asset using (2) and the VaR of the loss function corresponding to each asset for three probability levels using (7). In the next table we provide the computational results obtained applying Historical Simulation method for values of three probability level α : 0.99, 0.95 and 0.9.

α	ALBZ	ATB	BIO	BRK	IPRU	OLT	SIF5	SNP	TBM	TLV
0.99	0.06867	0.0777	0.0652	0.1001	0.0602	0.0568	0.0924	0.0523	0.0883	0.0481
0.95	0.05480	0.0357	0.0351	0.0703	0.0564	0.0558	0.0591	0.0401	0.0711	0.0438
0.90	0.04066	0.0296	0.0227	0.0247	0.0414	0.0485	0.0245	0.0299	0.0392	0.0361

Table 3. The VaR of the loss function corresponding to each asset for three probability levels and 30 days time horizon, estimated using the historical simulation method.

4.3. MONTE CARLO SIMULATION METHOD

Generating many thousand future price scenarios that reflect the joint behaviour of all relevant market variables, a large hypothetical time series of closing price for each asset is available.

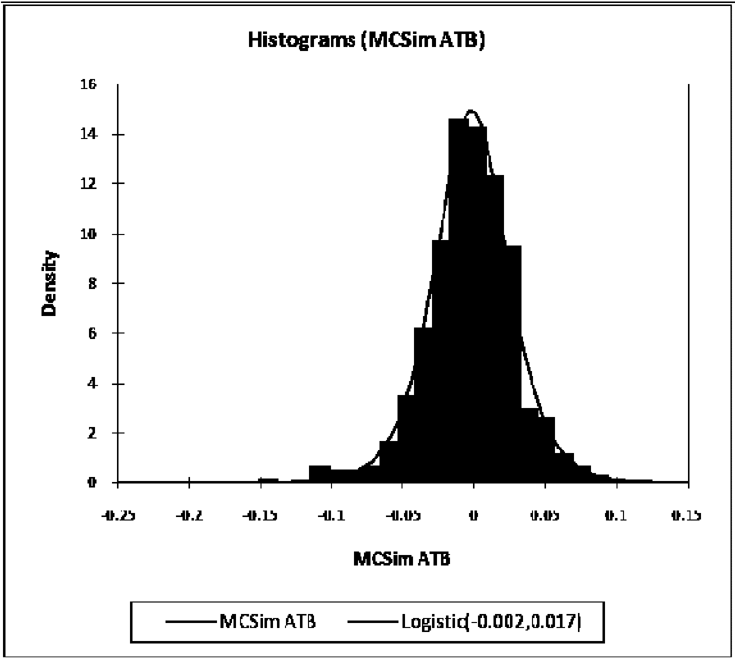

Figure 2. The histogram of the time series obtained using Monte Carlo simulation and the density of the modeling distribution

Next we report the results of the simulation as a price distribution and using it we will compute the values of VaR risk measure. In the next table we provide the estimated VaR for each asset and three probability levels, obtained running Monte Carlo simulation.

α	ALBZ	ATB	BIO	BRK	IPRU	OLT	SIF5	SNP	TBM	TLV
0.99	0.0770	0.0696	0.0624	0.0732	0.0594	0.0980	0.0642	0.0561	0.0716	0.0527
0.95	0.0542	0.0484	0.0417	0.0495	0.0435	0.0623	0.0491	0.0356	0.0561	0.0370
0.90	0.0380	0.0318	0.0302	0.0365	0.0338	0.0412	0.0322	0.0220	0.0435	0.0306

Table 4. The VaR of each asset corresponding to three probability levels and 30 days time horizon, obtained running Monte Carlo simulation.

In the next subsection we will compare the results presented above with those obtained using real data for the next period of 30 days.

4.4. COMPARISON BETWEEN VaR ESTIMATION METHODS

The following table presents a summary of the computational results obtained using the three estimation methods and a comparison of these results with the information concerning the evolution of the stock market during the following 30 days. The first 5 columns reveal, for each asset, the following indices: the VaR of the loss random variable for a 30 days time horizon and 0.95

probability level estimated by Historical method ($\hat{v}_j^H$), by Analytic method ($\hat{v}_j^A$) and by Monte Carlo simulation method ($\hat{v}_j^M$) and the maximal loss produced in the next 30 days next after VaR estimation (v_j^F) with probability 0.95. We can evaluate the performances of the three estimation methods computing the differences between $\hat{v}_j^H$, $\hat{v}_j^A$, $\hat{v}_j^M$ indices (which estimate the maximal loss a holder can suffer in the next 30 days with probability 0.95 using these methods) and v_j^F (which measures the maximal loss produced in the next 30 days).

