Ovidiu SONTEA, PhD Candidate
Faculty of Finance, Insurance, Banking and Stock Exchange
E-mail: **osontea@yahoo.com**
Professor Ion STANCU, PhD
The Bucharest Academy of Economic Studies

ABOUT MCSHANE'S DIFFERENTIAL EQUATION AND ITS APPLICATION IN FINANCE

Abstract. *In the literature of specialty, in order to establish the price of a variation model of an active, it is most often use Ito's theorem. In this article I will attempt to use another method of determining an equation for the price dynamics of a derived financial instrument. The article will present an introduction to stochastic calculus, used in McShane's equation and then, starting from McShane's model, an equation with applications in physics. By starting from the observation that the price equation provided by Black and Scholes is equivalent to the heat equation, I will try to expose another method for establishing the price of an option, and also a model for establishing the price of a bond.*
Key words*: Options pricing models, Bonds pricing models.*

JEL Classification: :C,G

1. INTRODUCTION

The stochastic calculus is the most ordinary instrument for studying natural models which have an randomly determined part. If we want to describe the Brownian motion, this can be done by using stochastic differential equations where a Wiener process formal derivate can be identified.

In stochastic calculus, as well as in classical differential equations, the hardest part is the interpretation and the approximation of the processes. Usually, the indeterminist part is approximated to something convenient for obtaining the desired result or, at best, the inferred one.

In finances, if the price of an active or a derivative must be decided, the most used instruments are the ones resulted as a consequence of Ito's theorem. This domain's pioneers were Black, Scholes, Merton, and Fischer who, in the 70s, attained at remarkable results in the establishing the financial instrument's prices by using mainly Ito's theorem. An alternative to this calculation technique, described by Stratonovich (1966), was analyzed by Sethi and Lehoczky (1981) who relized that for respecting the hypothesis of an "efficient market", the described process must be a martingale, which occurs in the case of Ito processes, but didn't occur in the case of the processes described by Stratonovich.

Brownian motion models studied up to date and described in specialty works do not cover all possible situations. Usually, the models that try to establish the prices of some financial instruments have restrictions regarding the variation of certain parameters, which makes them inefficient.

In this article I will use a result obtained by McShane (1974), the McShane integral, which is a particular case of a semi-martingale. The system proposed by McShane has the property that, in case of lipschtzien functions, the solution of the equation is the same as the solution inferred by a similar physical system.

The article will make an introduction to the stochastic calculus used in McShane's equation and then I will propose an application of this type of calculus for the variation of an option's price, similar to the Black-Scholes equation.

2. MCSHANE'S DIFFERENTIAL EQUATION

The equation proposed by McShane is the following:

$$x(t) = x(0) + \int_0^t f(s,x(s))ds + \int_0^t g(s,x(s))dz(s) + \int_0^t h(s,x(s))(dz(s))^2 \quad (1)$$

where f and g are functions having partial derivatives on each variable, required for the integral's and the differential's existence, z is a stochastic process, and $h(t,x) = \frac{1}{2} g_x(t,x) g(t,x)$ with the less used notation $g_x = \frac{\partial g}{\partial x}$.

While the first integral is a Riemann-Stieltjes integral, the last two must be introduced somehow because, although they look like a Riemann integral, they don't represent the same thing.

Let $\Delta_n = \left(0 = s_0^{(n)}, s_1^{(n)}, \ldots, s_{k_n}^{(n)} = t\right)$ be a series of divisions in the interval $[0,t]$, $t \in \mathbb{R}_+^*$ which has its norm tending to 0 when $n \to \infty$.

If the series

$$\sum_{i=1}^{k_n} g\left(s_{i-1}^{(n)}, x\left(s_{i-1}^{(n)}\right)\right)\left(z\left(s_i^{(n)}\right) - z\left(s_{i-1}^{(n)}\right)\right)$$

has a finite limit, then the limit is

$$\int_0^t g(s,x(s))dz(s)$$

and, similarly, if the series

$$\sum_{i=1}^{k_n} h\left(s_{i-1}^{(n)}, x\left(s_{i-1}^{(n)}\right)\right)\left(z\left(s_i^{(n)}\right) - z\left(s_{i-1}^{(n)}\right)\right)^2$$

