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ECONOMIC-FINANCIAL FORECASTING MODEL OF BUSINESSES USING FUZZY RELATIONS

Abstract *The traditional models of diagnosis economic financial have been characterized by its low formalism and subjectivity as the so called model of the ratios. Until evolving to more formal models know as bankruptcy prediction models. However both have a common denominator, the use of ratios economic financial, as key variables to determine the health of the company or its remoteness or proximity of insolvency. The work presented developed a model of financial economic diagnosis of companies, which on the basis of a simple scheme of cause-effect, simulate the action of the analyst in its task of diagnosis. In the same, the causes represent companies' diseases and the symptoms (represented by the ratios) the effects. The model determines a matrix of financial economic knowledge, on the use of resolution models of fuzzy binary equations, which gets a fuzzy relationship between symptoms (ratios) and diseases.*

Key words. *Matrix of economic-financial knowledge; Model of ratios; Economic-financial diagnosis; Equations in fuzzy binary relations; Symptoms and causes; Fuzzy relations; Finance.*

JEL Classification: G 17

1. Introduction

The economic context in which businesses develop is dynamic and changing and this has an impact on businesses that affects both their objectives and structure. Identifying and anticipating these changes allows businesses to take decisions that compensate for or eliminate their negative effects, and to strengthen themselves when these changes are positive, meaning that the business achieves its objectives in both situations. In any case, the multiplicity of elements and factors that determine whether a business achieves its objectives makes it difficult to predict all the events that make affect these. For this reason, it becomes very important to elaborate a model to help identify in time the problems that afflict a business and thus to be able to take appropriate decisions so that these problems have the least possible effect on the business achieving its goals.

The economic literature uses the term economic-financial diagnosis to refer to the act of identifying the problems that affect a business and this is carried out using a ratios model. Nevertheless, another large area of the economic literature has focused on identifying businesses with problems by using "Business Insolvency Prediction". The prediction models for business insolvency are not necessarily models for economic and financial diagnosis, although when they are applied, they do identify businesses with problems.

The two approaches have distinct characteristics, both in terms of the results they generate and their methodologies. The ratios model involves estimating a series of indicators obtained from the financial statements of the company. The ratios are evaluated by analysts and consultants from the firm, who compare them with financial statements that are considered normal. Their findings help the company's owners and shareholders to decide which paths to take in the future.

As stated above implied that the "ratios model" is highly subjective and that there is little chance of it being formalized, at least not using traditional methods. This is because the result depends on: a) the opinions of the analyst (expert) who is carrying out the diagnosis, b) the interpretation of normality used for the analysis, and c) the ratios that are selected for the evaluation.

The prediction models for business insolvency (also known as business failure prediction or bankruptcy prediction) originated at the end of the 1970s with the clear aim of incorporating more formal and analytical steps for diagnosing businesses. These analytical methods use ratios as explanatory variables and are based on a set of statistics-based procedures that try to specify a mechanism in order to "divide" a group of companies into those with the "greatest probability" of failure and those with the "greatest probability of success". Practically all the models developed use bankruptcy as a failure variable; therefore the models' central idea is to be able to predict, from a set of ratios, which companies have the greatest likelihood of going bankrupt, which have the greatest likelihood of not going bankrupt and which have some margin of doubt. In this way, the models choose a sample of bankrupt and non bankrupt companies and to assess the probability of bankruptcy. To do this, they use a time horizon prior to the moment of bankruptcy and a group of selected ratios.

The first models to be found in the literature were based on proposals from E. Altman (1968) and W. Beaver (1966), who formulated methods that used linear discriminating analysis techniques to estimate a certain "score", by which the company could be classified as healthy or unhealthy. The explanatory variables used by these models are the economic and financial ratios, the problem being to determine which ratios best "divide" the companies into healthy and unhealthy. Beaver's model uses a univariate analysis because it selects the "best" ratio for classifying companies as healthy or unhealthy. Altman's model uses multivariate analysis because this combines the best "set" of ratios. Meyers and Pifer (1970) developed a linear discriminating model with similar characteristics to that of Altman, with the difference that the deciding variable (the score) can be found in the interval [0,1]. In this way it becomes a Linear Probability Discriminating Model.

Among the models that use probability, authors such as Wilcox (1976), Santomero and Vinso(1977) and Scott (1981), among others, have developed “theoretical” models, most of which are based on the fact that predictions of bankruptcy originate from the probability that a company cannot meet its obligations. The firm fails if the value of the sale is less than the value of the company’s debt.

From a methodological point of view, perhaps the first innovation that can be found in the literature is the use of Conditional Probability Models (Olshon, J., 1980). These models raised the fundamental estimation problem: given that a firm belongs to a pre-specified population, what is the probability of the firm failing within a given period of time. The conditional probability model does not need prior suppositions to be made about these probabilities or about the distribution of the estimators’ probability (suppositions of the multiple discriminating models). The LOGIT and PROBIT schemes are the most commonly used specifications for conditional probability models.