Asset (j)	$\hat{v}_j^H$	$\hat{v}_j^A$	$\hat{v}_j^M$	v_j^F	$\hat{v}_j^H - v_j^F$	$\hat{v}_j^A - v_j^F$	$\hat{v}_j^M - v_j^F$
ALBZ	0.0850	0.0548	0.0542	0.0463	0.0387	0.0085	0.0079
ATB	0.0748	0.0357	0.0484	0.0282	0.0466	0.0075	0.0202
BIO	0.0562	0.0351	0.0417	0.0262	0.0300	0.0089	0.0155
BRK	0.0909	0.0703	0.0495	0.0419	0.0490	0.0284	0.0076
IPRU	0.0800	0.0564	0.0435	0.0364	0.0436	0.0200	0.0071
OLT	0.0866	0.0558	0.0623	0.0274	0.0592	0.0284	0.0349
SIF5	0.0798	0.0591	0.0491	0.0296	0.0502	0.0295	0.0195
SNP	0.0602	0.0401	0.0356	0.0235	0.0367	0.0166	0.0121
TBM	0.0971	0.0711	0.0561	0.0396	0.0575	0.0315	0.0165
TLV	0.0683	0.0438	0.0370	0.0284	0.0399	0.0154	0.0086
Total	0.7789	0.5222	0.4774	0.3275	0.4514	0.1947	0.1499

Table 5. The VaR of each asset corresponding to three probability levels obtained using three estimation methods, compared with the maximal loss in the next period

The results obtained show that the best method for VaR estimation and forecasting in this case is the Monte Carlo simulation method and the worst is the Historical method.

Next, we will apply the results of the estimation phase for solving an optimization problem which provides a sub-optimal portfolio compared to the minimal VaR portfolio. In order to simplify the computations and to increase the speed of solving the problem, we propose to use an objective function which represents, in the normal case, a majorant for the VaR of the portfolio. Applying this technique we obtain a portfolio whose VaR value is lower than the minimal value of our objective function. Next we will use the estimated VaR using the parametric method to solve the following optimization problem, which consists in finding the portfolio corresponding to the minimal total VaR of the assets such that the expected return of the portfolio exceeds a lower bound μ_0:

$$\begin{cases} \min\limits_{x} \sum\limits_{j=1}^{10} VaR_{\alpha}\left(L_j\right) \cdot x_j \\ \sum\limits_{j=1}^{10} \mu_j x_j \geq \mu_0 \\ \sum\limits_{j=1}^{10} x_j = 1 \\ x_j \geq 0,\ j = \overline{1,10} \end{cases} \tag{7}$$

Based on the results obtained in the previous phase, we can write the numerical form of the optimization problem as follows:

$$\min f = 0.085037x_1 + 0.074826x_2 + 0.056194x_3 + 0.090931x_4 + 0.080003x_5 + \\ + 0.086554x_6 + 0.0798x_7 + 0.060185x_8 + 0.097084x_9 + 0.0683x_{10}$$

$$\begin{cases} -0.058708x_1 + 0.019608x_2 + 0.039665x_3 + 0.154151x_4 - 0.147325x_5 + \\ + 0.0434856x_6 + 0.111704x_7 + 0.058336x_8 - 0.03774x_9 + 0.007067x_{10} \geq \mu_0 \\ \sum\limits_{i=1}^{10} x_i = 1 \\ x_i \geq 0,\ i = \overline{1,10} \end{cases}$$

for different values of the return threshold μ□ and using the VaR values corresponding to 0.95 probability level. In the next table we provide the results obtained solving the optimization problem.

Return (μ□)	**Optimal portfolio**									
	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_9}$	$\boldsymbol{x_{10}}$
0.03	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.04	0.00	0.00	0.98	0.00	0.00	0.00	0.00	0.02	0.00	0.00
0.05	0.00	0.00	0.45	0.00	0.00	0.00	0.00	0.55	0.00	0.00
0.06	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.98	0.00	0.00
0.07	0.00	0.00	0.00	0.12	0.00	0.00	0.00	0.88	0.00	0.00
0.08	0.00	0.00	0.00	0.23	0.00	0.00	0.00	0.77	0.00	0.00
0.09	0.00	0.00	0.00	0.33	0.00	0.00	0.00	0.67	0.00	0.00
0.1	0.00	0.00	0.00	0.43	0.00	0.00	0.00	0.57	0.00	0.00
0.11	0.00	0.00	0.00	0.54	0.00	0.00	0.00	0.46	0.00	0.00
0.12	0.00	0.00	0.00	0.64	0.00	0.00	0.00	0.36	0.00	0.00
0.13	0.00	0.00	0.00	0.75	0.00	0.00	0.00	0.25	0.00	0.00
0.14	0.00	0.00	0.00	0.85	0.00	0.00	0.00	0.15	0.00	0.00
0.15	0.00	0.00	0.00	0.96	0.00	0.00	0.00	0.04	0.00	0.00
0.151	0.00	0.00	0.00	0.97	0.00	0.00	0.00	0.03	0.00	0.00
0.152	0.00	0.00	0.00	0.98	0.00	0.00	0.00	0.02	0.00	0.00
0.153	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.01	0.00	0.00
0.154	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 6. The solution of the optimization problem computed using Matlab

6. CONCLUSIONS

In this paper we have developed an original model for risk estimation and optimization which has the advantage of being easy to be implemented and to be solved. In addition, we have presented three methods for estimating Value-at-Risk and we have analyzed their forecasting performances. In order to emphasize the importance and utility of these methods, we recall a class of the most important portfolio optimization problems whose solving process requires risk estimation as an essential phase. We have introduced and analyzed a new optimization problem. Our important finding relies in the fact that the minimization problem we have proposed can be solved using linear programming techniques. Even though it provides a sub-optimal portfolio, it proves good results in practice. In order to demonstrate its utility, we have applied our algorithm to estimate the risk corresponding to a set of assets and to solve an optimization problem using real data drawn from Bucharest Stock Exchange. The computational results provided illustrate the behavior and the advantages of our model.

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