has a finite limit, then the limit is

$$\int_0^t h(s,x(s))(dz(s))^2.$$

From equation (1) we deduce

$$dx = f(t,x)dt + g(t,x)dz + \frac{1}{2} g_x(t,x) g(t,x)(dz)^2 \quad (2)$$

Consider the continuous function F, having continuous partial derivatives F_t, F_x, F_{xx} and $x(t)$ being a stochastic differential process, then:

$$dF(t,x(t)) = \{F_t(t,x(t)) + F_x(t,x(t)) f(t,x(t))\}dt + F_x(t,x(t)) g(t,x(t)) dz(t) + \\ + \left\{\frac{1}{2} F_x(t,x(t)) g_x(t,x(t)) g(t,x(t)) + \frac{1}{2} F_{xx}(t,x(t)) g^2(t,x(t))\right\}(dz(t))^2$$

.(*)

We will analyze a particular case:

If g is independent of x, then $g_x = 0$ and so h=0, meaning

$$dx = f(t,x)dt + g(t)dz \quad (3)$$

An equation with a form usually used in finances.

In case of a Lipschitzen process z, meaning an L exists such that

$$|z(u) - z(v)| \leq L|u - v|$$

for any *u*, *v* in a compact interval, we can state that

$$\left(dz(t)\right)^2 = 0$$

and we have

$$dx = f(t,x)dt + g(t,x)dz \quad (4).$$

In other case, where z is a Wiener process, we have $\left(dz(t)\right)^2 = dt$ and equation (2) will become:

$$dx = f(t,x)dt + g(t,x)\,dz + \frac{1}{2}g_x(t,x)g(t,x)dt$$
$$= \left(f(t,x) + \frac{1}{2}g_x(t,x)g(t,x)\right)dt$$
$$+ g(t,x)dz \quad (5),$$

In the same context, equation (*) becomes

$$dF(t,x(t)) = \Big\{F_t(t,x(t)) + F_x\left(t,x(t)\right)f\left(t,x(t)\right)$$
$$+ \frac{1}{2}F_x(t,x((t))g_x(t,x(t))g(t,x(t))$$
$$+ \frac{1}{2}F_{xx}(t,x(t))g^2(t,x(t))\Big\}dt$$
$$+ F_x\left(t,x(t)\right)g\left(t,x(t)\right)dz(t) \quad (6)$$

3. AN APPLICATION FOR ESTABLISHING THE PRICE OF THE OPTIONS

If in the Black-Scholes formula, the Ito equation used is

$$dS = \sigma Sdt + \mu Sdz$$

with S (0) =C, C-constant, where μ is the instant efficiency of the active support, and σ its volatility, then, from McShane equation (6), we have the following model

$$dS = \left(\sigma + \frac{\mu^2}{2}\right)Sdt + \mu Sdz \qquad (7)$$

By letting down *C* (*S, t*) the value of a call-option using (6) and (7) we get:

$$dC = \left(\left(\sigma + \frac{\mu^2}{2}\right)C_S S + \frac{1}{2}C_{St}\mu C + C_t\right)dt + \mu C_S dz \qquad (8)$$

with the notations $C_S = \frac{\partial C}{\partial S}, C_{St} = \frac{\partial^2 C}{\partial S \partial t}$.

Acting similarly as in determining the Black-Scholes formula in which the condition for an active risk-free is set, that is

$$d\pi = r\pi dt \qquad (9)$$

where r is the risk-free rate, and π is the value of a portfolio formed from an option C and C_S shares, as follows:

$$\pi = C_S S - C \tag{10}$$

If we differentiate the last relation, we have:

$$d\pi = C_S dS - dC \tag{11}$$

which is introduced in (9) and by considering (7), (8) and (10) we will obtain the equation:

$$C_S rS + \frac{1}{2} C_{SS} \mu^2 S^2 + C_t - rC = 0 \tag{12}$$

Equation which resembles the Black-Scholes equation obtained in the situations given by Ito's Theorem.

We obtained a new way for deducting the equation determining the price of an option without considering Girsanov theorem of probability change, nor recalibering the Wiener process, nor using the infinitesimal operator of the diffusion process.

We know this equation with partial derivatives- by using a change of variable, this will lead us to the essence of heat. The McShane equation is mostly used in Physics; as many physical processes are similar to economic ones, the idea of using this deliverance method starting from known results in physics is not anything new and I reckoned on this thing in deducting the dynamic equation of the price of an option, the Black-Scholes equation.