Subsequently, models appeared that can be categorized as Non Parametrics, among them the Recursive Partitioning Algorithm (Marais et al., 1984, McKee et al., 2000, and Frydman et al. 1985); the A-Score model (Argenti, 1976); models that use Rough Sets, (Dimitras et al., 1999); and Neuronal Network Models.

The Recursive Partitioning Algorithm (RPA) is non parametric classification technique that is computerized using recognition patterns. These patterns are defined using univariate classification procedures, which in turn become multivariate. The model resulting from the application of the RPA is a binary classification tree that assigns the objects analyzed to a certain quantity of predefined groups. There are two predefined groups in the bankruptcy prediction models: bankrupt businesses and healthy businesses. At each step of the algorithm, the sample divides in two parts because of a classification rule that uses a univariate method (or a linear combination of variable), and at each step the independent variable is chosen that best assures the homogeneity of the groups resulting from the division.

Argenti’s A-Score Model (1976) used a different technique to deal with predicting insolvency. It can be said that his view is much more global and integrating In contrast to other authors, Argenti differentiates between symptoms and causes, although this differentiation is not as operative as it ought to be. The author demonstrates his analytical conception of the problem of business failure, and includes 12 items in it that he thinks important schematizing the problem. Argenti’s A-Score model still has a very low level of formalization and is based on the construction of an index by assigning weights to the different causes identified, such as insolvency and bankruptcy. An external analysts assigns these weights. When the value of the index gets above a certain threshold, the company is considered anomalous.

The use of the rough sets theory aims to define the problem of non-discernibility of objects in a set. On the basis that any set of all indiscernible objects is called elementary set and forms a basic part of knowledge. Then any set of objects is a union of some elementary sets defined by a set of real numbers, otherwise it is all rough. Then, each rough set has a dividing line of cases. Objects that can be classified with certainty within the set, called the upper approximation, and those objects which possibly belong to the set, called the lower approximation. The original version of this model assumes that the different values of a variable are, a priori, equally preferable when making a classification. However, what is really important is the predictive value of the variables revealed by the data and, thus, the attributes are factored into the model. When used to predict insolvency, this mechanism implies that the attributes are economic and financial ratios, and these are put in order for each business. In this way, if there is a set of healthy and bankrupt businesses, the values of each attribute for the companies that form part of each set allow us to determine the circumstances of a company in one set or the other.

Artificial Neuronal Models (Bell, 1998; Lam, 2004; Tam and Kiang, 1992) have been extensively used to predict insolvency. The application of this tool to resolving applied problems and in particular economics is relatively recent, although there are some articles dating back to the 1940s. The types of problem to be resolved are easily adapted to neuronal network methodology principally because neuronal networks do not require certain statistical determinants to be fulfilled which are necessary in parametric models and because these methods are able to filter certain sounds generated by incomplete, erroneous or adulterated information.

The last generation of models that can be found in the literature is related to the use of stochastic variable models (also known as survival models). According to these models, a standard model for predicting insolvency over time assumes that a company becomes insolvent when its assets fall to a sufficiently low level in relation to its total liabilities. For example, the models of Black and Scholes (1973), Merton (1974), Fisher, Heinkel and Zechner (1989), and Leland (1994) hold that the process of evaluating an asset follows a geometric Brownian motion. In these models, a firm's conditional probability of bankruptcy is totally determined according to how far it is from defaulting, this being the standard number of deviations from the annual growth of the asset by which the asset level (or the forecasted horizon of the asset level at a given moment) exceeds the firm's obligations.

It seems natural to include the distance from bankruptcy as an additional covariable for predicting bankruptcy. Duffie et all (2007) used a standard structural model for bankruptcy of this kind to demonstrate that the distance from defaulting is not an exact measurement, which means that filtering methods must be applied. This is because the intensity of the default depends on the distance from default and also covaries with other variables that reveal the prevailing conditions in the firm that it is heading towards defaulting. That is, the financial health of a firm is influenced by multiple factors such as the company itself, the sector and the macrovariables, all of which influence the earnings and the debt of the company.

The oldest of this generation of models is that of Lane, Looney and Wansley (1986), who developed a prediction model for bank failures which used a Cox proportional hazards model of time to failure. Shumway (2001) used a discrete duration model with time-dependent covariables. Finally, Duffie et all (2007) used stochastic covariables in a multi-period model for predicting failures. The particularity of this model was incorporated dynamically into both the firm specific variables and the macroeconomic variables.

The usefulness of the models described is intimately related to the objective of the analyst. Thus, these models will obtain their objective if the only information that we want is the value of the "score" that determines whether a company is healthy or not to take a decision. However, if we want to investigate a firm's problems and how these have arisen, we need to go back to the ratios model.

Describing the strengths and weaknesses of these models shows us that there is area that is favourable for developing a scheme containing the good qualities of the two procedures described.

In any case, it should be taken into account that by its own nature economic economic-financial diagnosis is very subjective because of the existence of value judgements from experts, qualitative variables and relative quantification variables. These elements turn this into a very uncertain procedure in which the tools and processes based on traditional methods cannot be fully applied. The area is suitable for tools from fuzzy logic which can be used to deal with qualitative variables and enables value judgements to be modelled.

The present study aims to develop a model for making economic-financial diagnoses of companies through a simple cause-effect scheme that will simulate an analyst making a diagnosis.

Among the applications found in medicine, the studies of E. Sanchez (1979) stand out because they propose a method that allows us to relate the occurrences of symptoms to illnesses, meaning that once a patient's symptoms have been recorded, the doctor can relate these to possible illnesses. The research of E. Sánchez is bases on the use of models to resolve fuzzy binary equations and thus determine a certain matrix of medical knowledge which is incorporated into the relations between symptoms and illness, these relations being the basis of the medical diagnosis.

If we accept that there is a certain parallel between patient and company, E. Sánchez's proposal may be extended to the economic-financial diagnosis of companies. In essence, the nucleus of this development is based on determining a "matrix of economic-financial knowledge" that expresses relations between symptoms and causes. In this case, the company's symptoms can be expressed using ratios which, in short, are the indicators that all analysts look at when making predictions or recommendation. On the other hand, the company's illnesses will be the causes that will have to determine and that will generate anomalies in the company. The possibility of obtaining a matrix of economic-financial knowledge implies be able to determine the intensity of the symptoms selected and thus the degree of the problems or illnesses that are affecting the profitability and efficiency of companies. A version of this study has already been presented in Vigier and Terceño (2008)

Section 2 presents some of the tools from logic and fuzzy mathematics used in the study; in particular, the fuzzy binary relations are defined, as are the solutions to equations carried out using them. Section 3 presents the model that will be developed in the following section, which sets out the model of economic-financial diagnosis based on the resolution of fuzzy binary equations. The objective of this section is to discuss and determine the most efficient method for obtaining the matrix of economic-financial knowledge. This matrix is the basis of the model because it provides the basic information needed to carry out the diagnostic process. This section thus sets out: the alternative ways of formulating this model, how the relations of occurrence between symptom and cause are obtained, the problems of constructing functions of belonging and of adding relations of occurrence. Last of all we present our conclusions and bibliographic references.

2. Instruments in fuzzy mathematics

We will first present a set of concepts and tools from fuzzy mathematics that are necessary to be able to understand our diagnosis model, as well as those instruments used to develop it. More complete overviews of fuzzy subset theory can be found in Kaufman-Gil Aluja-Terceño (1994), Klir-Bo (1995) and Dubois and Prade (1980), among others.

2.1. Fuzzy binary relations

Def. 1 Fuzzy binary relation

A fuzzy binary relation between two non empty sets X and Y is a fuzzy set R of X x Y if:

$$R:\ X \times Y \rightarrow [0\,,\,1\,] \qquad \textbf{(1)}$$

Let $F(XxY) = \{R : X \times Y \rightarrow [0\,,\,1\,]\,\}$

The value taken R(x,y) is interpreted as the degree or intensity of the relation R between x and y.

As binary relations are a type of fuzzy set, it is easy to extend to binary relations the usual operations between fuzzy sets. However, there are other operations that are only meaningful for these relations; among them, the most notable are:

Def. 2 Inverse relation of R

Given $R \in F(XxY)$ fuzzy binary relations, the inverse relation of R is a relation T $\in F(YxX)$, $T = R^{-1}$, so that:

$$T: Y \times X \rightarrow [0, 1] \quad \textbf{(2)}$$
$$T(y,x) = R(x,y)$$

Def. 3 Max-min composition of two fuzzy relations R and S

Given $R \in F(XxY)$, $S \in F(YxZ)$, we define T = R o S, where $T \in F(XxZ)$

So that $T(x,z) = (R \circ S)(x,z) = \max_{y} \{ \min \{ R(x,y), S(y,z) \} \}$ **(3)**

Relations can be composed in other ways; however, we will restrict ourselves to the max-min composition, as this is the one used by Sánchez (1979) in his model.

Def. 4 Relation α between fuzzy binary relations

The α relation is defined from the operation α between pairs of real numbers, a, b, $\in [0,1]$:

$$\alpha(a, b) = 1 \quad \text{if } a \leq b \quad \textbf{(4)}$$
$$\alpha(a, b) = b \quad \text{if } a > b$$

For the sake of notation, we will write a α b, to indicate $\alpha(a, b) \in [0,1]$. Notice that α operator is the Godel implication.