The solution from equation (12) in which we have the final condition $C(S,T) = \max(S_T - K, 0)$ is given by the representation theorem Feynman-Kac, namely $C = e^{-r\tau} E_{Q^*}[\max(S_T, 0)]$ under neutral probability at risk Q^*.

The expression of the theoretical price is known, namely:

$$C = SN(d_1) - Ke^{-rT} N(d_2),$$

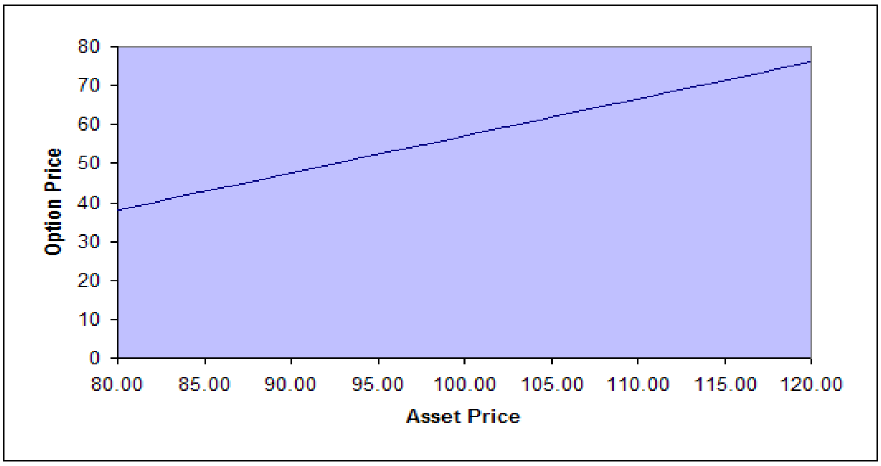
where $d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $N(d) = Prob(x \le d)$ is the repartition function of the standardized normal law. Let us consider an active which has six month maturity for an option, has the price of 42 euros, and the strike price of 40 euros and the risk-free rate is 10% per year, and the volatility 20% per year. We have $S = 42, K = 40, r = 0{,}1, \sigma = 0{,}2, T = 0{,}5,$

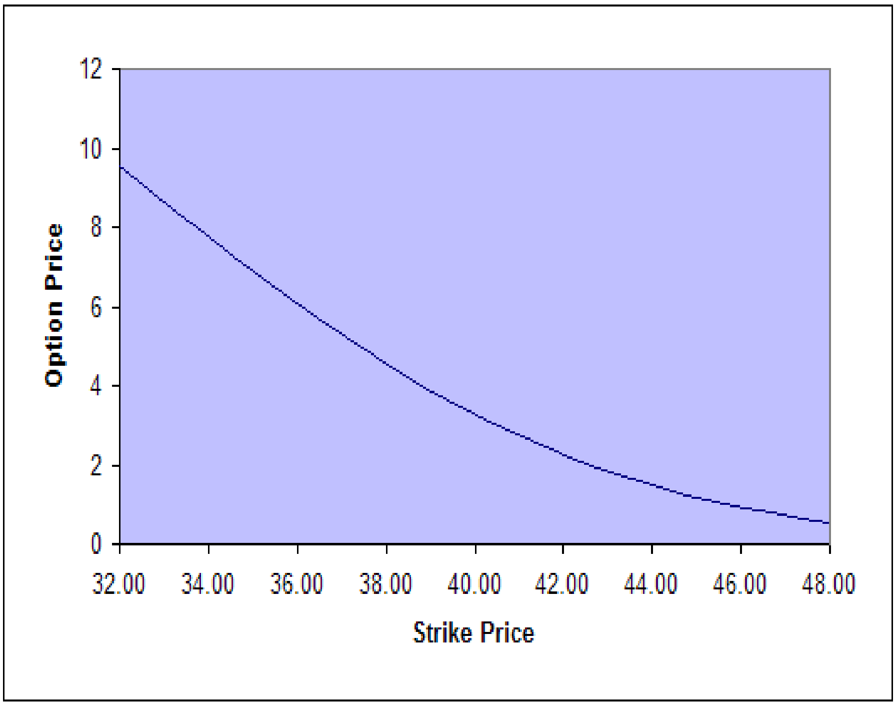
$$d_1 = \frac{\ln\frac{42}{40} + \left(0.1 + \frac{0.2^2}{2}\right) \cdot 0.5}{0.2\sqrt{0.5}} = 0.769,$$