Then, if $R \in F(XxY)$, $S \in F(YxZ)$, we define T = R α S, as:

$$T(x,z) = \min_{y} (R(x,y) \,\alpha\, S(y,z)) \quad \textbf{(5)}$$

As we will only consider binary relations between finite sets, we will be able to represent them as matrices. With these conditions, we can restate the above definitions as:

Let $X = \{ x_1, x_2, \ldots x_m \}$, $Y = \{ y_1, y_2, \ldots, y_n \}$, $Z = \{ z_1, z_2, \ldots z_p \}$ and $R \in F(XxY)$, $S \in F(YxZ)$, $T \in F(XxZ)$.

Then,

$$R = [r_{ij}], \text{ where } R(x_i, y_j) = r_{ij}, \quad \forall\ i \in \{1,\ldots,m\},\ j \in \{1,\ldots,n\} \quad \textbf{(6)}$$

And in a similar way we represent the matrices $S = [s_{jk}]$, and $T = [t_{ik}]$, where $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,p\}$

Then, if T = R o S:

$$t_{ik} = \max_{j} (\min (r_{ij}, s_{jk})), \quad \textbf{(7)}$$

And if T = R α S, then,

$$t_{ik} = \min_{j} [r_{ij} \,\alpha\, s_{jk}] \quad \textbf{(8)}$$

For the sake of clarity, we must note that in our notation, if $T=R^{-1}$ then $t_{ik}=r_{ki}$, that is, the matrix corresponding to relation R^{-1} is R's transposition matrix.

2.2. Solving equations with fuzzy binary relations

In this section we present the resolution of equations in fuzzy binary relations, based on the research and applications of E. Sanchez. Nevertheless, the most general presentations can be seen in Klir and Bo (1995). Given the fuzzy relations $R \in F(XxY)$, $S \in F(YxZ)$ and $T \in F(XxZ)$, the first problem which can be posed takes the form:

$$T = R \text{ o } S \qquad \textbf{(9)}$$

where "o" is a composition of two fuzzy relations R and S.

If R and S are given fuzzy relations, then the solution to the problem exists, is trivial, and is unique. Indeed, the problem arises when the unknown is R or S; in this case, neither the existence nor the uniqueness of a solution can be asserted. In this section we introduce a method for solving equations with fuzzy binary relations based on the research and applications of E. Sánchez (1979).

Let the sets $X = \{ x_1, x_2,...,x_m \}$,$Y = \{ y_1, y_2, ...,y_n\}$, and $Z = \{ z_1, z_2,....,z_p \}$ and the relations $R \in F(XxY)$, $S \in F(YxZ)$ be given. We have defined how to compute the max-min composition of these fuzzy binary relations.

$$T = R \text{ o } S \in F(XxZ)$$

The results known about the solution of inverse problems are:

(i) Given $R \in F(XxY)$, and $T \in F(XxZ)$, find, if it exists, a relation $S \in F(YxZ)$ so that:

$$R \text{ o } S = T \qquad \textbf{(10)}$$

(ii) Given $S \in F(YxZ)$, $T \in F(XxZ)$, find, if it exists, a relation $R \in F(XxY)$ so that:

$$R \text{ o } S = T. \qquad \textbf{(11)}$$

Theorem 1. If there exists a relation S so that R o S = T, the largest relation which is a solution to this equation is $S^* = R^{-1} \alpha\ T$.

Theorem 2. If there exists a relation R so that R o S = T, the largest relation which is a solution to this equation is $R^* = (S\ \alpha\ T^{-1})^{-1}$.

Sánchez discusses some limitations to these methods for solving equations, the main one being that the operations must conform a complete reticulate. It is also well known that equations (9) and (10) can have no solution. For example, a necessary condition for solving equation (9) is that each row in the matrix T should have at least one element in the corresponding row of the matrix R that is greater than all its elements.

3 Setting out the model of economic-financial diagnosis.

Following the models by Sánchez, our main goal from an analytical point of view is to determine the matrix R, which in this case will be a *matrix of economic-financial knowledge* to be represented as a fuzzy binary relation between symptoms and causes. Therefore, the elements of matrix R represent the degree to which the occurrence of a symptom implies the occurrence of a certain cause (disease). In many situations, the symptoms in economic-financial diagnosis can be represented as ratios, whereas the causes are the problems which generate the relative state of the symptoms used to diagnose them. However, in a clear analogy with medicine, at the time of diagnoses the causes are unknown, whereas the symptoms are known; thus, knowing the relative state of symptoms will serve to determine the relative state of causes.

The matrix of economic-financial knowledge, R, can be determined by evaluating the relative values of the symptoms for a great number of cases and combining them with the analyst's expertise and using the methods for solving equations in fuzzy binary relations.

In this way, we define S as the set of symptoms (where S is a classic set) composed by the diverse symptoms, $S_1, S_2, S_3,\ldots, S_n$, i.e., $S = \{ S_i \}$, where $i=1,2,\ldots,n$. On the other hand we define C as the set of causes (also a classic set), $C_1, C_2, C_3,\ldots C_p$, i.e., $C = \{ C_j \}$, where $j= 1,2,\ldots, p$. Similarly, we define a set T of years or periods during which we are able to identify symptoms and causes for a given firm, with $T_1, T_2, T_3,\ldots, T_t$, that is, $T = \{ T_k \}$, where $k=1,2,3,\ldots,t$. Finally, we define E as the set of firms being considered, which we represent as $E_1, E_2, E_3, \ldots, E_m$, i.e., $E = \{ E_h \}$, where $h=1,2,3,\ldots,m$.

The matrix R of economic-financial knowledge is supposed to be initially determined from a set E of firms. This way, through the analyst's knowledge and the information s/he possesses (either historical or prospective), R is determined as follows:

$$R \in F(S \times C)$$

where R, a matrix of order n x p, represents the fuzzy relation between the symptoms and the causes. Each element of matrix R represents to which degree (intensity level) a symptom S_i implies a cause C_j; and it is represented by means of a value r_{ij}, where $r_{ij} \in [0, 1]$.

Therefore, we can represent matrix R as:

$$R = [r_{ij}] \quad \text{with} \quad i = 1,\ldots,n \quad \text{and} \quad j = 1,\ldots,p \qquad \textbf{(12)}$$

The matrix R of economic-financial knowledge can be derived using two different methods:

1. By using the fuzzy relation between symptoms and causes for the m firms, at a given moment in time; that is, for a fixed T_k. This way a certain matrix R_k will be obtained.
2. By using the fuzzy relation between symptoms and causes for a given firm E_h taken from the set E. In this case, a matrix R_h will be obtained.

If method 1 is followed, if we use the t periods and can “aggregate” the t matrices R_k thus obtained to find a representative matrix of economic-financial knowledge. If method 2 is followed, a matrix R_h will be obtained and then, if we use the m firms from set E, we can aggregate the m matrices R_h corresponding to each firm to get, once again, another relation of economic-financial knowledge.

Estimating R from the R_h's has the advantage of making the model more stable and consistent with regard to the fuzzy relations, because all the information is taken from the same firm; on the other hand, when the matrix is derived by comparing firms, such stability and consistency may not be possible. Nevertheless, the

estimation of R from the R_h's, that is, from a single firm, is incomplete, because it is unlikely that a single firm will possess all the possible states of nature during the period under consideration in order to generate a matrix R with enough generality.
Moreover, a firm that is "healthy" within the time horizon under consideration (i.e. it exhibits a low intensity level of the symptoms) may generate low values for the r_{ij}'s and thus may not reflect the true relation between symptom and cause. This situation seems to be analogous to what happens in classic models for prediction of bankruptcy, when the discriminant function is estimated from sets of firms that are either bankrupted or not. That is, if we do not include some firms which are in trouble it will be very difficult (even impossible) to determine the relations between symptoms and causes.
The estimation of R using the R_k's has the advantage that, as information from many firms is taken into consideration, all possible states of nature can be incorporated into the model, thus allowing the matrix of economic-financial knowledge to be sufficiently general. However, estimating R from the R_k's can pose some inconsistency problems, chiefly due to the instability of some relations generated from firms (or from a single firm) that are not following the usual causality of most firms, and are thus deteriorating the true relation. However, there are techniques, such as filtering, that allow this problem to be alleviated or even eliminated.
Moreover, the estimation of R from the R_k's is done at a given moment in time, so it is unlikely to capture the time effects that can occur between symptoms and causes from one period to another. Similarly, the same can happen if relations are variable, that is, if the causal relations between symptoms and causes increase or decrease due to changes in the accepted economic standards. Nevertheless, as we will see, this problem can be solved by an appropriate choice of method for aggregating the matrices.

4. Determining the matrix R of economic-financial knowledge for a given year T_k of the time horizon

As we have noted in the preceding section, our goal is to construct a matrix R of economic-financial knowledge because this is the main element on which the diagnosis of the firm's situation will be based. To do this, we will analyze the data from the m firms of our sample corresponding to a certain year T_k, because, as we have already noted, this is the most stable choice.

4.1. Construction of the membership functions

In this first stage, once we have selected the symptoms, the causes, the firms, and the periods or years to be observed, the next step is to construct the membership functions (both for symptoms and causes), because the incidence relations will be established through these membership functions.

4.1.1 The membership function for symptoms

The occurrence of symptoms, which are objectively measurable, allows us to use certain methods to the construct membership functions (Xiadong, 1998; and Zhang, 1993) that can be related to the idea of relative frequency.