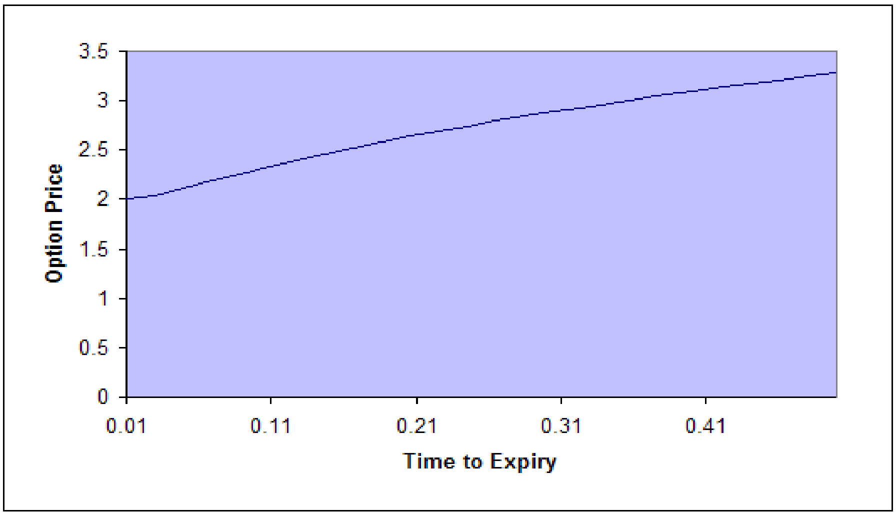
$$d_2 = 0.769 - 0.2 \cdot 0.5 = 0.627.$$

$$C = 42N(0.769) - 38.049N(0.627).$$

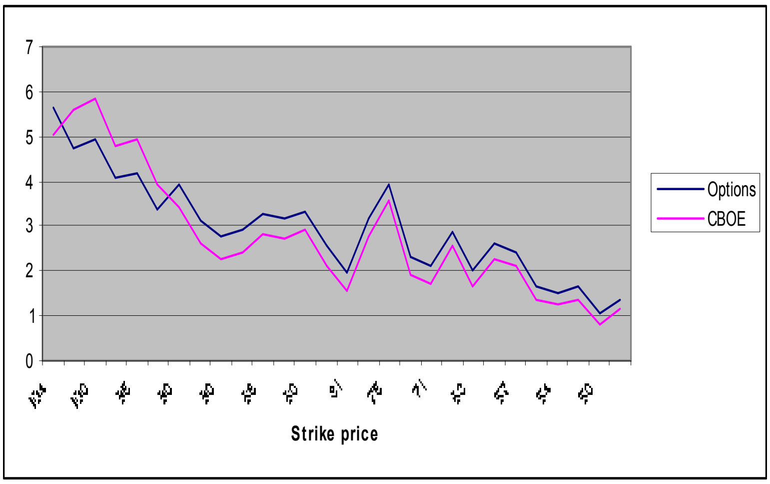
From the probability tables we obtain N(0.679)=0.779, N(0.627)= 0.734 and moreover, C=4.76. Here we see the evolution of the option price depending on asset price, strike price and time to expiry.


Option Price
80
70
60
50
40
30
20
10
0
80.00
85.00
90.00
95.00
100.00
105.00
110.00
115.00
120.00
Asset Price


Option Price
12
10
8
6
4
2
0
32.00
34.00
36.00
38.00
40.00
42.00
44.00
46.00
48.00
Strike Price



Next, we considered that a series of data are applied to the price formula obtained above, it being compared with a range of price options data obtained from www.cboe.com. We noted the price options as Options, obtained by the method above and the others as CBOE.



The viability of the options evaluation model formulated by Black and Scholes is analyzed by El Karoui N., Jeanblanc-Picque M., Shreve S., (1998) in an article from Mathematical Finance. From this article we can understand that the model is very robust and assures an efficient gestation of the risks the investors are exposed to. Even more, the error of this model is systematical, overestimating "out the money" options and underestimating "in the money" options.

We can thus say that the evaluation error of this model, being systematical, is the same as a small coverage error. Consequently, the investor has precise information about the error and acts accordingly to avoid losses.

The analysis of this data model offered by B.M.F.M from Sibiu cannot be done with very good results, because the transacted options are American and they have as a support future contracts, while the presented model is based upon the analysis of an European option having as a support a share that doesn't distribute dividends or the price of future contracts acts like the price of a share distributing dividends with a dividend rate equal to the risk-free rate. In conclusion, to evaluate an option on a future contract means evaluating an option on a share whose current price is $Fe^{-r\tau}$, where r is the risk-free rate, and τ is the duration of the contract.

4. APPLICATION FOR ESTABLISHING THE PRICE OF THE BONDS

Furthermore, I will demonstrate how to obtain the equation gives the price of a bond by using the principle of absence of arbitrage opportunities and McShane's equation.

I suppose that interest rate r is governed by a stochastic differential equation McShane type of the following form

$$dr_t = \mu(t,r)dt + \sigma(t,r)dz + \rho(t,r)(dz)^2 \tag{13}$$

To carry on, we will consider that the price of a bond depends only upon r the spot rate of the interest, t the current date and T the expiration date), date of payment.

Using the (*) equation for the price of a bond, we will obtain the following relation

$$dP = (P_t + \mu(t,r)P_r)dt + \sigma(t,r)P_r dz + \left(\rho(t,r)P_r + \frac{1}{2}\sigma^2(t,r)P_{rr}\right)(dz)^2 \tag{14}$$

If we consider that the price dynamics of a bond are given by the equation

$$dP = \mu_P(t,r)Pdt + \sigma_P(t,r)Pdz + \rho_P(t,r)P(dz)^2 \tag{15}$$

we will be able to deduce that

$$\begin{aligned} \mu_P(t,r) &= \frac{1}{P}(P_t + \mu(t,r)P_r) \\ \sigma_P(t,r) &= \frac{1}{P}(\sigma(t,r)P_r) \\ \rho_P(t,r) &= \frac{1}{P}\left(\rho(t,r)P_r + \frac{1}{2}\sigma^2(t,r)P_{rr}\right) \end{aligned} \tag{16}$$

In order to obtain a hedging operation for bonds of different maturity, we will create the following portfolio: we will buy a bond with a currency unit having a V_1 value and T_1 maturity and we will sell another bond with a currency unit having the V_2 value and T_2 maturity. The portfolio value Π is given by

$$\Pi = V_1 - V_2 \tag{17}$$

In relation to the price dynamics of a bond (15), the variation of the portfolio value is given by

$$d\Pi = (V_2\mu_P(r,t,T_2) - V_1\mu_P(r,t,T_1))dt + (V_2\sigma_P(r,t,T_2) - V_1\sigma_P(r,t,T_1))dz + (V_2\rho_P(r,t,T_2) - V_2\rho_P(r,t,T_1))(dz)^2 \tag{18}$$

I assume that the portfolio is risk-free,

$$d\Pi = r\Pi dt \tag{19}$$

Linking the relations (18) and (19), we obtain the following

$$\begin{gathered} V_2\mu_P(r,t,T_2) - V_1\mu_P(r,t,T_1) = rV_2 - rV_1 \\ V_2\sigma_P(r,t,T_2) - V_1\sigma_P(r,t,T_1) = 0 \\ V_2\rho_P(r,t,T_2) - V_1\rho_P(r,t,T_1) = 0 \end{gathered} \tag{20}$$

From relations (20) we obtain

$$\frac{\mu_P(r,t,T_2)-r}{\sigma_P(r,t,T_2)} = \frac{\mu_P(r,t,T_1)-r}{\sigma_P(r,t,T_1)} \tag{21}$$

which tells us that the above mentioned ratio does not depend on the maturity term. We will consider

$$\lambda(r,t) = \frac{\mu_P(r,t,T)-r}{\sigma_P(r,t,T)},$$

this term being called: market price for risk.
In a similar way, as in (20), I obtain

$$\frac{\rho_P(r,t,T_1)}{\sigma_P(r,t,T_1)} = \frac{\rho_P(r,t,T_2)}{\sigma_P(r,t,T_2)} = \eta(r,t),$$

which can be interpreted as bonus risk.
Following, the result

$$\begin{gathered} \mu_P(r,t,T) = r + \lambda(r,t)\sigma_P(r,t) \\ \rho_P(r,t) = \eta(r,t)\sigma_P(r,t) \end{gathered} \tag{22}$$