In this way, having determined the sets of firms and symptoms, we measure the nominal level of each symptom S_i at each of the m firms; then we use this measure to construct the matrix SE, where SE = [se_{hi}] is a matrix of order m x n (m firms by n symptoms) showing the nominal level of each symptom:

$$SE = \begin{array}{c|cccccc|} & S_1 & S_2 & --- & S_i & --- & S_n \\ E_1 & se_{11} & se_{12} & --- & se_{1i} & --- & se_{1n} \\ E_2 & se_{21} & se_{22} & --- & se_{2i} & --- & se_{2n} \\ --- & --- & --- & --- & --- & --- & --- \\ E_h & se_{h1} & se_{h2} & --- & se_{hi} & --- & se_{hn} \\ --- & --- & --- & --- & --- & --- & --- \\ E_m & se_{m1} & se_{m2} & --- & se_{mi} & --- & se_{mn} \end{array} \qquad \textbf{(13)}$$

Having the nominal levels of the symptoms at each firm (the se_{hi}'s) we then execute the following procedure to obtain the membership function:

1. First, we determine the sign of the property p_i with respect to symptom S_i. That is, if the symptom S_i is solvency, its sign is positive, as the higher the solvency, the better the financial condition of the firm. Therefore, the lowest membership level (that is, when a firm shows a low intensity level of the symptom) corresponds to the most solvent firm; in other words, the firm has a low level of solvency problems. Determining the sign of the diverse properties serves to establish a complete ordering on each column of matrix SE.
2. Then, we establish a complete ordering on each symptom S_i according to the sign determined in 1). That is, we arrange the se_{hi}'s for each i, so that if the sign of the property is positive, the elements are ordered from highest to lowest; and if the sign of the property is negative, the elements are ordered from lowest to highest.
3. Once the elements are ordered, we estimate the incidence level of symptom S_i at firm E_h, μ_{Si} (se_{hi}) as the ratio between the ordinal —within the order established in 2— of symptom S_i at that firm and the cardinal of the set, that is, the quantity of firms. That is:

$$\mu_{si}(se_{hi}) = q_{hi} = \frac{|se_{hi}|}{|E_h|} = \frac{|se_{hi}|}{m} \qquad \textbf{(14)}$$

4. After repeating this procedure for every symptom, a matrix is obtained that we will denote Q=[q_{hi}]. The order of this matrix is m x n (m firms by n symptoms) and it shows the intensity levels of each symptom at each firm.

$$Q = \begin{array}{c|cccccc|}
 & S_1 & S_2 & --- & S_i & --- & S_n \\
\hline
E_1 & q_{11} & q_{12} & --- & q_{1i} & --- & q_{1n} \\
E_2 & q_{21} & q_{22} & --- & q_{2i} & --- & q_{2n} \\
--- & --- & --- & --- & --- & --- & --- \\
E_h & q_{h1} & q_{h2} & --- & q_{hi} & --- & q_{hn} \\
--- & --- & --- & --- & --- & --- & --- \\
E_m & q_{m1} & q_{m2} & --- & q_{mi} & --- & q_{mn} \\
\hline
\end{array} \quad (15)$$

4.1.2. The membership function for causes

The main difference between causes and symptoms is that the former may not always be objectively measurable. Consequently, the construction of the membership functions must rely on other methods suitable for working with qualitative variables. Therefore, we take the set of causes C = { C_j } with j = 1,2,…,p, and we split it into two subsets, C_j^S with j=1,2,……s, corresponding to the s subjectively measurable causes, and C_j^O with j=s+1,s+2,……p, corresponding to the p – s objectively measurable causes.

We define CE as the matrix showing the nominal level (whether objectively measured or subjectively measured) of each cause C_j at each firm E_h. That is, the row h of matrix CE represents the nominal level of the p causes considered at the firm E_h. The order of CE is m x p (m firms by p causes)

Now, we can split the matrix CE into two submatrices according to whether the causes are measured objectively or subjectively: [CE] = [[CE^S] [CE^O].]

The order of the submatrix CE^S is m x s (m firms by s subjectively-measurable causes), while the order of the matrix CE^O is m x (p – s) (m firms by (p-s) objectively-measurable causes); we can represent these matrices as follows:

$$CE^S = \begin{array}{c|ccccc|}
 & C_1^S & C_2^S & C_3^S & --- & C_s^S \\
\hline
E_1 & ce_{11} & ce_{12} & ce_{13} & --- & ce_{1s} \\
E_2 & ce_{21} & ce_{22} & ce_{23} & --- & ce_{2s} \\
--- & --- & --- & --- & --- & --- \\
E_h & ce_{h1} & ce_{h2} & ce_{h3} & --- & ce_{hs} \\
--- & --- & --- & --- & --- & --- \\
E_m & ce_{m1} & ce_{m2} & ce_{m3} & --- & ce_{ms} \\
\hline
\end{array} \quad (16)$$

C_{s+1}^O C_{s+2}^O C_{s+3}^O --- C_p^O

	E_1	**$ce_{1\ s+1}$**	**$ce_{1\ s+2}$**	**$ce_{1\ s+3}$**	**---**	**ce_{1p}**	
	E_2	**$ce_{2\ s+1}$**	**$ce_{2\ s+2}$**	**$ce_{2\ s+3}$**	**---**	**$ce_{2\ p}$**	
CE^O =	**---**	**---**	**---**	**---**	**---**	**---**	**(17)**
	E_h	**$ce_{h\ s+1}$**	**$ce_{h\ s+2}$**	**$ce_{\ h\text{-}s+3}$**	**---**	**ce_{hp}**	
	---	**---**	**---**	**---**	**---**	**---**	
	E_m	**$ce_{m\ s+1}$**	**$ce_{m\ s+2}$**	**$ce_{m\text{-}s+3}$**	**---**	**ce_{mp}**	

By arranging both matrices [CE^S] [CE^O] one after the other we get the matrix [CE].

A) Membership function for the subjectively-measurable causes at each firm
When causes are subjectively measurable, a group of experts is invoked for each firm E_h, $G^h = \{G_e^h\}$ where e=1,2,...,g, to provide a valuation of the intensity level of each cause C_j at firm E_h.
Depending on the method used by each expert, s/he will provide a valuation as one of these:

1. linguistic labels
2. intervals $\subset [0,1]$
3. simple evaluation

Once we are given the evaluation, the next step is to perform the necessary conversion in order to get a single value within the interval [0,1], that is, a simple evaluation. If the valuation is given in the form of linguistic labels, its equivalent valuation in a previously determined scale [0,1] shall be found. If the valuation is given as intervals, the midpoint of each interval will be taken.
So we have "g" judgments for each cause and each firm. We will denote each judgment as ce_{ej}^h, that is, the judgment of expert G_e^h regarding the occurrence of cause C_j at firm E_h.
To estimate the value of the intensity of a cause C_j at a firm E_h, which we denote ce_j^h, we will take a weighted average of the judgments of the experts. Then, if we use ge_e^h to denote the weight assigned to expert G_e^h at firm h, the value of ce_j^h will be computed as follows:

$$ce_j^h = \frac{\sum_{e=1}^{g} ce_{ej}^h . ge_e^h}{g} \qquad \textbf{(18)}$$

where: $\sum_{e=1}^{g} ge_e^h = 1$. Clearly, if all experts are assigned the same weight, then $ge_e^h = \frac{1}{g}$ $\forall$e.

Once this procedure has been repeated for each firm and every subjectively-measurable cause, the construction of the submatrix CE^S is straightforward, as the elements ce_{hi}, by construction, take values between [0,1]; thus they correspond to the incidence level of cause C_j at firm E_h.

B) Membership function for the objectively-measurable causes at each firm

The belonging function of the objectively measurable causes is estimated in the same way as the symptoms. For each one of the m companies, the nominal level is evaluated for each cause C_j, and this is used to construct the submatrix CE^O. In this case, this obtains μ^O_{Cj} (ce_{hj}), which are the m companies' levels of belonging of the objectively measurable causes (p – s), and this provides the submatrix CE^S.

C) Membership function for the causes at each firm

Having obtained the submatrices [CE^S] and [CE^O], we can build the matrix P to determine the extent (intensity degree) of each cause at each one of the m firms. To do that, we just let ce^h_j be p_{hj} for j=1........s and $\mu^o_{Cj}(ce_{hj})$ be p_{hj} for j=(s+1)...........p. That is:

P = **(19)**

	C_1	C_2	---	C_j	---	C_p
E_1	p_{11}	p_{12}	---	p_{1j}	---	p_{1p}
E_2	p_{21}	p_{22}	---	p_{2j}	---	p_{2p}
---	---	---	---	---	---	---
E_h	p_{h1}	p_{h2}	---	p_{hj}	---	p_{hp}
---	---	---	---	---	---	---
E_m	p_{m1}	p_{m2}	---	p_{mj}	---	p_{mp}

4.2. Determining the matrix R for a specific year

Once the matrices Q and P are constructed to measure, respectively, the occurrence of symptoms and of causes at each one of the m firms for a given year, the next step is to find the matrix R, whose elements r_{ij} measure the incidence intensity of a symptom S_i with respect to a cause C_j. To find the matrix R we will operate on the equation:

$$P = Q \circ R \tag{20}$$

Because P and Q are known and R is unknown in this equation, we use the method for solving fuzzy equations developed by Sánchez (1979), which we introduced in Section 3, and we find that the largest solution is:

$$R = Q^{-1}\ \alpha\ P, \quad \text{where } Q^{-1} = [\,q_{hi}\,]^{-1} = [\,q_{ih}\,] \tag{21}$$

That is:

$$R = Q^{-1}\ \alpha\ P \quad = [\,q_{ih}\,]\ \alpha\ [\,p_{hj}\,] = [r_{ij}] \tag{22}$$

where, following Sánchez (1979), the operation $R = Q^{-1}\ \alpha\ P$ is defined as:

$$[\, r_{ij} \,] = \bigwedge_{h} [\, q_{ih} \;\alpha\; p_{hj} \,] \tag{23}$$

where

$$q_{ih} \;\alpha\; p_{hj} = \begin{cases} 1 & \text{si } q_{ih} \leq p_{hj} \\ p_{hj} & \text{si } q_{ih} > p_{hj} \end{cases} \tag{24}$$

In this way, we get the matrix R of economic-financial knowledge.