Now, we can use (16) and (22) to obtain

$$\begin{gathered} (P_t + \mu(t,r)P_r) = rP + \lambda(r,t)\,(\sigma(t,r)P_r) \\ \left(\rho(t,r)P_r + \frac{1}{2}\sigma^2(t,r)P_{rr}\right) = \eta(r,t)(\sigma(t,r)P_r) \end{gathered} \tag{23}$$

And this implies that the equation of structure term for pricing bond is

$$P_t + \left(\mu + \rho - \sigma(\lambda + \mu)\right)P_r + \frac{1}{2}\sigma^2 P_{rr} = rP \tag{24}$$

with condition $P(r,T) = 1$.

Finally, we obtain another way to deduce the equation of bond price dynamics from McShane's equation like as an alternative to the Ito equation.

The solution of this equation is

$$P = exp\left\{-(T-t)r - \frac{(T-t)^2}{2}[\mu + \rho - \sigma(\lambda + \eta)] + \frac{(T-t)^3}{6}\sigma^2\right\}.$$

We suppose the following inputs *t=0,T=0.5,r=10%,* $\mu = 15\%, \sigma = 20\%, \rho = 0.1$. We obtain immediately $\eta = 0.5$ and $\lambda = 0.25$. Plugging in these results, we obtain $P = e^{-0,16}$.

5.CONCLUSIONS

In this article we made reference to an application of McShane's differential equation in finances, namely to establish the dynamics equation for the price of an option, and then, for that of a bond. The result obtained by McShane (1974), the McShane integer, is a particular case of semi-martingale and I used this property in article. Throughout this article I wanted to show another way of reaching the result obtained by Black and Scholes starting from a differential equation with many applications in Physics, this being equivalent to the heat equation. The system proposed by McShane has the property that, in the case of lipschtzian functions, the solution of the proposed equation coincides with the solution intuited by a similar physical system, which, at least at an intuitive level, leads to applications for the price modeling of derived financial instruments.
What remains to be done is analyzing if the McShane equation can lead us to models in the case of stochastic volatility or the rate of the stochastic interest.

REFERENCES

[1] **Albanese, C., Campolieti, G. (2006),** ***Advanced Derivatives Pricing and Risk Management***. Elsevier Academic Press;
[2] **Constantin, G. (2005) ,** ***An Application of The Stochastic McShane's Equations in Financial Modeling***. *Conferences Telecom Bretagne*, www.conferences.telecom-bretagne.eu.;
[3] **Constantin, A. (1994),** ***On the Existence, Uniqueness and Parametric Dependence on the Coefficients of the Solution Processes in McShane's Stochastic Integral Equations*** . *Publicacions Matematiques*, 38, 11-24;
[4] **Hull, J. (2000),** ***Options ,Futures, and Other Derivatives***. *Prentice Hall*;
[5] **Moisă, A. (2002),** ***Inginerie Financiară*** . ASE, www.ase.ro ;
[6] **McShane, E.J. (1974),** ***Stochastic Calculus and Model***. *Academic Press*, New York;
[7] **Negrea ,R.** ***On The Extension of Merton and Vasicek Models Related to the Pricing of Zero-Coupon Bonds***. www.math.uvt.ro/eng/pubs/preprints;
[8] **Negrea, B. (2006), Evaluarea Activelor Financiare.** *Economică*

Publishing, Bucharest;
[9] **Stancu,I. (2007),** ***Finante*** . Economică Publishing, Bucharest;
[10] **Wiley,J. (2006),** ***Derivatives: Market, Valuation, and Risk Management***. *John Wiley and Sons, Inc.*;
[11] **Willmott, P. (2002),** ***Derivative. Inginerie financiară***. *Economică* Publishing, Bucharest;
[12] **El Karoui N., Jeanblanc-Picque M. Shreve S. (1998),** ***Robustness of Black and Scholes Formula*** . *Mathematical Finance*, 8.