4.3. Determining the matrix $\mathfrak{R}$ of economic-financial knowledge, by aggregation of the t matrices R*

Having found R*, that is, the matrix R for a given year, the next step is the aggregation of the different matrices R* computed for each period T_k. In this way we will obtain a matrix of economic-financial knowledge that would be representative of all firms and of every year under consideration. To carry out this aggregation, we propose using the operator of the generalized averages because this is the broadest possible operator. It should be mentioned that when selecting the operator, it is possible to analyze the existence of the matrices in the series for each year (for further discussion of this, see Vigier and Terceño, 2008).

Therefore, the operator for aggregation proposed is:

$$h_{\varphi}((r_{ij})_{1,} (r_{ij})_{2,} \ldots\ldots (r_{ij})_{k} \ldots\ldots (r_{ij})_{t}) = \left[\frac{\sum_{k=1}^{t} \left((r_{ij})_{k}\right)^{\varphi}}{t} \right]^{\frac{1}{\varphi}} \tag{25}$$

Having chosen the best operator for aggregation, we then aggregate the t matrices R* and thus get the matrix $\mathfrak{R}$ of economic-financial knowledge:

$$\mathfrak{R} = \begin{array}{c|cccccc|} & C_1 & C_2 & --- & C_j & --- & C_P \\ \hline S_1 & r_{11} & r_{12} & --- & r_{1j} & --- & r_{1P} \\ S_2 & r_{21} & r_{22} & --- & r_{2j} & --- & r_{2P} \\ --- & --- & --- & --- & --- & --- & --- \\ S_i & r_{i1} & r_{i2} & --- & r_{ij} & --- & r_{ip} \\ --- & --- & --- & --- & --- & --- & --- \\ S_n & r_{n1} & r_{n2} & --- & r_{nj} & --- & r_{np} \\ \hline \end{array} \tag{26}$$

5. Conclusions

The Introduction of this article clearly shows that the traditional analysis in which economic-financial diagnosis can be classified uses two types of model: ratios models and those that come under the heading prediction of business insolvency. As has been mentioned, both models have particular strengths and weaknesses. The strength of the ratios model is that it identifies the existence of symptoms and causes within the problem of the diagnosis, which lead to the obvious assumption that if a symptom adopts a value that is not desired, there must be a cause for this. However, the weakness of this model is that has a very low level of formalization, which means that is of relative value to academics. Likewise, these models, which are based on the "knowledge" of the analysts, possess a certain subjectivity which obviously originates in these analysts own experience. Furthermore, the strength of the models used for the prediction of insolvency is that most of them have been developed with an important level of formalization. This is the reason why academics are so interested in them and why so many articles can be found about them in the literature. However, the weakness of these models is that most of them "separate" or rank companies into healthy and unhealthy without ever determining the origin of the companies' state of health. That is, the reasons why a company becomes insolvent are never analyzed. Having looked at the problem from this angle, we are able to propose an alternative that combines the strengths of both approaches.

The economic-financial model proposed in this study combines the strengths of both models and uses a cause effect scheme to simulate a diagnosis carried out by an analyst. Elie Sanchez's (1979) model is based on resolving fuzzy binary equations and this idea can be used to identify the relations of occurrence between symptoms and illnesses (causes), meaning that, if the symptoms of a company's (ratios) relations of occurrence are known, the causes can be determined.

To summarize, the proposal allows us to construct a "matrix of economic-financial knowledge" which uses the intensity of the symptoms to determine the extent of the principal problems or illnesses. The symptoms correspond to the ratios and the illnesses generate anomalies and are the causes to be determined. The matrix of economic-financial knowledge, R, represents a fuzzy binary relation between causes and symptoms; thus, the elements of the matrix R represent to what extent the presence of a symptom signifies the existence of a specific cause (illness).

The virtue of this model is that, when the analysts make their diagnoses, the causes are unknown and whereas the symptoms are known, meaning that the matrix of economic-financial knowledge can be used to determine the relative state of the symptoms and which in turn can be used to determine the relative state of the causes. This is called a economic-financial "forecast"; that is, if the symptoms of a company are known and represented by ratios, then the matrix of economic-financial knowledge can be used to forecast the level of occurrence of each defined cause. Therefore it is possible to determine the "illnesses" of a company and the reasons behind these illnesses. Furthermore, it allows predictions to be made sufficienlty early so that the necessary decisions can be taken to mitigate problems or take advantage of benficial situations.

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